Consumption risk sharing with private information when earnings are persistent*

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Abstract
In this paper, we investigate the implications of a model of consumption risk sharing where infinitely-lived households face persistent idiosyncratic shocks to earnings, where the realizations of these shocks are private information and where there is limited enforcement of risk-sharing contracts. We compare and contrast these implications with the implications of other models of consumption risk sharing and with the data. We also investigate the implied effects of various changes in the environment in the context of our models and other models. We find that, in contrast to a model where the only friction is limited enforcement, our model has implications that are similar to those of a Bewley model and therefore broadly consistent with empirical observations. However, the implied effects of changes in the environment are noticeably different in our model compared to a Bewley model.

1 Introduction
This paper studies the quantitative implications of a general equilibrium model of consumption risk sharing where earnings are private information and there is limited enforcement of contracts in the sense that consumers can walk away from a dynamic contract (with a

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one-period lag), giving rise to period-by-period participation constraints as well as truth-telling constraints. Our main finding is that the testable implications of our model are similar—though not identical—to those of a Bewley (1977) model. This contrasts with the implications of models where the only friction is limited enforcement, as in However, our model has quite different implications from a Bewley (1977) model when we consider the response to an intervention such as an increase in idiosyncratic risk or the introduction of a compulsory social insurance scheme.

The spirit of our exercise is quite similar to that of Krueger and Perri (2006) and Krueger and Perri (2011). The question is: what if we model explicitly the friction that leads to imperfect risk sharing, as opposed to simply assuming that markets are exogenously incomplete; how does that affect the implied effects of hypothetical interventions, either by policy or by nature? It seems to us that this question needs to be answered in the context of a model whose implications are broadly in line with the facts. In this respect, as documented by Broer (2011), a model with limited enforcement alone, such as that of Krueger and Perri (2006), falls short. Specifically, such models imply a much stronger left skew of log consumption than of log earnings. This feature comes from the fact that, in these models, consumption always drifts down when the participation constraint does not bind, and then jumps up when it does bind, which happens whenever earnings increase sufficiently. This gives rise to an extreme and very counterfactual degree of left skewness in both consumption and log consumption.

In contrast, our model implies that consumption will gradually drift up when income is high as well as gradually drift down when income is low. This means that our model avoids the counterfactual implications of Krueger and Perri (2006) or Krueger and Uhlig (2006). Indeed, the implications of our model are quite similar, though not identical, to those of Huggett (1993) or Huggett (1997).

Our paper makes three sets of contributions. The first is to characterize, from a theoretical point of view, the properties of the equilibrium in our model. In this respect, we build on the results of Fernandes and Phelan (2000a) but go several steps further. Specifically, we
show that when the period utility function is bounded above by zero\(^1\) then, in the absence of period-by-period participation constraints, the feasible state space (the set of vectors of promised utilities for each type) is a convex cone, and there are very simple and explicit expressions for the boundaries of this cone.

Moreover, in contrast to Fernandes and Phelan (2000a) who impose exogenous upper and lower bounds on consumption, we impose period-by-period participation constraints which endogenize bounds on consumption. In the presence of these constraints, we provide a near-analytic approach to characterizing the lower bound of the feasible set; the upper bound, meanwhile, remains a straight line. This lower bound turns out to be the convex envelope of a function that has a countable number of differentiable segments, and for each segment the function values can be found by solving a non-linear equation at each point in the domain.

The computation has a certain recursivity to it in that the first segment can be computed without any knowledge of any of the other segments, the second segment can be computed using only the first etc. Meanwhile, in the case where earnings follow a two-state process, we show that the lower left-hand corner of the feasible set is just the point \((v_\text{AUT}^\ell, v_\text{AUT}^h)\), where \(v_\text{AUT}^i\) is the continuation utility for an agent who opts out of the dynamic contract and whose previous earnings level was \(i \in \{\ell, h\}\).\(^2\) This approach obviates the need to apply a fully-fledged Abreu et al. (1990) iterative approach.

The second, quantitative contribution consists in parameterizing our model based on empirical data, computing an approximation of the equilibrium, and then investigating the implications for the distribution of consumption. As in Bewley (1977) models, but in stark contrast to limited enforcement models such as that in Krueger and Uhlig (2006), consumption not only drifts down when earnings are low, it drifts up (rather than jumps up in one step) when earnings are high. In this sense it provides a more rigorous foundation of this property than models in the tradition of Bewley (1977) do. Though our model is not able to deliver log-normality of consumption, it avoids the counterfactual implication that log consumption is much more skewed to the left than log earnings are.

\(^1\)This is the case with CARA preferences and also CRRA preferences provided that the risk aversion parameter is strictly greater than 1.

\(^2\)This easily generalizes to an \(n\)-state earnings process; see Section 2.
Finally, our third contribution is to examine the implications in the context of our framework of various interventions. In particular, we look at an increase in idiosyncratic risk, the introduction (or strengthening) of a compulsory social insurance scheme and the introduction of a capital income tax. We find that [[to be written]].

Apart from the papers mentioned already, our work is related to that of Farhi and Werning (2011), who consider a dynamic Mirrlees economy with hidden actions as well as hidden information. The contrast with our work is that our model features hidden information but no hidden action, unless savings or consumption are considered actions. Also, they assume that productivity follows a random walk rather than a stationary Markov process and assume finite rather than infinite lives. Moreover, their exercise is to characterize optimal taxation rather than to compare their model’s implication with facts. Other recent related papers include Ales and Maziero (2009), who provide rigorous foundations for the riskless one-period bond environment of Bewley (1977) in the context of a private information model where agents are able to deal with multiple insurance providers who remain in the dark about the agent’s dealing with other providers. In our work, we assume that agents can only deal with one insurance provider at a time.

The paper is organized as follows. Section 2 lays out the model framework and establishes some important theoretical results. In Section 3, we talk about our data sources and our approach to modelling an earnings process. Section 4 presents a very preliminary confrontation between our model and the data, comparing and contrasting the implications of not just our model but also some other models of consumption risk sharing with some moments of the data. Section 5 concludes.

## 2 The Model Environment

Each household lives forever in discrete time. Endowments of dated, perishable goods follow a finite-state Markov chain with state space \( \{y^1, y^2, \ldots, y^N\} \) where \( y^1 < y^2 < \ldots < y^N \) and we simplify the analysis by assuming that the shocks are equispaced. The probability transition function is denoted by \( \pi(i|j) \) and the probability of a sequence of shocks \( h^t = (i_0, i_1, \ldots, i_t) \)
given an initial shock level \( i_{-1} = j \) is denoted by \( \pi^t(h^t|j) \). The set of possible sequences \( h^t \) is denoted by \( \mathcal{N}^t \). The initial “seed” value \( i_{-1} \) is given and known to everyone.

Each period, the agents report their current shock. The report is observed both by the government and the insurance providers. The government taxes income and consumption as follows. It chooses an income tax schedule \( s^y = \{ s^y_t(h^t) \}_{t=0}^{\infty} \), where \( s^y_t(h^t) \) is the tax that must be paid by an agent with reported history \( h^t \). In addition, the government imposes a flat tax on consumption \( s^c = \{ s^c_t \}_{t=0}^{\infty} \), where \( s^c_t \) is the tax rate in period \( t \). Throughout the paper we take the government policy \( s = (s^y, s^c) \) as exogenously given.

Insurance providers (principals) sign, at time \( t = 0 \), mutually agreeable risk sharing contracts with households (agents). They design a transfer scheme \( \tau = \{ \tau_t(h^t) \}_{t=0}^{\infty} \). The agents cannot save on their own, and so the principal’s transfer, together with government policies, completely determines the consumption of an agent that report her shock truthfully:

\[
c_t(h^{t-1}, i) = (1 - s^c_t) \left[ y^i - s^y_t(h^{t-1}, i) + \tau_t(h^{t-1}, i) \right].
\]

Note that the government is subject to the same private information friction as are the insurance providers. The only difference, as shall be clear later, is that the agents can default on their private contract, but they cannot default on their tax obligations. That is, the government policy will affect the value of autarky.

Insurance providers evaluate the transfer scheme according to the cost function

\[
C(\tau|i_{-1}) = \sum_{t=0}^{\infty} \sum_{N^{t+1}} R^{-t} \pi^t(h^t|i_{-1}),
\]

where \( R \) is the interest rate. The households rank transfer schemes \( \tau \) according to the lifetime utility function

\[
U(\tau, s|i_{-1}) = \sum_{t=0}^{\infty} \sum_{N^{t+1}} \beta^t U \left( c_t(h^t) \right) \pi^t(h^t|i_{-1}).
\]

where \( U : \mathbb{R}_+ \to \mathbb{R}_- \) is increasing, differentiable, and is such that \( U(c) < 0 \) for all \( c \geq 0 \) and
lim_{c \to \infty} U(c) = 0.

Without loss of generality, it is assumed that the transfer scheme induces agents to tell the truth. As shown by Fernandes and Phelan (2000b), one can equivalently impose the following temporary incentive constraints: for all histories \( h^{t-1} \in \mathcal{N}^{t-1} \) and all \( i, j \).

\[
U \left( c_t(h^{t-1}, i) \right) + \beta U \left( \tau(h^{t-1}, i), s(h^{t-1}, i) | i \right) \\
\geq U \left( c_t(h^{t-1}, j) + (1 - s^*_c)(y^j - y^i) \right) + \beta U \left( \tau(h^{t-1}, j), s(h^{t-1}, j) | i \right),
\]

where \( \tau(h^{t-1}, i) := \{ \tau_{t+j}(h^{t-1}, i, \tilde{h}^j) \}_{j=0}^\infty \) where \( \tilde{h}^j \in \mathcal{N}^j \) is a continuation of the transfer scheme after history \( h^{t-1} \), similarly for \( s(h^{t-1}, i) \), and so \( U(\tau(h^t), s(h^t) | i) \) is a continuation lifetime utility for someone who reported \( h_t \) and had a last period shock \( i \).

It is well known from the work of Thomas and Worrall (1990) and Atkeson and Lucas (1992) that, under full commitment, consumption converges over time to any exogenously imposed lower bound with probability one. To avoid this unpalatable result, we retain full commitment on the part of the principal but drop it for the agent. To prevent the agents from walking away from the contract, the transfers have to satisfy the limited commitment constraint:

\[
U(\tau(h^{t-1}), s(h^{t-1}) | j) \geq V^{AUT}(s(h^{t-1}) | j) \quad \forall j, \forall h^{t-1} \in \mathcal{N}^t,
\]

where \( V^{AUT}(s | j) \) is the expected value of autarky if the last period shock was \( y^j \). The specific value of autarky will be determined later. It suffices to say now, that it depends on the government policy \( s \), and is history dependent if the income policy is history dependent as well. That is, an individual can walk away from the principal, but cannot walk away from the government. Also, note that incentive compatibility does not necessarily apply in autarky, and the deviating agents may thus misreport their type.\(^3\) Finally, note the specific timing of moves defined by our limited commitment assumption: The agents are assumed to default before the value of the current shock is known. Note also that government taxes

\(^3\)One could extend the revelation principle to extend the off-equilibrium autarkic allocations as well, if the tax policy in autarky were allowed to be different from the non-autarkic tax policy.
are imposed on the autarchic allocations as well.

At the beginning of period $t$, the agent observes period $t$ earnings, makes a report to the principal, and receives (or pays) the appropriate transfer from (or to) the principal. At the end of the period, before knowing the next period’s realization of earnings, the agent may choose to opt out of the contract and live in autarky as of the next period.

### 2.1 The Effects of Government Policies

Consumption taxes and income taxes have a very different impact on the efficient allocations. In particular, the role of income taxes is very limited: they potentially matter only through the value of autarky:

**Proposition 1** Suppose that $V^{AUT} = -\inf$. If $\tau$ is optimal given income tax policy $s^y$ then

$$\tilde{\tau} = \tau - s^y + \tilde{s}^u$$

is optimal given income tax policy $\tilde{s}^y$.

**Proof.** The consumption under the income tax policy $\tilde{s}^y$ is

$$\tilde{c}_i(h^{t-1}, i) = (1 - s^c_i) \left[ y^i - \tilde{s}^y_t(h^{t-1}, i) + \tilde{\tau}_i(h^{t-1}, i) \right]$$

$$= (1 - s^c_i) \left[ y^i - s^y_t(h^{t-1}, i) + \tau_i(h^{t-1}, i) \right]$$

$$= c_i(h^{t-1}, i).$$

Thus, the agents rank $\tau$ under $s^y$ identically to $\tilde{\tau}$ under $\tilde{s}^y$. The planner’s costs are

$$C(\tilde{\tau}|i_{-1}) = \sum_{t=0}^{\infty} \sum_{N_{t+1}} R^{-t} \tilde{\tau}_i(h^t) \pi^t(h^t|i_{-1})$$

$$= C(\tau|i_{-1}) + \sum_{t=0}^{\infty} \sum_{N_{t+1}} R^{-t}(1 - s^c_i) \left( \tilde{s}^y_t(h^t) - s^y_t(h^t) \right) \pi^t(h^t|i_{-1}).$$

The last term is exogenous, and since $s^y$ does not enter the incentive constraint (2) directly and the promise keeping constraint (3) is slack, the principal ranks $\tau$ identically to $\tilde{\tau}$. ■
If, in addition, the income tax policy is history independent then the result can be strengthened. Variations in the income tax that do not change the minimum tax liability do not matter even for any autarky value:

**Proposition 2** Suppose that $s^y_t(h^{t-1}, i)$ is independent of $h^{t-1}$. Let $\tilde{s}^y$ be another history independent income tax policy satisfying

$$\min_i s^y_t(i) = \min_i \tilde{s}^y_t(i) \quad \forall t \geq 0.$$ 

If $\tau$ is optimal given $s^y$ then $\tilde{\tau} = \tau - s^y + \tilde{s}^y$ is optimal given $\tilde{s}^y$.

**Proof.** Since the income tax policy is independent of history, the agents in autarky maximize utility by minimizing tax liabilities in every state of the world. The consumption in autarky is therefore

$$\tilde{c}^{AUT}(h^t, i) = \max_j (1 - s^c_t) \left[ y^i - \tilde{s}^y_t(j) \right]$$

$$= \max_j (1 - s^c_t) \left[ y^i - s^y_t(j) \right]$$

$$= c^{AUT}(h^t, i),$$

since $\min_i s^y_t(i) = \min_i \tilde{s}^y_t(i)$. Thus, the value of autarky is the same under both policies, $V^{AUT}(s|i) = V^{AUT}(\tilde{s}|i)$. The rest of the proof is analogous to the proof of Proposition 1. ■

In general, if the value of autarky is finite and the tax policies are history dependent, the income tax policy will affect the value of autarky, and so the efficient allocations. This is similar to Krueger and Perri (2011). A more progressive income tax increases the value of autarky, and reduces the extent of private risk sharing, and the overall effect can go both ways. However, as shown in Proposition 1, if the tax policy is history independent then the results differ from Krueger and Perri (2011). In an environment with private information the income tax policy affects the value of autarky only through the value of the minimum tax liability. In Krueger and Perri (2011), the whole history independent income tax matters.
In contrast to income tax, the consumption tax matters in two different ways. It affects the value of autarky as well, but it also affects the incentive constraint directly. As seen from the right-hand side of (2), any hidden earnings are hit by a consumption tax before being consumed, and this reduces the incentives to deviate.

2.2 A Relaxed Problem

Having defined the environment, we now want to construct a state space that will render the cost minimization problem of the principal recursive. Fernandes and Phelan (2000a) establish that a sufficient state vector is given by the lifetime utilities as of the current period for each possible previous period earnings level, together with the agent’s report of what the last period’s earnings were. This means that there are $N$ continuous state variables and one discrete state variable.

Dealing with many continuous state variables is typically very computationally costly, and we therefore examine an alternative route by analyzing a relaxed problem, where only local one-step-down incentive compatibility constraints among all those in (2) are assumed to bind: for all $h^{t-1} \in N^t$ we impose, for all $i = 2, 3, \ldots, N$:

$$U(c_t(h^{t-1}, i)) + \beta U(\tau(h^{t-1}, i), s(h^{t-1}, i)|i)$$
$$\geq U((1 - s_t^i)(y^i - y^{i-1}) + c_t(h^{t-1}, i-1)) + \beta U(\tau(h^{t-1}, i-1), s(h^{t-1}, i-1)|i),$$  \hspace{1cm} (4)

and that all the other constraints are slack. A relaxed problem also assumes that the lifetime utility of the truth-teller and one-step deviator cannot exceed the autarkic value:

$$U(\tau(h^{t-1}, i), s(h^{t-1}, i)|i) \geq V^{\text{AUT}}(s(h^{t-1}, i)|i)$$  \hspace{1cm} (5)
$$U(\tau(h^{t-1}, i), s(h^{t-1}, i)|i+1) \geq V^{\text{AUT}}(s(h^{t-1}, i)|i+1).$$  \hspace{1cm} (6)

\footnote{For details, see Appendix B.}

\footnote{In Section 4, we confine our attention to the two-state case, which may appear to render the relaxed approach irrelevant. That isn’t quite the case, however, because if the agent’s report of last period’s earnings was high, there is only a single continuous state variable as opposed to two. In any case, our analysis of the relaxed problem lays the conceptual groundwork for allowing for more than two states.}
The relaxed problem has an advantage that its recursive formulation involves only two continuous dimensions, one for the truth-teller, and one for an agent whose shock is one step higher (since only this agent could have reported the same shock as the truth-teller). Anticipating the recursive formulation, we need to define what the state space is. We begin with the following definitions of promised utility \( v \) and threat utility \( \hat{v} \).

\[
v = U(\tau, s|i_-)
\]  \hspace{1cm} (7)

\[
\hat{v} = U(\tau, s|i_- + 1)
\]  \hspace{1cm} (8)

We are now in a position to define the set of promised and threat utilities that can be implemented by some transfer scheme.

**Definition 1** For each \( i_- = 1, 2, \ldots, N - 1 \), pair of utilities \((v, \hat{v})\) is said to be feasible if there exists a transfer scheme \( \tau \) such that Equations (4)-(8) hold. For \( i_- = N \), a promised utility \( v \) is said to be feasible if there exists a transfer scheme \( \tau \) such that Equations (4)-(7) hold.

For convenience we define \( V^*_j \), for \( j = 1, 2, \ldots, N - 1 \), as the set of feasible pairs \((v, \hat{v})\) when \( i_- = j \) and \( V^*_N \) as the set of feasible promised utilities \( v \) when \( i_- = N \). The entire profile (\( N \)-tuple) of feasible sets \( V^* \) is defined as follows.

\[
V^* = (V^*_1, V^*_2, \ldots, V^*_N).
\]

This \( N \)-tuple of sets \( V^* \) has a recursive structure in the style of Abreu et al. (1990), as demonstrated below.

### 2.3 Recursive Formulation of the State Space

In order to provide a recursive representation of the optimal contracting problem, we will simplify government policies as follows. We will assume that income tax is history and time independent. Given Proposition 2, we can restrict attention to a constant lump-sum income
tax (or, more plausibly, transfer) $s^y$. We will also assume that the consumption tax is time independent as well, and will be denoted by $s^c$.

We also introduce, as a state variable, the promised utility assured to an agent who was truthful in the previous period, denoted by $v$. We also need to keep track of the lifetime utility (as of the present period) promised to the agent who reported a shock $i_-$ last period, but in fact had a shock $i_- + 1$, for $i_- = 1, \ldots, N - 1$ (a “liar”). We denote the liar’s lifetime utility by $\hat{v}$. Finally, we need to keep track of the agent’s report $i_-$ itself.

Define an allocation rule by $(u, w, \hat{w}) = \{u_i, w_i, \hat{w}_i\}_{i=1}^N$, where $u_i = U((1 - s^c)(y^i - s^y + \tau_i))$ is the current utility of $i_-$ type agent who truthfully reports her shock, $\tau_i$ is the transfer from the principal to the agent, $w_i$ is the truthteller’s continuation utility (as of the next period), and $\hat{w}_{i-1}$ is the continuation utility of a current type $i$ who reports $i - 1$. By definition, $w_i$ and $\hat{w}_i$ are contingency plans for future values of $v$ and $\hat{v}$. If the current period realization of earnings is $y^i$, then the next period’s value of $v$ is $w_i$ and the next period’s value of $\hat{v}$ is $\hat{w}_i$. For each allocation rule there is an associated transfer rule defined via $\tau_i = (1 - s^c)^{-1}(U^{-1}(u_i) - y^i - s^y)$. Given an allocation rule, a transfer function $\tau$ can of course be defined recursively. The converse is however not the case; not all transfer schemes have a recursive representation. Nevertheless, we show in this section that to insist on a recursive representation involves no important loss of generality; every feasible pair of promised and threat utilities can be implemented by a transfer scheme that has a recursive representation.

To state this result, the following notation will prove useful. Let the current utility of a deviating agent who receives income $y^i$, but reports $y^{i-1}$ be denoted by $\psi(u_{i-1})$, where

$$\psi(u) := U(\delta + U^{-1}(u)),$$

where $\delta = (1 - s^c)(y^i - y^{i-1})$. The function $\psi$ is increasing in $u$, and satisfies $\psi(0) = 0$, $\psi(-\infty) = U(\delta)$ and $\psi(u) > u$ for any $u < 0$. It is differentiable, with its derivative $\psi'(u)$ converging to zero as $u$ goes to minus infinity, and to 1 as $u$ goes to zero. We further assume:
Assumption 1 The function $\varphi(c) := -\frac{U''(c)}{U'(c)} \cdot c$ is strictly positive and decreasing in $c$.

The function $\psi$ is then strictly convex, as stated in the following lemma.\footnote{One can show a similar result for a utility with strictly positive and decreasing absolute risk aversion.}

Lemma 1 If Assumption 1 holds then $\psi(u)$ is strictly convex in $u$.

Proof. See Appendix A.

The temporary incentive compatibility constraint of the relaxed problem is

$$ u_i + \beta w_i \geq \psi(u_{i-1}) + \beta \hat{w}_{i-1} \quad \forall i = 2 \ldots N. \quad (9) $$

Note that there is no incentive constraint for $i = 1$, because no agent has a lower income. Furthermore, it is required that reporting truthfully and choosing autarky at the beginning of the next period cannot be optimal for both the truthteller and the deviator:

$$ w_i \geq V^{AUT}(i), \quad i = 1, \ldots, N \quad (10) $$

$$ \hat{w}_{i-1} \geq V^{AUT}(i), \quad i = 2, \ldots, N, \quad (11) $$

where we have dropped the dependence of the value of autarky on $s^y$ and $s^c$. The definition of promised and threat utility is of course as follows.

$$ v = \sum_{i=1}^{N} (u_i + \beta w_i) \pi(i|i_-) \quad (12) $$

$$ \hat{v} = \sum_{i=1}^{N} (u_i + \beta \hat{w}_i) \pi(i|i_- + 1). \quad (13) $$

We are now in a position to define recursive feasibility, where, for each $i_- = 1, 2, \ldots, N - 1$ we will denote the set of recursively feasible pairs $(v, \hat{v})$ by $\mathcal{V}_{i_-}$ and for $i_- = N$ we denote the set of recursively feasible promised utilities $v$ by $\mathcal{V}_N$. 

Definition 2 For each $i_- = 1, 2, \ldots, N - 1$, a pair $(v, \hat{v})$ is said to be recursively feasible if
there is an allocation rule such that Equations (9)-(13) hold and, in addition, the promised and threat continuation utilities are themselves recursively feasible:

\[(w_i, \hat{w}_i) \in V_i \quad \forall i = 1, 2, \ldots, N - 1\]  \hspace{1cm} (14)

\[w_N \in V_N.\]  \hspace{1cm} (15)

The definition of \(V_N\) is the same, except that we drop Equation (13), and that the members of \(V_N\) are (negative) numbers, not pairs of numbers.

For convenience, we define the entire profile of feasible sets via

\[V = (V_1, V_2, \ldots, V_N).\]

Notice that this defines the set \(V\) in terms of itself; in this sense, the definition is recursive, as in Abreu et al. (1990).

The following theorem states that the \(N\)-tuple of sets \(V\) coincides with the \(N\)-tuple of sets \(V^*\) of Definition 1.

**Theorem 1** Recursive feasibility and feasibility are equivalent, i.e.

\[V = V^*.\]

**Proof.** See Appendix A.

The equivalence between both collections of sets validate the recursive formulation in the sense that any pair of promised and threat utilities that can be implemented by some transfer scheme can be achieved by an allocation rule (and its associated transfer rule). (The converse is trivial.)

A convenient property of the profile \(V\) is that its elements are related in a simple one-directional way. To solve for \(V_1\), no other set is needed. To solve for \(V_2\), only \(V_1\) is needed, and so on. \(V_i\) is also clearly nonempty, since \((0, 0) \in V_i\) for all \(i\). The following Proposition
shows that $\mathcal{V}_i$ is convex whenever the utility exhibits either decreasing absolute risk aversion, or decreasing relative risk aversion:

**Proposition 1.** If Assumption 1 holds then $\mathcal{V}_i$ is convex for all $i$.

**Proof.** See Appendix A.

From now on, we will assume that Assumption 1 holds. Note that the convexity result depends crucially on the first-order approach. If there is any incentive compatibility constraint that binds in the opposite direction (e.g. agent $i$ lying to be $i+1$), then the result will not hold.

To further characterize the set $\mathcal{V}_i$, we need to put more structure on the transition matrix. We assume that the transition matrix satisfies the *monotone likelihood ratio property*:

**Assumption 2 (MLRP)**

$$\pi(j|i + 1) / \pi(j|i)$$ is increasing in $j$.

MLRP in our context means that the difference between the deviator’s and the truthteller’s probability will be the largest for the largest shock, regardless of what the last period shock was. As we shall see, this property will be critical for determining the upper contour of $\mathcal{V}$. MLRP implies the following:

**Lemma 2** If Assumption 2 holds, then $\Pi(i|i_-) \geq \Pi(i|i_- + 1)$ $\forall i, i_-.$

**Proof.** See Appendix A.

MLRP thus implies a very natural sort of persistence: For any shock $i$, the probability of getting shocks lower than $i$ is decreasing in $i_-$. However, MLRP is stronger than persistence in this sense. For the results that follow it would not be enough to assume that $\Pi(i|i_-) \geq \Pi(i|i_- + 1)$ for all $i$ and $i_-$. 
2.3.1 Upper and Lower Contours

A very useful way of characterizing the set of feasible utilities $V_i$ is to characterize its lower and upper contour. Define them by

\[ V(v, i) = \min \{ \hat{v} | (v, \hat{v}) \in V_i \}, \]
\[ V(v, i) = \max \{ \hat{v} | (v, \hat{v}) \in V_i \}. \]

The next proposition provides a sharp characterization of the sets $V_i$.

**Proposition 2.** The upper and lower boundaries of the set $V_i$ satisfy for all $i = 1, \ldots, N-1$

\[ V(v, i) \geq v \]
\[ V(v, i) = q_i v, \]

where $q_i = \frac{\pi(1|i + 1)}{\pi(1|i)}$. Additionally, $V(V^{AUT}(i), i) = V^{AUT}(i + 1, s)$.

**Proof.** See Appendix A.

The green set in Figure 1 is a typical set of feasible utilities $V_i$ when the value of autarky is finite. While there is no closed form solution for the lower boundary, we can still characterize its value at the autarkic promised utility $V^{AUT}(i, s)$. The Proposition shows that the value of the lower boundary is the autarkic value of the deviator $V^{AUT}(i + 1, s)$. Remarkably, the result holds even if the constraint (11) is not explicitly imposed. Since the lower boundary is increasing, it follows that once the constraint (10) is imposed, it is never optimal to deviate jointly by misreporting in the current period and then choosing autarky and so constraint (11) can be ignored:

**Proposition 3.** The constraint (11) never binds.

**Proof.** See Appendix A.

Though the details of the proof of Proposition 2 are in an Appendix, we sketch an outline of the proof here. The idea is to ignore first the incentive compatibility constraint and solve
for the resulting upper bound. The solution is then to assign lifetime utility $u_i + \beta w_i$ as low as possible to a state where the deviator is the least likely to be relative to the truthteller, and zero otherwise. Given MLRP, this state is the lowest state $i = 1$. But such an allocation satisfies the incentive compatibility constraint (9) because agents with shock $i = 3, \ldots, N$ are indifferent between reporting their shock and a shock $i - 1$, while an agent with shock $i = 2$ strictly prefers to tell the truth to reporting $i = 1$. Hence the upper bound on the upper contour coincides with the upper contour. One can easily verify that the upper contour then takes the form given in Proposition 4.

Note that Proposition 2 depends critically on MLRP, but does not depend on the convexity assumption. On the other hand, Proposition 1 does not depend on MLRP, but depends on the convexity assumption. Finally, note that if the shocks are i.i.d., $\pi(1|1) = 1$ and the cone shrinks to a line with a slope of -1. That is, the deviator’s utility is always the same as the truthteller’s utility.

If the value of autarky is minus infinity, then the sets of implementable utilities simplify further and become cones. The lower contour is

**Proposition 4.** If $V^{AUT}(i, s) = -\infty$ for all $i = 1, \ldots, N$, then $V(v, i) = v$ for $i \in 1, \ldots, N - 1$.

**Proof.** See Appendix A.

The blue cone in Figure 1 is a typical set of feasible utilities $V_i$. when the value of autarky is infinite. The intuition behind the shape of the lower contour of $V_i$ is the following: Any feasible allocation that is independent of the report delivers $\hat{v} = v$. Hence it must be true that the lower bound is below $v$. On the other hand, the deviator has always the option of pretending he is of the lower type. If he does so, he consumes all the transfers of the lower type. However, since his past endowment is higher and MLRP holds, Lemma 5 implies that he can secure himself at least $\hat{v} \geq v$. Hence the lower contour is above $v$. Taken together, the lower contour of $V_i$ equals $v$. Note that it is larger than if the value of autarky is finite. That is intuitive: decreasing the autarkic value relaxes the constraints on the insurance provider, and allows him to achieve more.
We also have the following result regarding monotonicity of lifetime utilities:

**Proposition 5.** If \((u, w, \hat{w})\) implements some \((v, \hat{v}) \in \mathcal{V}\) then \(u_i + \beta w_i\) is increasing in \(i\).

**Proof.** See Appendix A.

Beyond the southwest corner of the feasible set under limited enforcement, we use a numerical approach to characterize the lower bound. The approach involves computing a sequence of segments that together constitute the lower bound. In the Appendix we describe the approach in the context of the two-state case where \(y_t\) can only take two values: \(y^\ell\) and \(y^h\) where \(y^\ell < y^h\). (A generalization is not hard but is a bit heavier in notation.)
2.4 Principal’s problem after the initial period

After the agent has signed up with the principal, the efficient contract after the initial period solves the following dynamic program. For each \( i = 1, \ldots, N \) the minimal cost function \( C \) satisfies

\[
C(v, \hat{v}, i_{-}) = \min_{u, w, \hat{w}} \sum_{i=1}^{N} \left[ U^{-1}(u_i) - y^i + s^i_i + \frac{1}{R} C(w_i, \hat{w}_i, i) \right] \pi(i | i_{-}) \tag{16}
\]

subject to the constraints (9)-(10), (12)-(15). Note that, due to Proposition 3, the constraint (11) does not need to be imposed. Also note that \( C(v, \hat{v}, N) \) is independent of \( \hat{v} \) since there is no threat keeping constraint if the last period report is \( N \).

The Principle of Optimality for the principal problem’s after the first period implies that \( C \) solves its corresponding sequence problem where the principal chooses a transfer scheme \( \tau \) minimizes the costs \( C(\tau | i_{-}) \) defined in (1) subject to the constraints (4) -(6), and a requirement that \( W(\tau | i) = v \) and \( U(\tau | i + 1) = \hat{v} \).

We show that the cost function is strictly convex jointly in the promised utility and the threat utility:

**Lemma 3** \( C(v, \hat{v}, i_{-}) \) is strictly convex in \((v, \hat{v})\) for all \( i_{-} = 1, \ldots, N \).

**Proof.** See Appendix A.

If the autarky value is equal to minus infinity, then one can also show that the incentive compatibility constraint is binding in the optimum:

**Lemma 4** Suppose that \( V^{AUT} = -\infty \). Then the incentive compatibility constraint (9) is binding.

**Proof.** See Appendix A.

This leads us to conjecture that the incentive compatibility constraint binds when \( V^{AUT} > -\infty \) as well; this is a useful starting point for the numerical computations.
2.5 Principal’s problem in the initial period

In the initial period problem the principal is not bound by any past promises and so chooses a threat utility that minimizes the costs among all feasible threat utilities.

\[ C^*(v, i_0) = \min_{\hat{v}} \{C(v, \hat{v}, i_0) \mid (v, \hat{v}) \in \mathcal{V}_{i_0} \}. \]

2.6 Validity of the Relaxed Problem

Once the relaxed problem has been computed, one can check numerically whether the relaxed problem is valid. Compute first the lifetime utility of an agent who has reported earnings \( y^j \) in the previous period, but in fact had earnings \( y^i \):

\[ \hat{V}(v, \hat{v}, j| i_-) = \sum_{i=1}^{N} [u_i(v, \hat{v}, j) + \beta w_i(v, \hat{v}, j)] \pi(i| i_-) \quad \forall j, i_- = 1 \ldots N. \]  

(17)

Note that, by construction, \( \hat{V}(v, \hat{v}, j| j) = v \) and \( \hat{V}(v, \hat{v}, j| j+1) = \hat{v} \). But \( \hat{V} \) is more general because it computes the continuation utility for all possible types.

The incentive compatibility constraint can be checked by computing for all \( i, j = 1, \ldots, N \) the following inequality:

\[ D^j_i = u_i + \beta w_i - U \left( y^i - y^j + U^{-1}(u_{i-j}) \right) - \beta \hat{V}(w_j, \hat{w}_j, j|i). \]  

(18)

The quantity \( D^j_i \) shows the gain of an agent \( i \) from telling the truth relative to reporting that his type is \( j \), under the relaxed problem. For the relaxed problem to work correctly, it better be the case that no report can deliver higher utility than truthtelling. We thus define the validity of the relaxed approach as follows:

**Definition 3** The relaxed approach is said to be valid if for all \( (v, \hat{v}, i_-) \in \mathcal{V} \),

\[ D^j_i \geq 0 \quad \forall i, j = 1, \ldots, N. \]  

(19)
While we cannot establish the validity of the relaxed approach in general, we can check it numerically in specific cases. For our findings in this respect, see Section 4.

2.7 General Equilibrium

It is assumed that the government budget is balanced:

\[ s^c E_c = s^y. \]

We close the model by endogenizing the interest rate \( R \). This simply means that average consumption in the stationary distribution is equal to average earnings:

\[ E_c = E_y. \]

However, we also consider other possible values of \( R \).

3 Data

The data sources used here are the Panel Study of Income Dynamics (PSID) and the Consumer Expenditure Survey (CEX), the main sources of income and consumption data for the United States. The appropriate measure of earnings for our model are household earnings net of taxes and government transfers, but excluding private transfers. Our consumption measure is expenditure on non-durables, and the frequency of measurement is annual. By considering logs of earnings and of consumption we exclude observations with values of consumption or earnings that are zero or negative. Moreover, we censor earnings and consumption observations that are so low as to cast doubt on the quality of the data.

For both consumption and earnings we identify the idiosyncratic component as the residuals
from a first-state regression on household observables

$$\ln X_{i,t} = Z_{i,t}' \varphi_t + y_{i,t}$$

where $X \in \{C, Y\}$, $t$ indexes time, $Z_{i,t}$ is a set of characteristics observable and known by household $i$ at time $t$, and a hat denotes residuals. Specifically, we consider $Z_{i,t}$ to comprise dummies describing whether the household is a married couple, a single man or a single woman and whether the adult members of the household have more than 12 years of education, time dummies and a polynomial in the age of the head of household. The covariates account for about 40 percent of the total variance of earnings in our PSID sample. The source is Heathcote et al. (2010), and our definition of earnings is post taxes and government transfers. Earnings in that dataset have already been cleansed of clearly unreasonable observations, for instance where labour earnings are strictly greater than zero but hours worked are not or where earnings and hours are such as to imply an hourly wage less than half the minimum wage for the relevant year. Yearly means are then subtracted from each observation of log earnings to obtain a measure of relative income. Only then are log earnings regressed on observables to obtain residuals. A histogram of these residuals can be seen in Figure 2.

Figure 2: Histogram for residual log earnings in the PSID 1967-2002
3.1 Estimation of the earnings process

Our approach to estimating the earnings process is designed to (1) capture the key statistical properties in micro data and (2) produce an earnings process that is consistent with our theoretical framework. We obtain the earnings process for each household $i$ by decomposing the residual idiosyncratic component $y_{i,t}$ of log earnings (whose unconditional mean is zero by construction) into three components as follows

$$y_{i,t} = \alpha_i + z_{i,t} + x_{i,t}.$$

where $\alpha_i$ is the permanent (unchanging) component, $z_{i,t}$ is the persistent, and $x_{i,t}$ is the transitory (i.i.d.) component. The idea here is to treat the purely transitory component as measurement error and the permanent component as inherently uninsurable. Therefore the only relevant component for our purposes is the persistent component. It satisfies

$$z_{i,t-1} = \rho z_{i,t-1} + \varepsilon_{i,t}$$

where $\varepsilon_{i,t}$ is an i.i.d. shock.\footnote{An alternative approach is taken by Guvenen (2007) who posits heterogeneity in the growth rate of earnings. Guvenen justifies it by referring to the implied behavior of consumption. While we find Guvenen’s argument interesting, it is done in the context of a financial market with riskless one-period bonds only and cannot be directly applied to our model. We regard our work as complementary to Guvenen’s.}

The parameters that characterize the income process are the variances of the shocks $\sigma^2_\varepsilon$, $\sigma^2_x$ and $\sigma^2_\alpha$, the autocorrelation of the persistent shock $\rho$ and the third central moments of $\alpha_i$, $x_{i,t}$ and $\varepsilon_{i,t}$.

We estimated these parameters by GMM where the moments are the autocovariances $\Gamma_k = \mathbb{E}[y_{i,t}y_{i,t+k}]$, where the $k$th covariance is computed as the average over all possible products $y_{i,t}y_{i,t+k}$ for which data is available and $k = 0, 1, \ldots, 11$ as well as the following third moments:

$$\Delta_{0,0} := \mathbb{E}[y_{i,t}^3],$$
$$\Delta_{0,1} := \mathbb{E}[y_{i,t}^2y_{i,t+1}].$$
and

$$\Delta_{1,1} := \mathbb{E}[y_{i,t}y_{i,t+1}^2].$$

The autocovariances are geometrically smoothed as described in Table 1.
Table 1: Moments used in estimation

<table>
<thead>
<tr>
<th></th>
<th>Sample moments</th>
<th>Smoothed moments</th>
<th>Theoretical moments</th>
<th>Number of obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_0$</td>
<td>0.1690</td>
<td>0.1690</td>
<td>0.1686</td>
<td>59550</td>
</tr>
<tr>
<td>$\Gamma_1$</td>
<td>0.1197</td>
<td>0.1126</td>
<td>0.1135</td>
<td>37828</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>0.1064</td>
<td>0.1055</td>
<td>0.1060</td>
<td>36791</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>0.0958</td>
<td>0.0988</td>
<td>0.0990</td>
<td>27202</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>0.0909</td>
<td>0.0926</td>
<td>0.0925</td>
<td>27178</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>0.0844</td>
<td>0.0868</td>
<td>0.0865</td>
<td>22712</td>
</tr>
<tr>
<td>$\Gamma_6$</td>
<td>0.0799</td>
<td>0.0813</td>
<td>0.0810</td>
<td>22074</td>
</tr>
<tr>
<td>$\Gamma_7$</td>
<td>0.0745</td>
<td>0.0762</td>
<td>0.0759</td>
<td>19492</td>
</tr>
<tr>
<td>$\Gamma_8$</td>
<td>0.0716</td>
<td>0.0715</td>
<td>0.0712</td>
<td>18061</td>
</tr>
<tr>
<td>$\Gamma_9$</td>
<td>0.0675</td>
<td>0.0670</td>
<td>0.0668</td>
<td>15972</td>
</tr>
<tr>
<td>$\Gamma_{10}$</td>
<td>0.0641</td>
<td>0.0628</td>
<td>0.0628</td>
<td>14861</td>
</tr>
<tr>
<td>$\Gamma_{11}$</td>
<td>0.0601</td>
<td>0.0589</td>
<td>0.0591</td>
<td>12106</td>
</tr>
<tr>
<td>$\Delta_{0,0}$</td>
<td>-0.0369</td>
<td>-0.0369</td>
<td>-0.0369</td>
<td>59550</td>
</tr>
<tr>
<td>$\Delta_{0,1}$</td>
<td>-0.0181</td>
<td>-0.0181</td>
<td>-0.0181</td>
<td>37828</td>
</tr>
<tr>
<td>$\Delta_{1,1}$</td>
<td>-0.0167</td>
<td>-0.0167</td>
<td>-0.0167</td>
<td>37828</td>
</tr>
</tbody>
</table>

Our GMM procedure simply involves matching these empirical moments to the theoretical moments implied by our statistical model. Our parameter estimates are $\hat{\sigma}_\varepsilon^2 = 0.0157$, $\hat{\sigma}_x^2 = 0.0469$ and $\hat{\sigma}_\alpha^2 = 0.0132$, and $\hat{\rho} = 0.9247$. These results are very similar to those of Klein and Telyukova (2013), implying that about 64 percent of the total variance of residual log earnings is accounted for by the persistent component.

For the CEX, the earnings estimation requires more elaborate techniques, because of the overlapping observations. This is discussed in Gervais and Klein (2010).

When mapping our estimates to the theoretical framework, we ignore the permanent component of the earnings. While this is a considerable simplification, it is worth noting that, unless people can be insured before they are born, insurance markets will not provide insur-
ance against those shocks. The transitory component we treat as measurement error; this leaves us with only the persistent component, which we approximate by a two-state Markov chain.

4 Confronting Theory with Data

In this section we examine the quantitative implications of our theory and compare and contrast those implications with the implications of the models of Huggett (1993), Krueger and Uhlig (2005) and Krueger and Perri (2004). For the purposes of this preliminary confrontation, we simplify the earnings process considerably compared to what we described above and assume simply a two-state Markov chain, so that $y_t \in \{y^f, y^h\}$. We do this to simplify the solution especially of our own model. Anyhow, let’s begin by looking at some facts.

4.1 Some features of the data

The standard deviation of the residuals from a first-stage regression on household observables are about 0.45 for log-earnings\(^8\) and 0.41 for log-consumption.\(^9\) A common measure of the degree of risk sharing is the regression coefficient of consumption changes and earnings changes. As discussed in Gervais and Klein (2010), there are some serious econometric issue involved with measuring it in U.S. data. The value that they report for this coefficient, on a quarterly rather than a yearly basis, is about 0.12. Using a more straightforward, but invalid, OLS approach, the annual number is about 0.07. This is evidence of significant risk sharing: consumption does not respond very strongly to earnings changes.

Another feature of the data worth noting is the autocorrelation of consumption and earnings, equal to 0.66 and 0.78 respectively. Thus consumption is apparently less autocorrelated than earnings. It remains an open question as to whether this is entirely due to measurement

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error. The fact that consumption has a smaller variance suggests, but certainly does not establish, that measurement error might not be the whole story.

A striking feature not only of U.S. micro data is that the cross-sectional distribution of (log) consumption is more symmetric than that of earnings. In fact, (log, residual) earnings are somewhat skewed to the left, with a skewness coefficient of about \(-0.4\).\textsuperscript{10} Meanwhile, the consumption distribution is virtually symmetric; see Figure 3 and contrast with Figure 2. This fact is documented for Canada in Brzozowski et al. (2010) and for the United Kingdom in Battistin et al. (2009).

![Figure 3: Histogram for residual log consumption in the CEX 1980-2003](image)

4.2 Computing the Kapička-Klein model

To find the optimal contract, we use value function iteration with finite-element interpolation. Interpolation in this case is somewhat complicated by the fact that the state space is not rectangular when \(i_\ell = \ell\). In that case, the set of feasible values for \(\hat{v}\) depends on the value of \(v\). Thus we cannot define a grid as a Cartesian product between two fixed one-dimensional grids. What we do instead is to define a grid for \(v\), and for each point on that grid we define a grid for \(\hat{v}\). The total number of points is 17250, and the range is more than wide enough in each dimension to cover the ergodic set as discovered in simulations.

\textsuperscript{10}Source: PSID.
The finite-element interpolation method works as follows. First we define the elements as triangles whose vertices are the gridpoints and where the union of these triangles coincides with the convex hull of the set of gridpoints.\textsuperscript{11} Clearly such a triangulation is not unique. The particular triangulation that we use is that defined by Delaunay (1934). It is illustrated for a case of four gridpoints in Figure 4. We do not claim that this particular triangulation is optimal in any sense, though optimality can be established in special cases; see, for instance Chen and Xu (2013).

Second, we define, once and for all given the gridpoints and the triangulation, coefficients that we will need to linearly interpolate within each triangle. To see what these coefficients are, let’s take one step ahead and suppose we have an arbitrary point \((x, y)\) in the domain and we want to find the interpolating value \(z\) given that we know what triangle \((x, y)\) is in. Call the vertices of this triangle \((x_0, y_0), (x_1, y_1)\) and \((x_2, y_2)\) and suppose the function values at these vertices are \(z_0, z_1\) and \(z_2\). Evidently there exist scalars \(t\) and \(u\) such that

\[
(x, y) = (x_0, y_0) + t \cdot (x_1 - x_0, y_1 - y_0) + u \cdot (x_2 - x_1, y_2 - y_1).
\]

Figure 5 illustrates this very simple idea. Once we have found the values of \(t\) and \(u\), our interpolating value is of course

\[
z = z_0 + t \cdot (z_1 - z_0) + u \cdot (z_2 - z_1).
\]

We need, then, an efficient way of finding \(t\) and \(u\) as a function of \(x\) and \(y\). It turns out that, for each triangle, the vector \([t \quad u]\) is a fixed affine function of the vector \([x \quad y]\). Thus what we need to do, once and for all given the gridpoints and the triangulation, is to compute, for each triangle, the constant term and the gradient of this affine function. Finding this constant term and this gradient requires us to invert a \(2 \times 2\) matrix, but the important thing is that this is only done \textit{once} for each triangle. Having computed these coefficients once and for all, finding the interpolating value at an arbitrary point is just a matter of (1) finding the right triangle and (2) evaluating an affine function. This can of course be done very quickly,

\textsuperscript{11}In this context, the convexity of the feasible set, as stated in Proposition 1, is evidently important.
and this is important because it is done many thousands of times during the value function iteration.

Figure 4: A Delaunay triangulation

Figure 5: Linear interpolation on a triangle

To compute moments of the stationary distribution, we simply use the optimal allocation rule to simulate 100000 observations. To find the general equilibrium, we choose the interest
rate $R$ that ensures that mean consumption is equal to available resources in a steady state. An aggregate steady state is defined by the following resource constraint:

$$C + \delta K = AK^\theta,$$  \hspace{1cm} (20)

the following profit maximizing condition on the part of competitive output producers:

$$R = 1 + A\theta K^{\theta-1} - \delta$$  \hspace{1cm} (21)

and the following normalization (saying that mean labour earnings equal 1):

$$A(1 - \theta)K^\theta = 1.$$  \hspace{1cm} (22)

The underlying assumption here is that the insurance providers own the capital. Going off to live in autarky means forgoing any access to this capital. The specific parameter values that we use are $\delta = .08$ and $\theta = 1/3$.

### 4.2.1 Validity of the relaxed approach

Once the equilibrium is computed, we can check if the “lying up” constraint is ever violated, i.e. whether the agent would gain at any part of the state space by claiming to have high earnings when in fact they are low. Unfortunately, this does happen in a region of the state space; however, no point of this region is a member of the ergodic set: the set of points where the “lying up” constraint is violated has measure zero under the stationary distribution.

### 4.3 Key moments across risk-sharing models

For the purpose of a very preliminary confrontation between the data and various models, including our own, we will consider environments with the following features. Earnings follow a two-state Markov chain with standard deviation of about 0.33, skewness $-0.57$ and autocorrelation of 0.92. The mean of earnings is normalized to 1 and the market interest
rate $R$ is either set arbitrarily (partial equilibrium) or set so as to satisfy Equations (20), (21), and (22) (general equilibrium).

Agents rank consumption streams according to

$$E\left[ \sum_{t=0}^{\infty} \beta^t \frac{c_1^{1-\sigma}}{1-\sigma} \right]$$

where $\beta = 0.96$ and $\sigma = 2$. Principals rank consumption streams according to

$$E\left[ \sum_{t=0}^{\infty} R^{-t} (y_t - c_t) \right].$$

**Implications for consumption of self-insurance**  Here we look at the implications of the Huggett (1993) model where agents face the following budget constraint

$$c_t + b_{t+1} = y_t + Rb_t$$

and a borrowing limit $b_{t+1} \geq \underline{b}$. We study two versions of the economy distinguished by the value of $\underline{b}$. First, we take the so-called *natural* borrowing limit, where $\underline{b} = -y_{\ell}/(R-1) \equiv b_{\text{nat}}$. And second, we look at an economy where $\underline{b} = b_{\text{nodef}}$, and $b_{\text{nodef}}$ is the maximum level of borrowing such that agents with assets equal to $b$ are indifferent between servicing their debt and moving to financial autarky. There, debt is forgiven, but agents cannot save or borrow and thus consume their income forever in the future. $b_{\text{nodef}}$ can thus be interpreted as resulting from a “no default” constraint that prevents agents from borrowing amounts that would make it optimal to default into financial autarky at any value of future income.

Rows 1 and 2 of Table 2 summarize the key moments of our model economies. With a natural borrowing limit, the standard deviation of log consumption in the stationary distribution is 0.41, which is slightly higher than that of earnings, while the regression coefficient of consumption changes on earnings changes is 0.34. With a higher interest rate (close to $\beta^{-1}$), consumption responds less to income changes, but the skewness of consumption remains

12A stricter borrowing limit does not change the results qualitatively with respect to the features considered here, though it certainly changes the results quantitatively.
large in magnitude.

With a no-default borrowing limit \( b_{\text{nodef}} \), which is less than a third of the natural limit, the results are not dramatically different. As shown in Table 2, the standard deviation of consumption is lowered a bit (by reducing that of wealth). The regression coefficient of consumption on earnings changes, on the other hand, increases a bit to 0.28 in general equilibrium, indicating lower risk-sharing at the no-default limit. Log consumption remains strongly skewed to the left.

In Figure 6 we can see the histograms resulting from both specifications of the self-insurance economy. The most-striking difference between the two is the large number of agents at the no-default borrowing limit. The natural limit, in contrast, where future consumption equals zero with strictly positive probability, is never binding for any consumer, as this would imply infinitely negative utility due to Inada conditions. Both distributions show a clear left-skew.

Figure 6: Histogram for log consumption in Huggett, no default borrowing limit, general equilibrium with capital
Table 2: Key Moments

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{\ln(c)}$</th>
<th>$\beta_{dc,dy}$</th>
<th>skew($\ln(c)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CEX data</td>
<td>0.41</td>
<td>0.12$^\dagger$(0.07)</td>
<td>0.13</td>
</tr>
<tr>
<td>SI $\tilde{b}_{\text{rat}}$ GE</td>
<td>0.41</td>
<td>0.34</td>
<td>–1.6</td>
</tr>
<tr>
<td>SI $\tilde{b}_{\text{rat}}$ high $R$</td>
<td>0.44</td>
<td>0.05</td>
<td>–0.91</td>
</tr>
<tr>
<td>SI $\tilde{b}_{\text{modef}}$ GE</td>
<td>0.28</td>
<td>0.38</td>
<td>–0.98</td>
</tr>
<tr>
<td>SI $\tilde{b}_{\text{modef}}$ high $R$</td>
<td>0.44</td>
<td>0.05</td>
<td>–0.90</td>
</tr>
<tr>
<td>LC high $R$</td>
<td>0.05</td>
<td>0.00</td>
<td>–2.9</td>
</tr>
<tr>
<td>KK GE</td>
<td>0.26</td>
<td>0.21</td>
<td>–1.0</td>
</tr>
<tr>
<td>KK high $R$</td>
<td>0.39</td>
<td>0.12</td>
<td>–0.30</td>
</tr>
</tbody>
</table>

For CEX data, the two self-insurance (SI) economies, the Krueger and Perri (2006) limited-commitment economy (LC), and the Kapička-Klein (KK) model, the table presents the standard deviation of log-consumption $\sigma_{\ln(c)}$, the regression coefficient of (percentage) consumption changes on (percentage) earnings changes $\beta_{dc,dy}$, the skewness of log consumption and the autocorrelation of consumption $\sigma_{\ln(c),\ln(c-1)}$. “GE” stands for general equilibrium with capital, and “high $R$” means $R = 1.0416$, a number close to $\beta^{-1}$. (The case of limited enforcement and general equilibrium is omitted because it turns out to imply perfect risk sharing.)

$^\dagger$This is the quarterly coefficient as derived in Gervais and Klein (2010). The number in brackets is the simple, but misspecified, annual coefficient.

Implications for consumption of limited enforcement  Here we consider the implications of the Krueger and Uhlig (2005) model where agents have the option to walk away in any period and sign up with another principal. In this context it is important to notice that the timing of the limited enforcement constraint is subtly different from what it is in our model, in addition to the outside option being different. In our model, the agent wakes up in the morning, observes the earnings shock, reports earnings to the principal, receives or pays the current period transfer according to the contract and can then choose to walk away and live in autarky. In the Krueger-Uhlig environment, the agent wakes up, observes earnings, and can then walk away immediately and sign up with a competing principal.

In this environment, the key features of the stationary distribution are described in Table 1. Figure 7 displays a histogram for log consumption in our calibration of the Krueger and Uhlig (2005) model (with an interest rate of $R = 1.04$) and the picture is quite stark. The tendency for consumption to jump up when the participation constraint binds for the high type and drift down whenever the participation constraint doesn’t bind for the low type, produces a very strong tendency for consumption to be skewed to the left. Clearly this is not a feature found in the data at all.
Implications for consumption of private information and limited enforcement

Finally, we consider the implications of our own model. General equilibrium considerations require us to set the gross interest rate $R$ equal to about 1.038, which is of course lower than $\beta^{-1}$ which is equal to about 1.042. The standard deviation of log consumption is about 0.26, which is only a bit less than the standard deviation of log earnings. The regression coefficient of consumption changes on income changes is 0.21. Meanwhile, the skewness of log consumption is -1.0, which is somewhat bigger in magnitude than the skewness of log earnings; in this respect, it is similar to the Huggett (1993) model. Nevertheless, if the interest rate is close to $\beta-1$, log consumption is more symmetric than (log) earnings. Moreover, at that interest rate, the response of consumption changes to income changes is in line with the data. In this sense, our model has at least the potential to do better than the Huggett (1993) model in capturing the near log-normality of consumption.
Figure 8: Histogram for log consumption in Kapička-Klein in CE with capital
To understand better how our model works, it is useful to look at an example sequence of consumption and earnings. As shown in Figure 10, consumption tends to drift down when earnings are low, until it hits the low earnings level; it never falls below that, otherwise the agent would walk away. That feature is shared with the limited enforcement model of Krueger and Uhlig (2005) and others. Meanwhile, and here is the key to avoiding the strong tendency for consumption to be skewed to the left, consumption tends to drift gradually up when earnings are high. In this sense, the consumption dynamics are richer in this model than in the pure limited enforcement environment.
5 Concluding remarks

Models of efficient consumption risk sharing with private information alone imply immiserization; models with limited enforcement alone imply that consumption is either constant,
drifts down or leaps up. Combining the two frameworks, the implications are much more reasonable; more in line with the data and in fact rather similar to those of models with exogenous incomplete markets such as Huggett (1993). Meanwhile, the mechanisms are of course quite different. In future work, we intend to contrast the effects of various policy interventions in our model versus that of Huggett (1993).

Appendix A: Proofs

Proof of Lemma 1. Differentiating $\psi(u)$ twice and rearranging, one gets that

$$\psi''(u) = \frac{U'(U^{-1}(u) + \delta)}{U^{-1}(u)U'(U^{-1}(u))} \times \left[ \frac{-U^{-1}(u)U''(U^{-1}(u))}{U'(U^{-1}(u))} - \frac{U^{-1}(u) - (U^{-1}(u) + \delta)U''(U^{-1}(u) + \delta)}{U'(U^{-1}(u) + \delta)} \right].$$

The assumption that the utility is strictly increasing, $-\frac{U''(u)}{U'(u)}$ is decreasing in $c$ and the fact that $\frac{c}{c+\delta} < 1$ together imply that $\psi''(u) > 0$. ■

Proof of Proposition 1. The result follows from the fact that the promise keeping constraint (12), the threat keeping constraint (13) and the left hand side of the incentive compatibility constraint (9) are all linear in $u$ and $w$, and that the right-hand side of (9) is convex in $u$ and $w$ by Lemma 1 if Assumption 1 holds. ■
Proof of Theorem 1. Let $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N)$ where $\mathcal{B}_{i_-} \subset \mathbb{R}^2$ for $i_- < N$ and, for $i_- = N$, $\mathcal{B}_N \subset \mathbb{R}_-$. Define an operator $\mathcal{T}$ by

$$\mathcal{T}\mathcal{B}_{i_-} = \{(v, \hat{\nu})| \text{ There exists an allocation rule } (u, w, \hat{\nu}) \text{ such that } (9)-(13) \text{ hold, } (w_i, \hat{w}_i) \in \mathcal{B}_i \ \forall \ i = 1 \ldots N - 1 \text{ and } w_N \in \mathcal{B}_N\}$$

for all $i_- = 1, \ldots, N - 1$ and, for $i_- = N$,

$$\mathcal{T}\mathcal{B}_{i_-} = \{v| \text{ There exists an allocation rule } (u, w, \hat{\nu}) \text{ such that } (9)-(12) \text{ hold, } (w_i, \hat{w}_i) \in \mathcal{B}_i \ \forall \ i = 1 \ldots N - 1 \text{ and } w_N \in \mathcal{B}_N\}.$$

The set $\mathcal{V}$ is by definition a fixed point of the operator $\mathcal{T}$. We will now show that $\mathcal{V}^* = \mathcal{T}\mathcal{V}^*$, implying $\mathcal{V} = \mathcal{V}^*$. Notice that we only have to prove $\mathcal{V}^*_{i_-} = \mathcal{T}\mathcal{V}^*_{i_-}$ for $i_- < N$, since it is evident that $\mathcal{V}_N = \mathcal{V}^*_N = \{x \in \mathbb{R} : x < 0\}$.

Let $(v, \hat{\nu}) \in \mathcal{V}^*_{i_-}$ for some $i_- < N$. Then there exists some transfer scheme $\tau$ such that (4)-(6) hold, $\mathcal{U}(\tau|i) = v$ and $\mathcal{U}(\tau|i+1) = \hat{\nu}$. Define the allocation rule $(u, w, \hat{\nu})$ by $u_i = U(y^i + \tau_0(i))$, $w_i = \mathcal{U}(\tau(\emptyset, i), i)$ and $\hat{w}_i = \mathcal{U}(\tau(\emptyset, i), i + 1)$. The allocation rule $(u, w, \hat{\nu})$ satisfies (9), since

$$u_i + \beta w_i = U(y^i + \tau_0(i)) + \beta \mathcal{U}(\tau(\emptyset, i), i) \geq U(y^i + \tau_0(i - 1)) + \beta \mathcal{U}(\tau(\emptyset, i - 1), i) = \phi(u_i) + \beta \hat{w}_i.$$ 

The allocation rule $(u, w, \hat{\nu})$ clearly satisfies constraints (10)-(13) as well. Because $\tau$ satisfies (4) after all histories, $(w_i, \hat{w}_i) \in \mathcal{V}^*_i$ for all $i = 1, \ldots, N - 1$, and so (15) holds. Hence, $(v_i, \hat{v}_i) \in \mathcal{T}\mathcal{V}^*_{i_-}$ and so $\mathcal{V}^*_{i_-} \subset (\mathcal{T}\mathcal{V}^*)_{i_-}$. Since $i_- < N$ is arbitrary, $\mathcal{V}^* \subset \mathcal{T}\mathcal{V}^*$.

To prove the reverse set inclusion, suppose that $(v, \hat{\nu}) \in \mathcal{T}\mathcal{V}^*_i$. Then there exists some allocation rule $(u, w, \hat{\nu})$ such that (9)-(15) holds. Define a transfer scheme $\tau$ as follows. Let
\[ \tau_0(i) \text{ solve } u_i = U(y^i + \tau_0(i)). \] Since \((w_i, \hat{w}_i) \in \mathcal{V}_i^*\), there is some continuation transfer scheme \(\tau(\emptyset, i)\) such that \(U(\tau(\emptyset, i)|i) = w_i\) and \(U(\tau(\emptyset, i-1)|i) = \hat{w}_i\) for each \(i = 1, \ldots, N-1\). It is easy to see that the transfer scheme \(\tau\) satisfies (4)-(6), and that \(U(\tau|i) = v\) and \(U(\tau|i-1) = \hat{v}\). Thus, \((v, \hat{v}) \in \mathcal{V}_i^*\), and so \(\mathcal{TV}_i^* \subset \mathcal{V}_i^*\). Since \(i\) is arbitrary, \(\mathcal{TV}^* \subset \mathcal{V}^*\).

**Proof of Lemma 2.** Suppose that \(j_1 \geq j_0\). Then
\[ \pi(j_1|i+1)\pi(j_0|i) \geq \pi(j_1|i)\pi(j_0|i+1) \]

Summing those inequalities over \(j_0 = 1 \ldots j_1\), one gets
\[ \pi(j_1|i+1)\Pi(j_1|i) \geq \pi(j_1|i)\Pi(j_1|i+1). \]

Similarly, summing over \(j_1 = j_0 + 1 \ldots N\),
\[ (1 - \Pi(j_0|i+1))\pi(j_0|i) \geq (1 - \Pi(j_0|i))\pi(j_0|i+1). \]

Since both equations hold for any \(j_0\) and \(j_1\), we can write, for any \(j\),
\[ \frac{1 - \Pi(j|i+1)}{1 - \Pi(j|i)} \geq \frac{\pi(j|i+1)}{\pi(j|i)} \geq \frac{\Pi(j|i+1)}{\Pi(j|i)}, \]

hence
\[ \frac{1 - \Pi(j|i+1)}{\Pi(j|i+1)} \geq \frac{1 - \Pi(j|i)}{\Pi(j|i)}, \]

and so \(\Pi(j|i) \geq \Pi(j|i+1)\) for any \(j\).

To prove Proposition 4, the following lemma is needed.

**Lemma 5** If MLRP holds, then for any increasing vector \((f_1, \ldots, f_N)\) and any \(i_- = 1, \ldots, N - 1\),
\[ \sum_{i=1}^{N} f_i \pi(i|i_- + 1) \geq \sum_{i=1}^{N} f_i \pi(i|i_-). \]
Proof. Suppose, in order to yield a contradiction, that

\[ \sum_{i=1}^{N} f_i \pi(i|i_- + 1) < \sum_{i=1}^{N} f_i \pi(i|i_-). \]  

(23)

Fix \( i_- \) and define \( g_i = \frac{\pi(i|i_++1)}{\pi(i|i_-)} \). Since MLRP holds, the vector \( g \) is increasing. Now rewrite (23) using \( g \):

\[ \sum_{i=1}^{N} f_i g_i \pi(i|i_-) < \sum_{i=1}^{N} f_i \pi(i|i_-) = \sum_{i=1}^{N} f_i \pi(i|i_-) \sum_{i=1}^{N} g_i \pi(i|i_-). \]

That is, \( E(fg) < E(f)E(g) \), and so \( f \) and \( g \) are strictly negatively correlated. This is a contradiction, since both \( f \) and \( g \) are increasing. ■

Proof of Proposition 2. i) The proof of \( V(v, i_-) \geq v \) uses induction, and a limit argument. Consider a truncated problem where the agents live only for \( T + 1 < \infty \) periods \( t = 1, \ldots, T \). Let \((u_{i,t}^{(T)}, w_{i,t+1}^{(T)})\) be an allocation of the truncated problem in period \( t = 0, \ldots, T \), with \( w_{i,T+1}^{(T)} = 0 \) by truncation. Let also \( V_{T}^{(T)}(v, i_-) \) be the lower contour of the truncated problem in period \( t = 0, \ldots, T \). The incentive compatibility constraint (9) in the last period \( T \) imply that

\[ u_{i,T} \geq \psi(u_{i-1,T}) \geq u_{i-1,T}, \]

and so \((u_{1,T}, \ldots, u_{N,T})\) is an increasing vector. By Lemma 5, for any \( i_- = 1, \ldots, N - 1, \)

\[ V_{T}^{(T)}(v, i_-) = \sum_{i=1}^{N} u_{i,T} \pi(i|i_- + 1) \geq \sum_{i=1}^{N} u_{i,T} \pi(i|i_-) = v. \]

Now assume that \( V_{T+1}^{(T)}(v, i_-) \geq v \) for some \( t \in (1, \ldots, T) \), for all \( i_- = 1, \ldots, N - 1. \) The incentive compatibility constraint in period \( t \) implies

\[ u_{i,t} + \beta w_{i,t+1} \geq \psi(u_{i-1,t}) + \beta V_{T+1}^{(T)}(w_{i-1,t+1}, i - 1) \geq u_{i-1,t} + \beta w_{i-1,t+1}. \]
Hence \( u_{i,t} + \beta w_{i,t+1} \) increases in \( i \). By Lemma 5, for any \( i_- = 1, \ldots, N - 1 \),

\[
V_t^{(T)}(v, i_-) = \sum_{i=1}^{N} (u_{i,t} + \beta w_{i,t+1}) \pi(i|i_- + 1) \geq \sum_{i=1}^{N} (u_{i,t} + \beta w_{i,t+1}) \pi(i|i_-) = v.
\]

Hence, by induction, \( V_t^{(T)}(v, i_-) \geq v \) for all \( t = 1, \ldots, T \), all \( i_- = 1, \ldots, N - 1 \). Since \( T \) was arbitrary we have

\[
V(v, i_-) = \lim_{T \to \infty} V_t^{(T)}(v, i_-) \geq v.
\]

ii) Consider an upper bound on the upper contour that ignores the incentive compatibility constraint. The solution is such that the principal assigns the lowest possible utility to a state \( i \) such that

\[
i \in \arg\min_{k=1, \ldots, N} \frac{\pi(k|i_- + 1)}{\pi(k|i_-)}.
\]

MLRP implies that \( i = 1 \). Thus

\[
u_1 + \beta w_1 = \frac{v}{\pi(1|i_-)}
\]

\[
u_i + \beta w_i = 0, \quad i \neq 1.
\]

It follows that \( u_i = w_i = 0 \) for all \( i > 1 \). Set \( w_1 = 0 \). Then \( u_1 < 0 \). Since \( \psi(0) = 0 \) and \( \psi \) is increasing, the incentive compatibility constraint (9) is satisfied for all \( i = 2, \ldots, N \). The upper contour is then given by

\[
V(v, i_-) = (u_1 + \beta w_1) \pi(1|i_- + 1) = \frac{\pi(1|i_- + 1)}{\pi(1|i_-)} v,
\]

where the last equality follows from substituting in the expression for \( u_1 + \beta w_1 \).

iii) Ignoring both constraints (10) and (11), consider the following allocation: \( u_i = 0 \) for all \( i = 1 \ldots N \) and \( w_i = \frac{v}{\beta} \). This allocation is trivially incentive compatible since it is independent of the report, and delivers \( \hat{v} = v \). Given that \( V(v, i_-) \geq v \), this allocation is optimal along the lower boundary. However, the allocation violates (10) at \( v = V^{AUT}(i_-) \).
since \( w_i = \beta^{-1} V^{AUT}(i_-) < V^{AUT}(i_-) \). Hence (10) must bind for all \( i = 1, \ldots, N-1 \) at \( v = V^{AUT}(i_-) \), i.e.

\[
  w_i = V^{AUT}(i) \quad \forall i = 1, \ldots N.
\]

It then follows from (12) and the definition of autarky that the period utility has to satisfy

\[
  \sum_{i=1}^{N} u_i \pi(i|i_-) = \sum_{i=1}^{N} U(y^i) \pi(i|i_-).
\]

Since the continuation utility is independent of the current report, the allocation is incentive compatible only if \( u_i = U(y^i) \) for all \( i = 1, \ldots, N \). It then follows that

\[
  V^*(V^{AUT}(i_-), i_-) = \sum_{i=1}^{N} (u_i + \beta w_i) \pi(i|i_- + 1)
  = \sum_{i=1}^{N} (U(y^i) + \beta V^{AUT}(i)) \pi(i|i_- + 1)
  = V^{AUT}(i_- + 1).
\]

In addition, it follows that the constraint (11) is satisfied automatically at \( v = V^{AUT}(i_-) \), and hence does not bind. ■

**Proof of Proposition 3.** The result follows from the fact that constraint (11) was not used in the proof of Proposition 2 and the lower contour is increasing in \( v \). ■

**Proof of Proposition 4.** We prove that \( \underline{V}(v, i_-) \leq v \) which, together with Proposition 2, implies the result. Consider an allocation that assigns \( u_i = 0 \) for all \( i = 1 \ldots N \) (by setting \( \tau_i = \infty \)) and \( w_i = \frac{v}{\beta} \). This allocation is trivially incentive compatible since it is independent of the report. It also delivers \( \hat{v} = v \). Because the lower bound is minus infinity, inequalities (10) (11) trivially hold. Hence \( \underline{V}(v, i_-) \leq v \). ■
Proof of Proposition 5. We have for all $i = 2, \ldots, N$,

$$u_i + \beta w_i \geq \psi(u_{i-1}) + \beta \hat{w}_{i-1}$$

$$\geq \psi(u_{i-1}) + \beta V(w_{i-1}, i - 1)$$

$$\geq u_{i-1} + \beta w_{i-1}$$

where the first inequality follows from (9), the second one from definition of the lower contour, and the third one from Proposition (4) and the properties of $\psi$. ■

Proof of Lemma 3. Fix $i_-$ and define an operator $T$ by the right-hand side of the Bellman equation (16):

$$T f(v, \hat{v}, i_-) = \min_{u,w,\hat{w}} \sum_{i=1}^{N} [U^{-1}(u_i) - y_i + \frac{1}{R} f(w_i, \hat{w}_i, i)] \pi(i|i_-)$$

subject to

$$(u, w, \hat{w}) \text{ implements } (v, \hat{v}) \text{ given } V \text{ and } i_-.$$  

Suppose that $(u^a, w^a, \hat{w}^a)$ solves the principal’s problem for a promised and threat utility pair $(v^a, \hat{v}^a)$. Similarly, let $(u^b, w^b, \hat{w}^b)$ solve the principal’s problem for a promised and threat utility pair $(v^b, \hat{v}^b)$. Let also $v^\lambda = \lambda v^a + (1 - \lambda) v^b$ and $\hat{v}^\lambda = \lambda \hat{v}^a + (1 - \lambda) \hat{v}^b$ for some $\lambda \in (0, 1)$. Define $(u^\lambda, w^\lambda, \hat{w}^\lambda)$ similarly.

We first show that if $(u^a, w^a, \hat{w}^a)$ implements $(v^a, \hat{v}^a)$, and $(u^b, w^b, \hat{w}^b)$ implements $(v^b, \hat{v}^b)$ given $V$ and $i_-$ then $(u^\lambda, w^\lambda, \hat{w}^\lambda)$ implements $(v^\lambda, \hat{v}^\lambda)$ given $V$ and $i_-$. Clearly, the allocation $(u^\lambda, w^\lambda, \hat{w}^\lambda)$ delivers a promised utility and threat utility pair $(v^\lambda, \hat{v}^\lambda)$, since (12) and (13) are both linear in all the variables. The incentive compatibility constraint (9) is satisfied as
well, since for all $i = 1, \ldots N$,

$$u_i^\lambda + \beta w_i^\lambda = \lambda (u_i^a + \beta w_i^a) + (1 - \lambda) (u_i^b + \beta w_i^b) \geq \lambda [\psi(u_{i-1}^a) + \beta \tilde{w}_{i-1}^a] + (1 - \lambda) [\psi(u_{i-1}^b) + \beta \tilde{w}_{i-1}^b] = \lambda \psi(u_{i-1}^a) + (1 - \lambda) (\psi(u_{i-1}^b) + \beta \tilde{w}_{i-1}^\lambda) \geq \psi(u_{i-1}^\lambda) + \beta \tilde{w}_{i-1}^\lambda,$$

where the first inequality follows from the fact that last inequality follows from the fact that (9) holds for both $(u^a, w^a, \tilde{w}^a)$ and $(u^b, w^b, \tilde{w}^b)$, and the last inequality follows from the fact that $\psi$ is convex by Lemma 1. Finally, since $\mathcal{V}$ is convex by Proposition 1, $(w_i^\lambda, \tilde{w}_i^\lambda) \in \mathcal{V}_i$ for all $i = 1, \ldots N$.

Suppose now that $f(v, \tilde{v}, i_-)$ is convex in $(v, \tilde{v})$ for all $i = 1, \ldots, N$. Then

$$T f(v^\lambda, \tilde{v}^\lambda, i_-) \leq \sum_{i=1}^N [ U^{-1}(u_i^\lambda) - y^i + \frac{1}{R} f(w_i^\lambda, \tilde{w}_i^\lambda, i) ] \pi(i|i_-) \leq \lambda T f(v^a, \tilde{v}^a, i_-) + (1 - \lambda) T f(v^b, \tilde{v}^b, i_-),$$

where the first inequality follows from the fact that $(w_i^\lambda, \tilde{w}_i^\lambda)$ is feasible but not necessarily optimal. The second inequality follows from convexity of $U^{-1}$ and $f$, and from the fact that $(u^a, w^a, \tilde{w}^a)$ is optimal given $(v^a, \tilde{v}^a)$, and $(u^b, w^b, \tilde{w}^b)$ is optimal given $(v^b, \tilde{v}^b)$. Hence $T f$ is convex in $(v, \tilde{v})$, and by the contraction mapping theorem the fixed point $C$ is also convex in $(v, \tilde{v})$.

By a corollary to the contraction mapping theorem and by strict convexity of $U^{-1}$, the fixed point $C$ is strictly convex. ■

**Proof of Lemma 4.** The proof is by contradiction. Fix $(v, \tilde{v}, i_-)$ and suppose that the solution to the cost minimization problem is such that there is some $i$ with

$$\Delta \equiv u_i + \beta w_i - \psi(u_{i-1}) - \beta \tilde{w}_{i-1} > 0. \quad (24)$$

Suppose without loss of generality that there is no other shock such that the incentive
compatibility constraint (9) is slack. Define new continuation utilities \((m, \hat{m})\) as follows: for \(j = 1, \ldots, i - 1\), let \(m_j = w_j + \Delta \left[1 - \Pi(i - 1|i_\cdot)\right]\) and \(\hat{m}_j = \hat{w}_j + \Delta \left[1 - \Pi(i - 1|i_\cdot)\right]\). Then for \(j = i, \ldots, N\) define \(m_j = w_j - \Delta \Pi(i - 1|i_\cdot)\) and \(\hat{m}_i = \hat{w}_j - \Delta \Pi(i - 1|i_\cdot)\).

By construction, the allocation \((u, m, \hat{m})\) then satisfies (9) with equality for all \(i = 2, \ldots, N\). In addition, (12) and (13) hold, since

\[
\sum_{i=1}^{N} (u_i + \beta m_i)\pi(i|i_\cdot) = \sum_{i=1}^{N} (u_i + \beta w_i)\pi(i|i_\cdot) + \sum_{i=1}^{i-1} \Delta \left[1 - \Pi(i - 1|i_\cdot)\right] - \sum_{i=1}^{N} \Delta \Pi(i - 1|i_\cdot)
\]

\[
= \sum_{i=1}^{N} (u_i + \beta w_i)\pi(i|i_\cdot)
\]

\[
= v,
\]

and similarly for (13). Proposition 4 and Proposition 1 imply that \((m_j, \hat{m}_j) \in \mathcal{V}_j\). Thus, \((u, m, \hat{m})\) implements \((v, \hat{v})\) given \(\mathcal{V}\) and \(i_\cdot\).

Since the cost function is strictly convex by Lemma 3 and \((m, \hat{m})\) is a mean preserving decrease in spread, it strictly reduces the expected costs. Hence \((u, w, \hat{w})\) cannot be a solution, a contradiction. ■

**Appendix B: Fernandes-Phelan Recursive Formulation**

Except for the initial period, the principal is constrained by a vector of promised utilities \((v_1, \ldots, v_N)\). An allocation is given by \(\{\tau_j, w^i_j\}_{i,j=1}^{N}\), where \(\tau_j \geq 0\) denotes a transfer to an agent who currently reports shock \(j\), and \(w^i_j \leq 0\) is his continuation utility, where \(j\) is the report of the agent and \(i\) is the shock. The incentive compatibility requires that the agents prefer to tell the truth about their shock to any other report:

\[
U(y^i + \tau_i) + \beta w^i_i \geq U(y^j + \tau_j) + \beta w^i_j \quad \forall i, j = 1 \ldots N.
\]

(25)
Let $\mathcal{V} \subseteq \mathbb{R}^N$. An allocation is said to be admissible with respect to $\mathcal{V}$ if it satisfies the incentive compatibility constraint (25), and the continuation utilities are drawn from the set $\mathcal{V}$:

$$(w_i^1, \ldots, w_i^N) \in \mathcal{V} \quad \forall i = 1 \ldots N.$$ 

An allocation generates a lifetime utility $v_{i-}$ to an agent who has received a shock $i-$ last period, given by

$$v_{i-} = \sum_{i=1}^{N} \left[ U(y_i + \tau_i) + \beta w_i^i \right] \pi(i| i-) \quad \forall n = 1 \ldots N. \tag{26}$$

If an allocation admissible with respect to $\mathcal{V}$ satisfies (26), it is said to support $(v_1, \ldots, v_N)$ given $\mathcal{V}$. The set of all lifetime utility vectors that are supported by some allocation that is admissible with respect to $\mathcal{V}$ defines an operator $\mathcal{T}$:

$$\mathcal{T}\mathcal{V} = \{(v_1, \ldots, v_N) | \text{ There exists an allocation that supports } (v_1, \ldots, v_N) \text{ given } \mathcal{V} \}.$$ 

Let $\mathcal{V}^*$ be the largest fixed point of the operator $\mathcal{T}$. The efficient contract can be found as follows. The social planner’s cost function $C : \mathcal{V}^* \times N \rightarrow \mathbb{R}$ satisfies the following Bellman equation:

$$C(v_1, \ldots, v_N, i-) = \min_{(\tau_j, w_j^i)_{i,j=1}^N} \sum_{i=1}^{N} \left[ \tau_i + \frac{1}{R} C(w_i^1, \ldots, w_i^N, i) \right] \pi(i| i-)$$

subject to a constraint that

$$(\tau_j, w_j^i)_{i,j=1}^N \text{ supports } (v_1, \ldots, v_N) \text{ given } \mathcal{V}^*.$$ 

The value function therefore has $I$ continuous dimensions. That is in general intractable except for a case when $I$ is very small.
Appendix C: Computing the Set of Implementable Utilities

Denote the probability of staying in the low state by \( p \) and the probability of staying in the high state by \( q \). The definition of the lower bound on \( \hat{v} \) given \( v \), denoted by \( \Omega(v) \), is as follows.

\[
\Omega(v) = \min \left\{ (1 - q)(u_L + \beta w_L) + q(u_H + \beta w_H) \right\}
\]
subject to

\[
v = p(u_L + \beta w_L) + (1 - p)(u_H + \beta w_H),
\]
\[
 u_H + \beta w_H \geq \psi(u_L) + \beta \Omega(w_L)
\]
\[
w_L \geq v_L^{\text{aut}}
\]
\[
w_h \geq v_H^{\text{aut}}
\]

The last inequality can be ignored by Proposition 2. With utility bounded above by zero, we also require \( u_L \leq 0, u_H \leq 0, w_L \leq 0 \) and \( w_H \leq 0 \). However, as we shall see, those constraints will not bind, and so we omit them.

To simplify the expression for \( \Omega(v) \), notice that \( u_H + \beta w_H \) can be eliminated by using the promise keeping constraint. Rearranging terms, the lower bound can be written as

\[
\Omega(v) = \min \left\{ \frac{1 - p - q}{1 - p} (u_L + \beta w_L) + \frac{q}{1 - p} v \right\}
\]
subject to

\[
v \geq p(u_L + \beta w_L) + (1 - p)(\psi(u_L) + \beta \Omega(w_L))
\]
\[
w_L \geq v_L^{\text{aut}}.
\]

The solution to this dynamic program will be denoted by \( u_L(v) \) and \( w_L(v) \). The following

\[13\]We will denote the low state 1 with subscript \( L \) and the high state 2 with subscript \( H \). In terms of previous notation, \( p = \pi(1,1), q = \pi(2,2) \) and \( q_1 = \frac{1 - q}{p} \). The transition matrix satisfies Assumption 2 if \( p + q - 1 > 0 \).
two lemmas are easily obtained:

**Lemma 6** The incentive constraint (29) binds along the lower boundary.

**Proof.** Suppose it doesn’t bind. Then one can increase $u_L$ by $\epsilon > 0$. This decreases the objective function. ■

**Lemma 7** The function $\Omega(v)$ is strictly convex. The optimal policy functions $u_L(v)$ and $w_L(v)$ are continuous.

**Proof.** Take any convex function $\Omega(v)$, and any $v^1$ and $v^2$. Let $u^1_L, w^1_L, u^2_L, w^2_L$ be the corresponding optimal policies. Let also $T$ be an operator defined by the right-hand side of (27). Consider a promised utility $v^\alpha = \alpha v^1 + (1 - \alpha)v^2$, and policies $u^\alpha_L$ and $w^\alpha_L$, defined analogously. Since both $\psi$ and $\Omega(v)$ are convex, $u^\alpha_L$ and $w^\alpha_L$ is a feasible choice at $v = v^\alpha$, and so

$$T\Omega(v^\alpha) \leq \alpha T\Omega(v^1) + (1 - \alpha) T\Omega(v^2).$$

Thus, $T\Omega(v^\alpha)$ is convex, and so is the fixed point of $T$. Moreover, since $\psi$ is strictly convex, the constraint (29) is slack when $u^\alpha_L$ and $w^\alpha_L$ is chosen. One can therefore increase $u_L$ by $\epsilon$, decreasing the costs. Hence the fixed point is strictly convex. ■

Assume for now that the lower bound is differentiable. Then the first order conditions for the problem imply

$$\psi'(u_L) \leq \Omega'(w_L), \quad \text{if } w_L > v^{\text{aut}}_L. \tag{31}$$

We consider two cases, depending on whether the lower bound binds or not.

**Case 1:** Lower bound on the promised utility binds.
In this case, the policy functions and the value function solve equations

\[ w_L(v) = v_{aut} \]
\[ v = p (u_L(v) + \beta v_{aut}^L) + (1 - p) (\psi(u_L) + \beta v_{aut}^H) \]
\[ \Omega(v) = \frac{1 - p - q}{1 - p} (u_L(v) + \beta v_{aut}^L) + \frac{q}{1 - p} v, \]

where we have used the fact that, as follows from Proposition 2, \( \Omega(w_L(v)) = v_{aut}^H \).
Case 2: Lower bound on the promised utility does not bind.

In this case, $w_L(v)$, $u_L(v)$ and $\Omega(v)$ solve

$$v = p(u_L(v) + \beta w_L(v)) + (1 - p)(\psi(u_L(v)) + \beta \Omega(w_L(v)))$$

$$\psi'(u_L(v)) = \Omega'(w_L(v))$$

$$\Omega(v) = \frac{1 - p - q}{1 - p} (u_L(v) + \beta w_L(v)) + \frac{q}{1 - p} v.$$ 

**Lemma 8** The function $\Omega(v)$ is differentiable, with its derivative given by

$$\Omega'(v) = \frac{1 - q + q \psi'(u_L(v))}{p + (1 - p) \psi'(u_L(v))}$$  \hspace{1cm} (35)

In addition, $\Omega'(v) \in (\psi'(u_L(v)), 1]$ with $\Omega'(v) = 1$ only when $v = 0$.

**Proof.** Suppose that case 1 applies. Then one can calculate the derivative directly to be (35). Suppose that case 2 applies. By Benveniste-Scheinkman formula its derivative exists, and is given again by (35).

It follows from the fact that $\psi'(u) \in (0, 1]$, with $\psi'(u) = 1$ only at $u = 0$ and that $p + q - 1 > 0$ that $\Omega'(v) = 1$ only when $u_L(v) = 0$. However, if $u_L(v) = 0$ then the first order condition (31) gives $\Omega'(w_L) \geq \psi'(0) = 1$, implying $w_L = 0$, which in turn implies that $v = 0$. 

As a corollary to the previous lemma, we have $u_L(v) = w_L(v) = 0$ only if $v = 0$. Thus, the inequality constraints $u_L \leq 0$ and $w_L \leq 0$ are slack for all $v < 0$. Also, the incentive constraint implies that $u_H + \beta w_H = \psi(u_L) + \beta \Omega(w_L) < 0$ for $v < 0$. This validates the exclusion of the constraints $u_L \leq 0$, $u_H \leq 0$, $w_L \leq 0$ and $w_H \leq 0$ from the dynamic program.

We now determine where case 1 applies, and where case 2 applies. Consider first $v = v_L^{\text{aut}}$. Then we know that $u_L = U(y^L)$, and one can easily obtain that $\psi'(u_L(v_L^{\text{aut}})) = \frac{U'(y^H)}{U'(y^L)}$. Thus, the right-derivative of $\Omega(v)$ is given by

$$\Omega'(v_L^{\text{aut}}) = \frac{(1 - q)U'(y^L) + q U'(y^H)}{p U'(y^L) + (1 - p) U'(y^H)}.$$
Hence
\[
\Omega'(w_L(v_{L}^{\text{aut}})) = \Omega'(v_{L}^{\text{aut}}) > \psi'(u_L(v_{L}^{\text{aut}})),
\]
That is, the lower bound constraint is binding at \(v = v_{L}^{\text{aut}}\). By continuity of the optimal policy functions, it is binding in the neighborhood of \(v_{L}^{\text{aut}}\) as well.

Define \(v^*\) to be a “breakpoint” promised utility such that the lower bound constraint is binding, and the first order condition holds with equality:
\[
\frac{(1-q)U'(y^{L}) + qU'(y^{H})}{pU'(y^{L}) + (1-p)U'(y^{H})} = \psi'(u_L(v^*)).
\]
where \(u_L(v)\) solves (33) and the left-hand side is the value of \(\Omega'(w_L)\) at \(w_L = v_{L}^{\text{aut}}\). Since \(u_L(v)\) is increasing in \(v\) and converges to 0 as \(v\) converges to zero, there is a unique value of \(v^*\) such that the lower bound constraint binds for \(v < v^*\), and is slack for \(v \geq v^*\). Note that \(v^*\) is very easy to compute numerically: compute \(u_L^* = u_L(v^*)\) that solves the previous inequality. Then use (33) to solve for the implied value of \(v^*\).
References


