General extensions of population principles to infinite-horizon social evaluation*

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Abstract

This paper examines evaluation criteria for streams of utility vectors of generations with variable population size. Specifically, we axiomatically analyze how an evaluation criterion applied to a single generation is extended between generations. First, we show that the axioms of finite anonymity, weak existence of critical levels, and existence independence are jointly equivalent to the existence of an ordering of utility profiles of finite generations satisfying the properties corresponding to the axioms that we can use to rank streams of utility vectors with a common tail. Then, adding strong Pareto and consistency axioms, we axiomatize three generalized evaluation criteria for streams of utility vectors including generalized overtaking and catching-up criteria. Further, adding minimal inequality aversion, we show that among the generalized overtaking and catching-up criteria, only those associated with a positive critical level avoid an infinite-horizon version of the repugnant conclusion. Also, we apply the results of the generalized criteria to axiomatizing infinite-horizon extensions of the critical-level leximin principle and examine their population ethics properties.

Keywords: Intergenerational equity, Variable population social choice, Overtaking criteria, Population ethics, Critical-level leximin principle

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1 Introduction

Evaluation criteria that we can use to compare possible social states with different population sizes are indispensable for evaluating public policies that will affect population sizes of the present and/or future generations. For example, when we evaluate public policies on childcare, reproductive health, immigration and refugees and so on, we need evaluation criteria that can deal with different population sizes. Also, the evaluation of climate change policy requires taking into account possible changes in population sizes of the present and future generations.

Evaluation criteria for possible social states with different population sizes have been developed in variable population social choice, where evaluation orderings of variable dimensional utility vectors are presented and axiomatized; see Blackorby, Bossert, and Donaldson (2002, 2005) for comprehensive reviews of the literature. In the standard framework of variable population social choice, population size is assumed to be finite and no distinction between generations is made. Consequently, evaluation orderings developed in that framework cannot be used to compare intergenerational utility distributions involving different population sizes of the present and infinitely many future generations such as

$$\begin{array}{rcl} & & \quad\text{1st generation} \qquad\quad \text{2nd generation} \qquad\qquad\quad t\text{-th generation} \\ \boldsymbol{u} & = & \Big((\underbrace{6,\ldots\ldots,6}_{\text{5 million people}}), \quad (\underbrace{7,\ldots\ldots,7}_{\text{3 million people}}), \quad \ldots, \quad (\underbrace{8,\ldots\ldots,8}_{\text{1 million people}}), \quad \ldots \Big) \\ \boldsymbol{v} & = & \Big((\underbrace{4,\ldots\ldots\ldots,4}_{\text{10 million people}}), \quad (\underbrace{3,\ldots\ldots\ldots,3}_{\text{11 million people}}), \quad \ldots, \quad (\underbrace{2,\ldots\ldots\ldots,2}_{\text{11.5 million people}}), \quad \ldots \Big), \end{array}$$

where an intragenerational utility distribution is represented by a utility vector, which is an element of infinite streams. The issue of how we should evaluate streams of variable dimensional utility vectors like these actually arises when we consider intertemporal economic models with endogenous population. The streams $\boldsymbol{u}$ and $\boldsymbol{v}$ described above will be attainable in a very simple economic model where each generation is endowed with a fixed amount of resources to be consumed by people alive and each potential people is assumed to have the same strictly concave utility function. See Boucekkine and Fabbri (2003), Boucekkine, Fabbri, and Gozzi (2011, 2014), Palivos and Yip (1993), and Razin and Yuen (1995) for more complicated dynamic economic models with production and endogenous population.

The framework for analyzing evaluation criteria for streams of utility vectors was presented in Kamaga (2016). This framework is different for the standard framework of variable population social choice in two respects. First, it allows us to examine evaluation criteria for utility distributions for infinitely many generations. Second, it explicitly distinguishes different generations describing each intragenerational utility distribution as an element of stream. When we deal with infinitely many generations, the explicit use of the notion of generation is important since without the distinction between different generations, we may end up with the evaluation regarding $\boldsymbol{u}$ and $\boldsymbol{v}$ described above that there are infinitely many people alive in both $\boldsymbol{u}$ and $\boldsymbol{v}$ and every individual in $\boldsymbol{u}$ has a higher utility level than people alive in $\boldsymbol{v}$, so that $\boldsymbol{u}$ is better than $\boldsymbol{v}$. It would be hard to agree with this evaluation immediately since there is the quality-quantity trade-off in each generation between $\boldsymbol{u}$ and $\boldsymbol{v}$.

There have been proposed three evaluation criteria for streams of utility vectors in Kamaga (2016), which are called a critical-level generalized utilitarian social welfare relation (SWR), a critical-level generalized overtaking SWR, and a critical-level generalized catching-up SWR, respectively.[1] Each of them is an infinite-horizon extension of critical-level generalized utilitarianism that was introduced by Blackorby and Donaldson (1984) in the finite-horizon framework of variable population social choice; see also Blackorby, Bossert, and Donaldson (1995). Specifically, they have a common feature that they apply critical-level generalized utilitarianism to the utilities of individuals alive in finite generations. In other words, they apply critical-level generalized utilitarianism within a generation and also extend its application between generations.

The purpose of this paper is to clarify through axiomatic analysis how an evaluation criterion applied within a generation is extended between generations and also to axiomatize generalized evaluation criteria for streams of utility vectors that can represent infinite-horizon extensions of critical-level generalized utilitarianism and some other finite-horizon evaluation criteria as specific examples. The importance of analyzing generalized evaluation criteria for streams of utility vectors is analogous to that of the analysis done for generalized evaluation criteria for infinite utility streams where the well-being of each generation is represented by a single utility value. In the literature on ranking infinite utility streams, a lot of researches have been done on generalized evaluation criteria to make possible to apply existing results obtained for the finite case to constructing and axiomatizing

[1] An SWR is an intratemporally anonymous and finitely complete quasi-ordering. The precise definition is given in Sect. 2.

specific evaluation criteria for infinite utility streams. See, for example, Asheim, d'Aspremont, and Banerjee (2010), Asheim and Banerjee (2010), d'Aspremont (2007), Kamaga and Kojima (2009, 2010), and Sakai (2010).[2] Analogously, the analysis of generalized criteria for streams of utility vectors allows us to make use of a large body of literature on variable population social choice developed for the finite case.

Our first result is a characterization of a class of SWRs satisfying three axioms, namely, finite anonymity, weak existence of critical levels, and existence independence. Finite anonymity formalizes equal treatment of finitely many generations. Weak existence of critical levels is a very weak assumption of the existence of a utility level such that the addition of an individual at that utility level does not change the goodness of stream. Existence independence requires the evaluation be independent of the addition of utility-unconcerned individuals. We show that the three axioms are jointly equivalent to the existence of an ordering of utility profiles of finite generations satisfying the properties corresponding to the axioms that we can use to rank streams of utility vectors with a common tail. That is, an SWR satisfying these axioms must apply an ordering of variable dimensional utility vectors satisfying the properties corresponding to the axioms not only within a generation but between generations. Examples of such orderings include critical-level generalized utilitarianism, the critical-level leximin principle in Blackorby, Bossert, and Donaldson (1996), and their lexicographic composition.

The generalized evaluation criteria for streams of utility vectors we propose and axiomatize are the *dominance-in-tails* criterion, the *generalized overtaking* criterion, and the *generalized catching-up* criterion. The dominance-in-tails criterion applies an ordering of variable dimensional utility vectors to the utilities of individuals alive in finite generations and also applies the Suppes–Sen grading principle to each of the subsequent generations. The generalized overtaking and the generalized catching-up criteria consecutively applies an ordering of variable dimensional utility vectors to the utilities of individuals alive in finite generations in the same way as the overtaking and catching-up criteria in Atsumi (1965) and von Weizsäcker (1965), respectively. Specific representations of these generalized criteria are given by employing, for example, critical-level generalized utilitarianism, the critical-level leximin principle, or their lexicographic composition as an ordering applied to finite generations. We axiomatize the dominance-in-tails criterion by

[2]Reviews of the literature on ranking infinite utility streams are presented by Asheim (2010) and Lauwers (2016).

adding the strong Pareto principle to the three axioms and the generalized overtaking and generalized catching-up criteria by adding consistency axioms. Further, we show the consequence of adding a weak distributional equity axiom called minimal inequality aversion.

We also evaluate population ethics properties of the generalized criteria. In the literature on population ethics, one of the most fundamental issue is how we can avoid the repugnant conclusion discussed by Parfit (1976, 1982, 1984); namely, the possibility of avoiding an ethically unacceptable preference for overpopulation in dealing with the quality-quantity trade-off. Using the axiom of avoidance of the repugnant conclusion, we present characterizations of the generalized overtaking and generalized catching-up criteria associated with an ordering for variable dimensional utility vectors that has a positive critical level of utility.

Our results of the generalized criteria can be applied to the analysis of their specific representations obtained by the choice of an ordering applied to finite generations. Indeed, if we additionally impose a restricted continuity axiom, we obtain the axiomatizations of the three infinite-horizon extensions of critical-level generalized utilitarianism presented in Kamaga (2016). Another application can be considered for the case of the critical-level leximin principle. Adding the Hammond equity axiom to the axiomatizations of the generalized criteria, we axiomatize the three infinite-horizon extensions of the critical-level leximin principles that correspond to specific representations of the generalized criteria. We also examine their population ethics properties using an infinite-horizon reformulation of the axiom of priority for lives worth living in Blackorby, Bossert, and Donaldson (2005).

The rest of the paper is organized as follows. Sect. 2 presents notation and basic definitions. Sect. 3 presents the characterization of a class of SWRs satisfying the three basic axioms. In Sect. 4, we provide the axiomatizations of the three generalized criteria. In Sect. 5, we evaluate population ethics properties of the generalized criteria. Sect. 6 presents the application of the general results to the critical-level leximin principle. Sect. 7 concludes the study.

2 Notation and definitions

Let $\mathbb{R}$ (resp. $\mathbb{R}_{++}$ and $\mathbb{R}_{--}$) be the set of all (resp. all positive and all negative) real numbers and $\mathbb{N}$ be the set of all positive integers. For all $n \in \mathbb{N}$, $\mathbf{1}_n$ is the vector consisting of n ones. The notation for the vector inequality is as follows: for all $n \in \mathbb{N}$ and all $(u_1, \ldots, u_n), (v_1, \ldots, v_n) \in \mathbb{R}^n$, $(u_1, \ldots, u_n) \geq (v_1, \ldots, v_n)$

if and only if $u_i \geq v_i$ for all $i = 1, \ldots, n$; and $(u_1, \ldots, u_n) > (v_1, \ldots, v_n)$ if and only if $(u_1, \ldots, u_n) \geq (v_1, \ldots, v_n)$ and $(u_1, \ldots, u_n) \neq (v_1, \ldots, v_n)$. Further, for all $(u_1, u_2, \ldots), (v_1, v_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, $(u_1, u_2, \ldots) \gg (v_1, v_2, \ldots)$ if and only if $u_i > v_i$ for all $i \in \mathbb{N}$. For all $n \in \mathbb{N}$ and for all $(u_1, \ldots, u_n) \in \mathbb{R}^n$, $(u_{[1]}, \ldots, u_{[n]})$ denotes a non-decreasing rearrangement of $(u_1, \ldots, u_n)$, ties being broken arbitrarily. For any sets A and B, we write $A \subseteq B$ to mean that A is a subset of B and $A \subset B$ to mean $A \subseteq B$ and $A \neq B$. The empty set is denoted by $\varnothing$. Negation of a statement is indicated by the symbol $\neg$.

We consider the welfarist framework of infinite-horizon variable population social choice presented in Kamaga (2016). Let $\Omega = \cup_{n \in \mathbb{N}} \mathbb{R}^n$, and let $\Omega^{\mathbb{N}}$ be the set of all streams of utility vectors $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots)$. For all $\boldsymbol{u} \in \Omega$ and all $t \in \mathbb{N}$, $\mathbf{n}(\boldsymbol{u}^t)$ is the number of components in $\boldsymbol{u}^t$, and thus, $\boldsymbol{u}^t = \left(u_1^t, \ldots, u_{\mathbf{n}(\boldsymbol{u}^t)}^t\right)$. For all $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ and all $t \in \mathbb{N}$, we interpret $\boldsymbol{u}^t$ as the utility distribution among $\mathbf{n}(\boldsymbol{u}^t)$ individuals in the t-th generation and we ignore the identities of individuals in each generation. This simplification does not affect the analysis since the evaluation relations we consider do not depend on the identities of individuals. We employ the convention in population ethics that a utility level of zero represents neutrality and a utility level above zero represents her life is worth living.[3]

For any $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ and any $t \in \mathbb{N}$, let $\boldsymbol{u}^{-t}$ denote $(\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t) \in \Omega^t$ and $\boldsymbol{u}^{+t}$ denote $(\boldsymbol{u}^{t+1}, \boldsymbol{u}^{t+2}, \ldots) \in \Omega^{\mathbb{N}}$. Thus, $\boldsymbol{u} = (\boldsymbol{u}^{-t}, \boldsymbol{u}^{+t}) = (\boldsymbol{u}^{-(t-1)}, \boldsymbol{u}^t, \boldsymbol{u}^{+t})$. We refer to $\boldsymbol{u}^{-t}$ as the head of a stream of utility vectors and $\boldsymbol{u}^{+t}$ as the tail of a stream of utility vectors. For any $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and any $t \in \mathbb{N}$, we write $[\boldsymbol{u}^t, \boldsymbol{v}^t]$ as $[\boldsymbol{u}^t, \boldsymbol{v}^t] = \left(u_1^t, \ldots, u_{\mathbf{n}(\boldsymbol{u}^t)}^t, v_1^t, \ldots, v_{\mathbf{n}(\boldsymbol{v}^t)}^t\right) \in \Omega$. Extending this notation to heads of a stream of utility vectors, for any $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ and any $t \in \mathbb{N}$, let $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t]$ denote the vector in Ω defined by $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t] = \left(u_1^1, \ldots, u_{\mathbf{n}(\boldsymbol{u}^1)}^1, \ldots, u_1^t, \ldots, u_{\mathbf{n}(\boldsymbol{u}^t)}^t\right)$.

A binary relation on $\Omega^{\mathbb{N}}$ is generically denoted by R. The asymmetric and symmetric parts of R is denoted by P and I, respectively. A binary relation on $\Omega^{\mathbb{N}}$ is quasi-ordering if it is reflexive and transitive. A binary relation R on $\Omega^{\mathbb{N}}$ is *intratemporally anonymous* if and only if, for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$, $\boldsymbol{u} I \boldsymbol{v}$ if, for all $t \in \mathbb{N}$, there exists a bijection $\pi^t : \{1, \ldots, \mathbf{n}(\boldsymbol{u}^t)\} \to \{1, \ldots, \mathbf{n}(\boldsymbol{v}^t)\}$ such that $\boldsymbol{u}^t = \left(v_{\pi^t(1)}^t, \ldots, v_{\pi^t(\mathbf{n}(\boldsymbol{u}^t))}^t\right)$. A binary relation R on $\Omega^{\mathbb{N}}$ is *finitely complete* if and only if $\boldsymbol{u} R \boldsymbol{v}$ or $\boldsymbol{v} R \boldsymbol{u}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{+t} = \boldsymbol{v}^{+t}$ for some $t \in \mathbb{N}$. An SWR on $\Omega^{\mathbb{N}}$ is an intratemporally anonymous and finitely complete quasi-ordering. Given binary relations R_1 and R_2 on $\Omega^{\mathbb{N}}$, we say that R_1 is a *subrelation* of R_2 if $I_1 \subseteq I_2$ and $P_1 \subseteq P_2$.

[3]For a discussion of neutrality and its normalization to zero, see Broome (1993).

We also consider a binary relation on Ω, which we generically denote by $\succsim$. The asymmetric and symmetric parts of $\succsim$ are denoted by $\succ$ and $\sim$, respectively. A binary relation on Ω is a quasi-ordering if it is reflexive and transitive. A binary relation on Ω is an ordering if it is a complete quasi-ordering.

3 Basic extension result

We begin by characterizing SWRs satisfying three axioms in Kamaga (2016), namely, finite anonymity, weak existence of critical levels, and existence independence. Finite anonymity asserts that the relative ranking of any two streams of utility vectors should be invariant with respect to reordering two generations.

Finite Anonymity (FA): For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z} \in \Omega^{\mathbb{N}}$, if there exist $t_1, t_2 \in \mathbb{N}$ such that $\boldsymbol{u}^{t_1} = \boldsymbol{w}^{t_2}$, $\boldsymbol{u}^{t_2} = \boldsymbol{w}^{t_1}$, $\boldsymbol{v}^{t_1} = \boldsymbol{z}^{t_2}$, $\boldsymbol{v}^{t_2} = \boldsymbol{z}^{t_1}$, and, for all $t \neq t_1, t_2$, $\boldsymbol{u}^t = \boldsymbol{w}^t$ and $\boldsymbol{v}^t = \boldsymbol{z}^t$, then $\boldsymbol{u}R\boldsymbol{v} \Leftrightarrow \boldsymbol{w}R\boldsymbol{z}$.

According to Asheim, d'Aspremont, and Banerjee (2010), this axiom should be called *relative finite anonymity*. However, we refer to the axiom as finite anonymity since for a finitely complete and transitive relation, imposing this axiom is equivalent to requiring the stronger property that is analogous to the finite anonymity axiom in the context of ranking infinite utility streams, i.e., for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$, if there exist $t_1, t_2 \in \mathbb{N}$ such that $\boldsymbol{u}^{t_1} = \boldsymbol{v}^{t_2}$, $\boldsymbol{u}^{t_2} = \boldsymbol{v}^{t_1}$, and $\boldsymbol{u}^t = \boldsymbol{v}^t$ for all $t \neq t_1, t_2$, then $\boldsymbol{u}I\boldsymbol{v}$.

To present the axiom fo weak existence of critical levels, we define the notion of a critical level of utility. For any $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ and any $t \in \mathbb{N}$, $\alpha \in \mathbb{R}$ is said to be a critical level for $\boldsymbol{u}$ at the t-th generation if $\boldsymbol{u}I(\boldsymbol{u}^{-(t-1)}, [\boldsymbol{u}^t, \alpha], \boldsymbol{u}^{+t})$. That is, a critical level of utility is the utility level such that the addition of an individual with that utility level does not change the goodness of a stream of utility vectors. Weak existence of critical levels asserts that a critical level of utility exists for at least one stream of utility vectors at at least one generation.

Weak Existence of Critical Levels (WECL): There exist $t \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ such that $\boldsymbol{u}I(\boldsymbol{u}^{-(t-1)}, [\boldsymbol{u}^t, \alpha], \boldsymbol{u}^{+t})$.

The existence independence axiom formalizes an independence property with respect to the existence of utility unconcerned individuals. It requires that the evaluation for streams of utility vectors be independent of any addition of individuals at all generations as long as the streams have a common tail.

Existence Independence (EI): For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Omega^{\mathbb{N}}$, if there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t = \boldsymbol{v}^t$ for all $t > T$, then $\boldsymbol{u} R \boldsymbol{v} \Leftrightarrow ([\boldsymbol{u}^t, \boldsymbol{w}^t])_{t\in\mathbb{N}} R ([\boldsymbol{v}^t, \boldsymbol{w}^t])_{t\in\mathbb{N}}$.

We state two lemmas that show the implications of the three axioms The first one shows that finite anonymity together with existence independence imply that any transposition of individuals across generations does not change the goodness of streams of utility vectors.

Lemma 1. *Suppose that an SWR R on $\Omega^{\mathbb{N}}$ satisfies finite anonymity and existence independence. For all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ for all $t \in \mathbb{N}$, if there exist $t_1, t_2 \in \mathbb{N}$, $i \in \{1, \dots, \mathbf{n}(\boldsymbol{u}^{t_1})\}$, and $j \in \{1, \dots, \mathbf{n}(\boldsymbol{u}^{t_2})\}$ such that $u_i^{t_1} = v_j^{t_2}$, $u_j^{t_2} = v_i^{t_1}$, and $u_k^t = v_k^t$ for all $(k, t) \neq (i, t_1), (j, t_2)$, then $\boldsymbol{u} I \boldsymbol{v}$.*

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ for all $t \in \mathbb{N}$, and suppose that there exist $t_1, t_2 \in \mathbb{N}$, $i \in \{1, \dots, \mathbf{n}(\boldsymbol{u}^{t_1})\}$, and $j \in \{1, \dots, \mathbf{n}(\boldsymbol{u}^{t_2})\}$ such that $u_i^{t_1} = v_j^{t_2}$, $u_j^{t_2} = v_i^{t_1}$, and $u_k^t = v_k^t$ for all $(k, t) \neq (i, t_1), (j, t_2)$. To show that $\boldsymbol{u} I \boldsymbol{v}$, we first consider $\bar{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ defined by

$$\bar{\boldsymbol{v}}^{t_1} = \boldsymbol{u}^{t_2},\ \bar{\boldsymbol{v}}^{t_2} = \boldsymbol{u}^{t_1},\ \text{and } \bar{\boldsymbol{v}}^t = \boldsymbol{u}^t \text{ for all } t \neq t_1, t_2.$$

By **FA**, $\boldsymbol{u} I \bar{\boldsymbol{v}}$. Next, we define $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by

$$\begin{cases} \tilde{\boldsymbol{u}}^{t_1} = [\boldsymbol{u}^{t_1}, u_i^{t_1}],\ \tilde{\boldsymbol{u}}^{t_2} = [\boldsymbol{u}^{t_2}, u_j^{t_2}],\ \text{and } \tilde{\boldsymbol{u}}^t = [\boldsymbol{u}^t, \boldsymbol{u}^t] \text{ for all } t \neq t_1, t_2; \\ \tilde{\boldsymbol{v}}^{t_1} = [\boldsymbol{u}^{t_2}, u_i^{t_1}],\ \tilde{\boldsymbol{v}}^{t_2} = [\boldsymbol{u}^{t_1}, u_j^{t_2}],\ \text{and } \tilde{\boldsymbol{v}}^t = [\boldsymbol{u}^t, \boldsymbol{u}^t] \text{ for all } t \neq t_1, t_2. \end{cases}$$

By **EI**, $\boldsymbol{u} I \bar{\boldsymbol{v}}$ implies $\tilde{\boldsymbol{u}} I \tilde{\boldsymbol{v}}$. Now, define $\check{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by

$$\check{\boldsymbol{v}}^{t_1} = [\boldsymbol{u}^{t_1}, u_j^{t_2}],\ \check{\boldsymbol{v}}^{t_2} = [\boldsymbol{u}^{t_2}, u_i^{t_1}],\ \text{and } \check{\boldsymbol{v}}^t = [\boldsymbol{u}^t, \boldsymbol{u}^t] \text{ for all } t \neq t_1, t_2.$$

By **FA**, $\tilde{\boldsymbol{v}} I \check{\boldsymbol{v}}$. Since $\tilde{\boldsymbol{u}} I \tilde{\boldsymbol{v}}$ and R is transitive, it follows $\tilde{\boldsymbol{u}} I \check{\boldsymbol{v}}$. Next, define $\hat{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by

$$\hat{\boldsymbol{v}}^{t_1} = [\boldsymbol{v}^{t_1}, u_i^{t_1}],\ \hat{\boldsymbol{v}}^{t_2} = [\boldsymbol{v}^{t_2}, u_j^{t_2}],\ \text{and } \hat{\boldsymbol{v}}^t = [\boldsymbol{u}^t, \boldsymbol{u}^t] \text{ for all } t \neq t_1, t_2.$$

Since R is intratemporally anonymous, we obtain $\check{\boldsymbol{v}} I \hat{\boldsymbol{v}}$. Further, since $\tilde{\boldsymbol{u}} I \check{\boldsymbol{v}}$ and R is transitive, we obtain $\tilde{\boldsymbol{u}} I \hat{\boldsymbol{v}}$. Thus, by **EI**, $\boldsymbol{u} I \boldsymbol{v}$ follows. ∎

The next lemma is a replication of the result presented by Blackorby, Bossert, and Donaldson (2005, Theorem 6.9 (i)) in the framework of finite-horizon variable population social choice. It shows that in the presence of finite anonymity and existence independence, weak existence of critical levels implies that the existence of a utility level which is a critical level for all streams and for all generations.

Lemma 2. *If an SWR R on $\Omega^{\mathbb{N}}$ satisfies finite anonymity, weak existence of critical levels, and existence independence, then there exists $\alpha \in \mathbb{R}$ such that*

$$\boldsymbol{u} I(\boldsymbol{u}^{-(t-1)}, [\boldsymbol{u}^t, \alpha], \boldsymbol{u}^{+t}) \textit{ for all } t \in \mathbb{N} \textit{ and for all } \boldsymbol{u} \in \Omega^{\mathbb{N}}. \tag{1}$$

Proof. By **WECL**, there exist $t^* \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $\boldsymbol{v} \in \Omega^{\mathbb{N}}$ such that $\boldsymbol{v} I(\boldsymbol{v}^{-(t^*-1)}, [\boldsymbol{v}^{t^*}, \alpha], \boldsymbol{v}^{+t^*})$. Let $\boldsymbol{u} \in \Omega^{\mathbb{N}}$. We show that $\boldsymbol{u} I(\boldsymbol{u}^{-(t^*-1)}, [\boldsymbol{u}^{t^*}, \alpha], \boldsymbol{u}^{+t^*})$. Let $\tilde{\boldsymbol{v}} = (\boldsymbol{v}^{-(t^*-1)}, [\boldsymbol{v}^{t^*}, \alpha], \boldsymbol{v}^{+t^*})$ and $\tilde{\boldsymbol{u}} = (\boldsymbol{u}^{-(t^*-1)}, [\boldsymbol{u}^{t^*}, \alpha], \boldsymbol{u}^{+t^*})$. By **EI**, $\boldsymbol{v} I \tilde{\boldsymbol{v}}$ implies $([\boldsymbol{v}^t, \boldsymbol{u}^t])_{t\in\mathbb{N}} I([\tilde{\boldsymbol{v}}^t, \boldsymbol{u}^t])_{t\in\mathbb{N}}$. Note that $[\tilde{\boldsymbol{v}}^{t^*}, \boldsymbol{u}^{t^*}]$ is a rearrangement of $[\boldsymbol{v}^{t^*}, \tilde{\boldsymbol{u}}^{t^*}]$ since $[\tilde{\boldsymbol{v}}^{t^*}, \boldsymbol{u}^{t^*}] = [\boldsymbol{v}^{t^*}, \alpha, \boldsymbol{u}^{t^*}]$. Further, for all $t \in \mathbb{N}\backslash\{t^*\}$, $[\boldsymbol{v}^t, \boldsymbol{u}^t] = [\tilde{\boldsymbol{v}}^t, \boldsymbol{u}^t] = [\boldsymbol{v}^t, \tilde{\boldsymbol{u}}^t]$. Since R is transitive and intratemporally anonymous, $([\boldsymbol{v}^t, \boldsymbol{u}^t])_{t\in\mathbb{N}} I([\tilde{\boldsymbol{v}}^t, \boldsymbol{u}^t])_{t\in\mathbb{N}}$ implies $([\boldsymbol{u}^t, \boldsymbol{v}^t])_{t\in\mathbb{N}} I([\tilde{\boldsymbol{u}}^t, \boldsymbol{v}^t])_{t\in\mathbb{N}}$. By **EI**, we obtain $\boldsymbol{u} I \tilde{\boldsymbol{u}} = (\boldsymbol{u}^{-(t^*-1)}, [\boldsymbol{u}^{t^*}, \alpha], \boldsymbol{u}^{+t^*})$. Since R is transitive and it satisfies **FA**, we can extend this result (established for t^*) to any $t \in \mathbb{N}$. We omit the easy proof of it for the sake of brevity. ■

To state the characterization of an SWR satisfying the three axioms, we need some additional definitions. Given an ordering $\succsim$ on Ω, we say that an SWR R on $\Omega^{\mathbb{N}}$ is an *extension* of $\succsim$ if for all $T \in \mathbb{N}$ and for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{+T} = \boldsymbol{v}^{+T}$,

$$[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T] \Leftrightarrow \boldsymbol{u} R \boldsymbol{v}, \tag{2}$$

that is, if R is an extension of $\succsim$, R ranks streams of utility vectors with a common tail applying $\succsim$ to vectors consisting of utilities of individuals in finite generations. Next, we define the properties of an ordering $\succsim$ on Ω corresponding to the finite anonymity and existence independence axioms and the existence of critical levels proved in Lemma 2. Throughout the paper, properties of an ordering $\succsim$ on Ω are labeled using an asterisk to distinguish them from axioms for an SWR.

Anonymity* **(A*)**: For all $n \in \mathbb{N}$ and all $\boldsymbol{u}^t, \boldsymbol{v}^t \in \mathbb{R}^n$, if there exists a permutation μ on $\{1, \dots, n\}$ such that $\boldsymbol{u}^t = \left(v^t_{\mu(1)}, \dots, v^t_{\mu(n)}\right)$, then $\boldsymbol{u}^t \sim \boldsymbol{v}^t$.

Existence of Constant Critical Levels* **(ECCL*)**: There exists $\alpha \in \mathbb{R}$ such that for all $\boldsymbol{u}^t \in \Omega$, $[\boldsymbol{u}^t, \alpha] \sim \boldsymbol{u}^t$.

Existence Independence* **(EI*)**: For all $\boldsymbol{u}^t, \boldsymbol{v}^t, \boldsymbol{w}^t \in \Omega$, $\boldsymbol{u}^t \succsim \boldsymbol{v}^t \Leftrightarrow [\boldsymbol{u}^t, \boldsymbol{w}^t] \succsim [\boldsymbol{v}^t, \boldsymbol{w}^t]$.

We are ready to state the characterization of an SWR that satisfies finite anonymity, weak existence of critical levels, and existence independence. The following theorem shows that an SWR satisfying the three axioms is an extension of an ordering on Ω satisfying anonymity*, existence of constant critical levels*, and existence independence*.

Theorem 1. *An SWR R on $\Omega^{\mathbb{N}}$ satisfies finite anonymity, weak existence of critical levels, and existence independence if and only if there exists an ordering $\succsim$ on Ω satisfying anonymity*, existence of constant critical levels*, and existence independence* such that R is an extension of $\succsim$.*

Proof. 'If.' To show that R satisfies **FA**, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that there exist $t_1, t_2 \in \mathbb{N}$ such that $\boldsymbol{u}^{t_1} = \boldsymbol{v}^{t_2}$, $\boldsymbol{u}^{t_2} = \boldsymbol{v}^{t_1}$, and $\boldsymbol{u}^t = \boldsymbol{v}^t$ for all $t \in \mathbb{N}\backslash\{t_1, t_2\}$. Let $T = \max\{t_1, t_2\}$. Since R is an extension of $\succsim$, it follows that

$$\boldsymbol{u}R\boldsymbol{v} \Leftrightarrow [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T].$$

Since $\succsim$ satisfies **A***, we have $[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$. Since R is an extension of $\succsim$, $\boldsymbol{u}I\boldsymbol{v}$ follows.

Next, we show that R satisfies **WECL**. Since $\succsim$ satisfies **ECCL***, there exists $\alpha \in \mathbb{R}$ such that $\boldsymbol{u}^1 \sim [\boldsymbol{u}^1, \alpha]$ for all $\boldsymbol{u}^1 \in \Omega$. Let $\boldsymbol{v} \in \Omega^{\mathbb{N}}$. Since R is an extension of $\succsim$, we obtain $(\boldsymbol{u}^1, \boldsymbol{v}^{+1})I([\boldsymbol{u}^1, \alpha], \boldsymbol{v}^{+1})$. Thus, R satisfies **WECL**.

Finally, to show that R satisfies **EI**, let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Omega^{\mathbb{N}}$ and $T \in \mathbb{N}$, and suppose that $\boldsymbol{u}^t = \boldsymbol{v}^t$ for all $t > T$. Since R is an extension of $\succsim$, we obtain

$$\boldsymbol{u}R\boldsymbol{v} \Leftrightarrow [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$$

and

$$([\boldsymbol{u}^t, \boldsymbol{w}^t])_{t\in\mathbb{N}}R([\boldsymbol{v}^t, \boldsymbol{w}^t])_{t\in\mathbb{N}} \Leftrightarrow [\boldsymbol{u}^1, \boldsymbol{w}^1, \dots, \boldsymbol{u}^T, \boldsymbol{w}^T] \succsim [\boldsymbol{v}^1, \boldsymbol{w}^1, \dots, \boldsymbol{u}^T, \boldsymbol{w}^T].$$

Since $\succsim$ is transitive and it satisfies **A*** and **EI***, it follows that

$$[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T] \Leftrightarrow [\boldsymbol{u}^1, \boldsymbol{w}^1, \dots, \boldsymbol{u}^T, \boldsymbol{w}^T] \succsim [\boldsymbol{v}^1, \boldsymbol{w}^1, \dots, \boldsymbol{u}^T, \boldsymbol{w}^T].$$

Thus, combining the above equivalence assertions, we obtain

$$\boldsymbol{u}R\boldsymbol{v} \Leftrightarrow ([\boldsymbol{u}^t, \boldsymbol{w}^t])_{t\in\mathbb{N}}R([\boldsymbol{v}^t, \boldsymbol{w}^t])_{t\in\mathbb{N}}.$$

'Only if.' We first show the existence of an ordering $\succsim$ on Ω such that R is an extension of it. Given $\boldsymbol{w} \in \Omega^{\mathbb{N}}$, define $\Omega^{\mathbb{N}}_{\boldsymbol{w}}$ by $\Omega^{\mathbb{N}}_{\boldsymbol{w}} = \{\boldsymbol{u} \in \Omega^{\mathbb{N}} : \boldsymbol{u}^{+1} = \boldsymbol{w}^{+1}\}$. Since there exists a bijection from $\Omega^{\mathbb{N}}_{\boldsymbol{w}}$ to Ω, we can define the binary relation $\succsim$ on Ω as follows: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega_{\boldsymbol{w}}^{\mathbb{N}}$,

$$\boldsymbol{u}^1 \succsim \boldsymbol{v}^1 \Leftrightarrow \boldsymbol{u} R \boldsymbol{v}. \tag{3}$$

Since R is an SWR, $\succsim$ is an ordering. We show that $\succsim$ satisfies (2) if $T = 1$. To show this, let $\boldsymbol{z} \in \Omega^{\mathbb{N}}$ and $\boldsymbol{u}, \boldsymbol{v} \in \Omega_{\boldsymbol{w}}^{\mathbb{N}}$. Then, we obtain, by (3) and **EI**, that

$$\boldsymbol{u}^1 \succsim \boldsymbol{v}^1 \Leftrightarrow \boldsymbol{u} R \boldsymbol{v} \Leftrightarrow ([\boldsymbol{u}^t, \boldsymbol{z}^t])_{t \in \mathbb{N}} R ([\boldsymbol{v}^t, \boldsymbol{z}^t])_{t \in \mathbb{N}} \Leftrightarrow (\boldsymbol{u}^{-1}, \boldsymbol{z}^{+1}) R (\boldsymbol{v}^{-1}, \boldsymbol{z}^{+1}).$$

Thus, $\succsim$ satisfies (2) if $T = 1$.

Next, we show that $\succsim$ satisfies (2) for $T > 1$. Let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and $T \in \mathbb{N} \backslash \{1\}$ and suppose $\boldsymbol{u}^{+T} = \boldsymbol{v}^{+T}$. Let $\ell(\boldsymbol{u})$ denote $\ell(\boldsymbol{u}) = \sum_{t=1}^{T} \mathbf{n}(\boldsymbol{u}^t)$ for all $\boldsymbol{u} \in \Omega^{\mathbb{N}}$. By Lemma 2, there exists $\alpha \in \mathbb{R}$ that satisfies (1). Define $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by $\bar{\boldsymbol{u}}^{+T} = \bar{\boldsymbol{v}}^{+T} = \boldsymbol{u}^{+T}$,

$$\bar{\boldsymbol{u}}^t = [\boldsymbol{u}^t, \alpha \mathbf{1}_{\ell(\boldsymbol{u}) - \mathbf{n}(\boldsymbol{u}^t)}] \text{ for all } t \leq T,$$

and

$$\bar{\boldsymbol{v}}^1 = [\boldsymbol{v}^1, \alpha \mathbf{1}_{\ell(\boldsymbol{v}) - \mathbf{n}(\boldsymbol{v}^1)}] \text{ and } \bar{\boldsymbol{v}}^t = [\boldsymbol{v}^t, \alpha \mathbf{1}_{\ell(\boldsymbol{u}) - \mathbf{n}(\boldsymbol{v}^t)}] \text{ for all } t = 2, \ldots, T.$$

By (1) and the transitivity of R, $\boldsymbol{u} I \bar{\boldsymbol{u}}$ and $\boldsymbol{v} I \bar{\boldsymbol{v}}$. Thus, by transitivity,

$$\boldsymbol{u} R \boldsymbol{v} \Leftrightarrow \bar{\boldsymbol{u}} R \bar{\boldsymbol{v}}. \tag{4}$$

Next, define $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ as follows: $\tilde{\boldsymbol{u}}^{+T} = \tilde{\boldsymbol{v}}^{+T} = \boldsymbol{u}^{+T}$,

$$\tilde{\boldsymbol{u}}^1 = [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \text{ and } \tilde{\boldsymbol{v}}^1 = [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T],$$

and

$$\tilde{\boldsymbol{u}}^t = \tilde{\boldsymbol{v}}^t = \alpha \mathbf{1}_{\ell(\boldsymbol{u})} \text{ for all } t = 2, \ldots, T.$$

By Lemma 1 and the transitivity of R, we obtain $\bar{\boldsymbol{u}} I \tilde{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}} I \tilde{\boldsymbol{v}}$. Thus, by transitivity,

$$\bar{\boldsymbol{u}} R \bar{\boldsymbol{v}} \Leftrightarrow \tilde{\boldsymbol{u}} R \tilde{\boldsymbol{v}}. \tag{5}$$

Further, by the definitions of $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$, we obtain

$$\tilde{\boldsymbol{u}} R \tilde{\boldsymbol{v}} \Leftrightarrow [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]. \tag{6}$$

Combining (4), (5), and (6), we complete the proof that $\succsim$ satisfies (2). Thus, R is an extension of $\succsim$.

Finally, we show that $\succsim$ satisfies $\mathbf{A}^*$, $\mathbf{ECCL}^*$, and $\mathbf{EI}^*$. Since R is an extension of $\succsim$ and intratemporally anonymous, $\succsim$ satisfies $\mathbf{A}^*$. Further, since R satisfies **EI**, $\succsim$ satisfies $\mathbf{EI}^*$. From Lemma 2, there exists $\alpha \in \mathbb{R}$ that satisfies (1). Since $\succsim$ is an extension of $\succsim$, it follows that for all $\boldsymbol{u}^t \in \Omega$, $[\boldsymbol{u}^t, \alpha] \sim \boldsymbol{u}^t$. Thus, $\succsim$ satisfies $\mathbf{ECCL}^*$. ■

From Theorem 1, if we require an SWR to satisfy the three axioms in the theorem statement, a permissible SWR must be an extension of an ordering $\succsim$ on Ω satisfying the three properties in the theorem statement. Examples of an ordering $\succsim$ on Ω satisfying those properties are given by a *critical-level generalized utilitarian ordering* in Blackorby and Donaldson (1984), a *critical-level leximin ordering* in Blackorby, Bossert, and Donaldson (1996), and the *lexicographic composition of these orderings* that applies a critical-level generalized utilitarian ordering first.

Given $\alpha \in \mathbb{R}$ and a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$, the critical-level generalized utilitarian ordering associated with α and g is defined as the following ordering $\succsim_{U,\alpha}$ on Ω: for all $n, m \in \mathbb{N}$, for all $\boldsymbol{u}^t \in \mathbb{R}^n$, and for all $\boldsymbol{v}^t \in \mathbb{R}^m$,

$$\boldsymbol{u}^t \succsim_{U,\alpha} \boldsymbol{v}^t \Leftrightarrow \sum_{i=1}^{n}(g(u_i^t) - g(\alpha)) \geq \sum_{i=1}^{m}(g(v_i^t) - g(\alpha)).$$

Given $\alpha \in \mathbb{R}$, the critical-level leximin ordering associated with α is defined as the following ordering $\succsim_{L,\alpha}$ on Ω: for all $n, m \in \mathbb{N}$, for all $\boldsymbol{u}^t \in \mathbb{R}^n$, and for all $\boldsymbol{v}^t \in \mathbb{R}^m$,

$$\boldsymbol{u}^t \succ_{L,\alpha} \boldsymbol{v}^t \Leftrightarrow \begin{cases} \boldsymbol{u}^t \succ_L^n [\boldsymbol{v}^t, \alpha\mathbf{1}_{n-m}] & \text{if } n \geq m \\ [\boldsymbol{u}^t, \alpha\mathbf{1}_{m-n}] \succ_L^m \boldsymbol{v}^t & \text{if } n < m, \end{cases}$$
$$\boldsymbol{u}^t \sim_{L,\alpha} \boldsymbol{v}^t \Leftrightarrow \begin{cases} \boldsymbol{u}^t \sim_L^n [\boldsymbol{v}^t, \alpha\mathbf{1}_{n-m}] & \text{if } n \geq m \\ [\boldsymbol{u}^t, \alpha\mathbf{1}_{m-n}] \sim_L^m \boldsymbol{v}^t & \text{if } n < m, \end{cases}$$

where for all $n \in \mathbb{N}$, $\succsim_L^n$ denotes the leximin ordering on $\mathbb{R}^n$; that is, for all $\boldsymbol{u}^t, \boldsymbol{v}^t \in \mathbb{R}^n$, (i) $\boldsymbol{u}^t \succ_L^n \boldsymbol{v}^t$ if and only if there exists $m \leq n$ such that $u_{[m]}^t > v_{[m]}^t$ and $u_{[i]}^t = v_{[i]}^t$ for all $i < m$; and (ii) $\boldsymbol{u}^t \sim_L^n \boldsymbol{v}^t$ if and only if $u_{[i]}^t = v_{[i]}^t$ for all $i = 1, \ldots, n$.

Given $\alpha \in \mathbb{R}$ and a continuous and increasing function $g\colon \mathbb{R} \to \mathbb{R}$, the lexicographic composition of $\succsim_{U,\alpha}$ and $\succsim_{L,\alpha}$ that applies $\succsim_{U,\alpha}$ first is defined as the following ordering $\succsim_{UL,\alpha}$ on Ω: for all $n, m \in \mathbb{N}$, for all $\boldsymbol{u}^t \in \mathbb{R}^n$, and for all

$\boldsymbol{v}^t \in \mathbb{R}^m$,

$$\boldsymbol{u}^t \succsim_{UL,\alpha} \boldsymbol{v}^t \Leftrightarrow \text{ (i) } \boldsymbol{u}^t \succ_{U,\alpha} \boldsymbol{v}^t \text{ or (ii) } \boldsymbol{u}^t \sim_{U,\alpha} \boldsymbol{v}^t \text{ and } \boldsymbol{u}^t \succsim_{L,\alpha} \boldsymbol{v}^t.$$

Note that each of these orderings constitutes a class of orderings with respect to a parameter α and a function g considered. Any member of these three classes of orderings on Ω satisfies the strong Pareto property as well.

Strong Pareto* **(SP*)**: For all $n \in \mathbb{N}$ and all $\boldsymbol{u}^t, \boldsymbol{v}^t \in \mathbb{R}^n$, if $\boldsymbol{u}^t \geq \boldsymbol{v}^t$ and $\boldsymbol{u}^t \neq \boldsymbol{v}^t$, then $\boldsymbol{u}^t \succ \boldsymbol{v}^t$.

Depending on the choice of the value of α and a function g, the associated three classes of orderings $\succsim_{U,\alpha}$, $\succsim_{L,\alpha}$, and $\succsim_{UL,\alpha}$ satisfies additional properties. If $\succsim_{U,\alpha}$, $\succsim_{L,\alpha}$, and $\succsim_{UL,\alpha}$ are associated with a positive α, these orderings satisfy the following property, which is stronger than existence of constant critical levels*.

Existence of Positive Constant Critical Levels* **(EPCCL*)**: There exists $\alpha \in \mathbb{R}_{++}$ such that for all $\boldsymbol{u}^t \in \Omega$, $[\boldsymbol{u}^t, \alpha] \sim \boldsymbol{u}^t$.

If a continuous and increasing function $g \colon \mathbb{R} \to \mathbb{R}$ is concave, the associated $\succsim_{U,\alpha}$ and $\succsim_{UL,\alpha}$ exhibits inequality aversion. To retain the scope of general analysis in the following sections as large as possible, we consider the very weak inequality aversion property considered in Blackorby, Bossert, and Donaldson (2005) and Blackorby, Bossert, Donaldson, and Fleurbaey (1998).

Minimal Inequality Aversion* **(MIA*)**: For all $n \in \mathbb{N}$ and for all $\boldsymbol{u}^1, \boldsymbol{v}^1 \in \Omega$, if $u_i^1 = (1/n)\sum_{j=1}^n v_j^1$ for all $i \in \{1, \ldots, n\}$, then $\boldsymbol{u} \succsim \boldsymbol{v}$.

Not only $\succsim_{U,\alpha}$ and $\succsim_{UL,\alpha}$ associated with a continuous, increasing, and concave function g but $\succsim_{L,\alpha}$ satisfies minimal inequality aversion* regardless of the choice of α.

The three classes of orderings presented above can be used to give a specific representation of the generalized evaluation criteria that we will analyze in the following sections.

4 Axiomatizations of generalized evaluation criteria

4.1 Dominance-in-tails criterion

In this section, we present and axiomatize three generalized evaluation criteria for streams of utility vectors, which we call *dominance-in-tails criterion*, *generalized overtaking criterion*, and *generalized catching-up criterion*. Each of them is an extension of an ordering on Ω and is given a specific representation according to the choice of an ordering on Ω that is applied to the heads of streams, for example, a critical-level generalized utilitarian ordering $\succsim_{U,\alpha}$, a critical-level leximin ordering $\succsim_{L,\alpha}$, and a lexicographic composition $\succsim_{UL,\alpha}$ of them.

To present the definition of the dominance-in-tails criterion, we define the quasi-ordering on Ω called the Suppes-Sen grading principle due to Sen (1970) and Suppes (1966). The Suppes-Sen grading principle $\succsim_S$ is the quasi-ordering on Ω defined as follows: for all $\boldsymbol{u}^t, \boldsymbol{v}^t \in \Omega$,

$$\boldsymbol{u}^t \succsim_S \boldsymbol{v}^t \Leftrightarrow \mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t) \text{ and } (u^t_{[1]}, \ldots, u^t_{[\mathbf{n}(\boldsymbol{u}^t)]}) \geq (v^t_{[1]}, \ldots, v^t_{[\mathbf{n}(\boldsymbol{u}^t)]}).$$

It is easy to check that, for all $\boldsymbol{u}^t, \boldsymbol{v}^t \in \Omega$, (i) $\boldsymbol{u}^t \succ_S \boldsymbol{v}^t$ if and only if $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ and $(u^t_{[1]}, \ldots, u^t_{[\mathbf{n}(\boldsymbol{u}^t)]}) > (v^t_{[1]}, \ldots, v^t_{[\mathbf{n}(\boldsymbol{u}^t)]})$, and (ii) $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ if and only if $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ and $(u^t_{[1]}, \ldots, u^t_{[\mathbf{n}(\boldsymbol{u}^t)]}) = (v^t_{[1]}, \ldots, v^t_{[\mathbf{n}(\boldsymbol{u}^t)]})$.

The dominance-in-tails criterion associated with an ordering $\succsim$ on Ω ranks streams of utility vectors applying $\succsim$ to the heads of streams and $\succsim_S$ to each generation in the tails of streams. Formally, given an ordering $\succsim$ on Ω, the dominance-in-tails criterion associated with $\succsim$ is defined as the following binary relation R^D on $\Omega^{\mathbb{N}}$: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{u} R^D \boldsymbol{v} \Leftrightarrow {} & \text{there exists } T \in \mathbb{N} \text{ such that } \boldsymbol{u}^t \succsim_S \boldsymbol{v}^t \text{ for all } t > T \text{ and} \\ & [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]. \end{aligned} \tag{7}$$

The dominance-in-tails criterion $\succsim^D$ associated with a critical-level generalized utilitarian ordering $\succsim_{U,\alpha}$ is called critical-level generalized utilitarian SWR in Kamaga (2016). While R^D associated with $\succsim_{U,\alpha}$ is an SWR, there is no guarantee that R^D associated with an arbitrary ordering $\succsim$ on Ω is an SWR. Meanwhile, the following lemma shows that R^D is well defined as an SWR if it is associated with an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, and existence independence*. It also provides the characterization of the asymmetric and symmetric parts of R^D. The proof is relegated to Appendix.

Lemma 3. *Let $\succsim$ be an ordering on Ω satisfying strong Pareto*, anonymity*, and existence independence*. Then, R^D associated with $\succsim$ is an SWR on $\Omega^{\mathbb{N}}$ and for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,*

$$\begin{aligned} \boldsymbol{u} P^D \boldsymbol{v} \Leftrightarrow \text{ there exists } T \in \mathbb{N} \text{ such that } \boldsymbol{u}^t \succsim_S \boldsymbol{v}^t \text{ for all } t > T \text{ and} \\ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T], \end{aligned} \tag{8a}$$

$$\begin{aligned} \boldsymbol{u} I^D \boldsymbol{v} \Leftrightarrow \text{ there exists } T \in \mathbb{N} \text{ such that } \boldsymbol{u}^t \sim_S \boldsymbol{v}^t \text{ for all } t > T \text{ and} \\ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]. \end{aligned} \tag{8b}$$

To present an axiomatization of the dominance-in-tails criterion R^D, we use the strong Pareto axiom, which requires an SWR must to be positively sensitive to individuals' utilities.

Strong Pareto (SP): For all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ such that $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ for all $t \in \mathbb{N}$, if $\boldsymbol{u}^t \geq \boldsymbol{v}^t$ for all $t \in \mathbb{N}$ and there exists $t' \in \mathbb{N}$ such that $\boldsymbol{u}^{t'} > \boldsymbol{v}^{t'}$, then $\boldsymbol{u} P \boldsymbol{v}$.

The following theorem shows that adding strong Pareto to the axioms in Theorem 1, the dominance-in-tails criterion R^D associated with an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of constant critical levels*, and existence independence* is characterized in terms of subrelation. That is, the class of SWRs satisfying the axioms coincides with the class of SWRs that include R^D associated with $\succsim$ satisfying the four properties as a subrelation.

Theorem 2. *An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, and existence independence if and only if there exists an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of constant critical levels*, and existence independence* such that R^D associated with $\succsim$ is a subrelation of R.*

Proof. 'If.' First, we show that R^D is an extension of $\succsim$. Let $T \in \mathbb{N}$ and $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{+T} = \boldsymbol{v}^{+T}$. By (7), $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ implies $\boldsymbol{u} R^D \boldsymbol{v}$, which in turn implies $\boldsymbol{u} R \boldsymbol{v}$ since $R^D \subseteq R$. Next, assume $\boldsymbol{u} R \boldsymbol{v}$. We show $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ by contradiction. Suppose $\neg[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. Since $\succsim$ is complete, we obtain $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. By (8a), $\boldsymbol{v} P^D \boldsymbol{u}$. Since $P^D \subseteq P$, we obtain $\boldsymbol{v} P \boldsymbol{u}$. This is a contradiction since $\boldsymbol{u} R \boldsymbol{v}$.

Next, we show that R satisfies the axioms in the theorem statement. Since R is the extension of $\succsim$, it follows from Theorem 1 that R satisfies **FA**, **WECL**, and **EI**. Since $\succsim$ satisfies **SP*** and R^D is a subrelation of R, it follows from (8a) that R satisfies **SP**.

'Only if.' From Theorem 1, there exists an ordering $\succsim$ on Ω satisfying $\mathbf{A}^*$, $\mathbf{ECCL}^*$, and $\mathbf{EI}^*$ such that R is an extension of $\succsim$. Since R satisfies **SP**, it follows from (2) that $\succsim$ satisfies $\mathbf{SP}^*$. We show that R^D associated with $\succsim$ is a subrelation of R. To show that $P^D \subseteq P$, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose $\boldsymbol{u} P^D \boldsymbol{v}$. By (8a), there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t \succsim_S \boldsymbol{v}^t$ for all $t > T$ and $[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$. Let $\boldsymbol{w} = (\boldsymbol{u}^{-T}, \boldsymbol{v}^{+T})$. Since R is an SWR and satisfies **SP**, $\boldsymbol{u} R \boldsymbol{w}$. Since R is an extension of $\succsim$, we obtain $\boldsymbol{w} P \boldsymbol{v}$. By transitivity, $\boldsymbol{u} P \boldsymbol{v}$. Next, to show that $I^D \subseteq I$, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose $\boldsymbol{u} I^D \boldsymbol{v}$. By (8b), there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ for all $t > T$ and $[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$. Let $\boldsymbol{w} = (\boldsymbol{u}^{-T}, \boldsymbol{v}^{+T})$. Since R is intratemporally anonymous, $\boldsymbol{u} I \boldsymbol{w}$ follows. Since R is an extension of $\succsim$, we obtain $\boldsymbol{w} I \boldsymbol{v}$. By transitivity, $\boldsymbol{u} I \boldsymbol{v}$. ■

To axiomatize the dominance-in-tails criterion associated with an ordering $\succsim$ on Ω that also satisfies minimal inequality aversion*, we define a direct reformulation of minimal inequality aversion* for an SWR as follows.

Minimal Inequality Aversion: There exists $t \in \mathbb{N}$ such that for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{t'} = \boldsymbol{v}^{t'}$ for all $t' \neq t$, if $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ and $u_i^t = (1/\mathbf{n}(\boldsymbol{u}^t)) \sum_{j=1}^{\mathbf{n}(\boldsymbol{u}^t)} v_j^t$ for all $i \in \{1, \dots, \mathbf{n}(\boldsymbol{u}^t)\}$, then $\boldsymbol{u} R \boldsymbol{v}$.

In the following theorem, we state the consequence of adding minimal inequality aversion to the axioms in Theorem 2. The proof is analogous to Theorem 2.

Theorem 3. *An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, and minimal inequality avresion if and only if there exists an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of constant critical levels*, existence independence*, and minimal inequality aversion* such that R^D associated with $\succsim$ is a subrelation of R.*

From Theorems 2 and 3, the dominance-in-tails criterion associated with an ordering $\succsim$ satisfying the properties in the theorem statements is the least element with respect to set inclusion in the class of all SWRs satisfying the axioms in the theorem statements. By Arrow's (1963) variant of Szpilrajn's (1930) lemma, there exists an ordering extension of R_L in these class. However, it is non-constructible object since the impossibility of explicit construction of a Paretian and finitely anonymous ordering for infinite utility streams proved by Dubey (2011), Lauwers (2010), and Zame (2007) carries over to the current framework.

4.2 Generalized overtaking and catching-up criteria

We next present the generalized overtaking and catching-up criteria, which are generalized reformulations of the overtaking and catching-up criteria in Atsumi (1965) and von Weizsäcker (1965). These evaluation criteria can compare (not all but some) streams of utility vectors even if the streams have different population sizes in their tails. We will show that their axiomatic characterizations are obtained by adding consistency axioms to the axioms in Theorem 1.

The generalized overtaking and generalized catching-up criteria associated with an ordering $\succsim$ on Ω rank streams of utility vectors by consecutively applying $\succsim$ to the heads of streams in slightly different ways. Given an ordering $\succsim$ on Ω, the generalized overtaking criterion associated with $\succsim$ is defined as the following binary relation R^O on Ω: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{u} P^O \boldsymbol{v} \Leftrightarrow & \text{ there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ & [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T], \end{aligned} \tag{9a}$$

$$\begin{aligned} \boldsymbol{u} I^O \boldsymbol{v} \Leftrightarrow & \text{ there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ & [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]. \end{aligned} \tag{9b}$$

Analogously, given an ordering $\succsim$ on Ω, we define the generalized catching-up criterion associated with $\succsim$ as the following binary relation R^C on $\Omega^{\mathbb{N}}$: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{u} R^C \boldsymbol{v} \Leftrightarrow & \text{ there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ & [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]. \end{aligned} \tag{10}$$

The generalized overtaking and generalized catching-up criteria associated with a critical-level generalized utilitarian ordering $\succsim_{U,\alpha}$ are called critical-level generalized overtaking SWR and critical-level generalized catching-up SWR, respectively, in Kamaga (2016).

The following lemma shows that the generalized overtaking and generalized catching-up criteria associated with an ordering $\succsim$ are well defined as an SWR on $\Omega^{\mathbb{N}}$ if $\succsim$ satisfies anonymity* and existence independence*. It also characterizes the asymmetric and symmetric parts of R^C. We relegate the proof to Appendix.

Lemma 4. *Let $\succsim$ be an ordering on Ω satisfying anonymity* and existence independence*.*

(i) R^O is an SWR.

(ii) R^C *is an SWR and for all* $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\left.\begin{aligned} \boldsymbol{u}P^C\boldsymbol{v} \Leftrightarrow & \text{ there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ & [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \gtrsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \\ & \text{and for all } T' \in \mathbb{N}, \text{ there exists } T > T' \text{ such that} \\ & [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]; \end{aligned}\right\} \tag{11a}$$

$$\begin{aligned} \boldsymbol{u}I^C\boldsymbol{v} \Leftrightarrow & \text{ there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ & [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]. \end{aligned} \tag{11b}$$

From Lemmas 3 and 4, it follows that if an ordering $\gtrsim$ on Ω satisfies strong Pareto*, anonymity*, and existence independence*, the dominance-in-tails criterion R^D associated with $\gtrsim$ is a subrelation of the generalized overtaking criterion R^O associated with $\gtrsim$, which in turn is a subrelation of the generalized catching-up criterion R^C associated with $\gtrsim$.

To provide axiomatic characterizations of R^O and R^C, we consider the three consistency axioms presented in Kamaga (2016). The first two axioms assert, in weak and strong forms, that the strict preference relation of the evaluation must be consistent with the evaluations obtained for streams with a common tail.

Weak Preference Consistency (WPC): For all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$, if $(\boldsymbol{u}^{-t}, \boldsymbol{w}^{+t})P(\boldsymbol{v}^{-t}, \boldsymbol{w}^{+t})$ for all $t \in \mathbb{N}$ and all $\boldsymbol{w} \in \Omega^{\mathbb{N}}$, then $\boldsymbol{u}P\boldsymbol{v}$.

Strong Preference Consistency (SPC): For all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$, if, for all $\boldsymbol{w} \in \Omega^{\mathbb{N}}$, $(\boldsymbol{u}^{-t}, \boldsymbol{w}^{+t})R(\boldsymbol{v}^{-t}, \boldsymbol{w}^{+t})$ for all $t \in \mathbb{N}$ and, for all $t' \in \mathbb{N}$, there exists $t > t'$ such that $(\boldsymbol{u}^{-t}, \boldsymbol{w}^{+t})P(\boldsymbol{v}^{-t}, \boldsymbol{w}^{+t})$, then $\boldsymbol{u}P\boldsymbol{v}$.

Note that strong preference consistency is stronger than weak preference consistency since the former allows weak preference relations in its premise.

The indifference consistency axiom requires the consistency property of the indifference relation analogously to weak preference consistency.

Indifference Consistency (IC): For all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$, if $(\boldsymbol{u}^{-t}, \boldsymbol{w}^{+t})I(\boldsymbol{v}^{-t}, \boldsymbol{w}^{+t})$ for all $t \in \mathbb{N}$ and all $\boldsymbol{w} \in \Omega^{\mathbb{N}}$, then $\boldsymbol{u}I\boldsymbol{v}$.

The following theorem shows (i) that adding weak preference consistency and indifference consistency to the axioms in Theorem 1, the generalized overtaking criterion R^O associated with an ordering $\succsim$ satisfying anonymity*, existence of constant critical levels*, and existence independence* is characterized in terms of subrelation and (ii) that if we strengthen weak preference consistency to strong preference consistency, we obtain an axiomatization of the generalized overtaking criterion R^O associated with the ordering $\succsim$.

Theorem 4. *(i) An SWR R on $\Omega^{\mathbb{N}}$ satisfies finite anonymity, weak existence of critical levels, existence independence, weak preference consistency, and indifference consistency if and only if there exists an ordering $\succsim$ on Ω satisfying anonymity*, existence of constant critical levels*, and existence independence* such that R^O associated with $\succsim$ is a subrelation of R.*

(ii) An SWR R on $\Omega^{\mathbb{N}}$ satisfies finite anonymity, weak existence of critical levels, existence independence, strong preference consistency, and indifference consistency if and only if there exists an ordering $\succsim$ on Ω satisfying anonymity, existence of constant critical levels*, and existence independence* such that R^C associated with $\succsim$ is a subrelation of R.*

Proof. (i) 'If.' First, we show that R is an extension of $\succsim$. Let $T \in \mathbb{N}$ and $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{+T} = \boldsymbol{v}^{+T}$. Suppose $[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$, and we show that $\boldsymbol{u}R\boldsymbol{v}$. Since $\succsim$ satisfies **EI***, we obtain that, for all $T' \geq T$,

$$[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T] \Leftrightarrow [\boldsymbol{u}^1, \dots, \boldsymbol{u}^{T'}] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^{T'}].$$

Thus, by (9a) and (9b), we have $\boldsymbol{u}R^O\boldsymbol{v}$. Since R^O is a subrelation of R, $\boldsymbol{u}R\boldsymbol{v}$. Next, assume $\boldsymbol{u}R\boldsymbol{v}$, and we show $[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$. Suppose, by way of contradiction, $\neg[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \dots, \boldsymbol{v}^T]$. Since $\succsim$ is complete, we have $[\boldsymbol{v}^1, \dots, \boldsymbol{v}^T] \succ [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T]$. Since $\succsim$ satisfies **EI***, we obtain that, for all $T' \geq T$, $[\boldsymbol{v}^1, \dots, \boldsymbol{v}^{T'}] \succ [\boldsymbol{u}^1, \dots, \boldsymbol{u}^{T'}]$. By (9a), $\boldsymbol{v}P^O\boldsymbol{u}$. Since R^O is a subrelation of R, we obtain $\boldsymbol{v}P\boldsymbol{u}$. This is a contradiction since $\boldsymbol{u}R\boldsymbol{v}$.

Since R is an extension of $\succsim$. it follows from Theorem 1, R satisfies **FA**, **WECL**, and **EI**. To show that R satisfies **WPC**, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that $(\boldsymbol{u}^{-t}, \boldsymbol{w}^{+t})P(\boldsymbol{v}^{-t}, \boldsymbol{w}^{+t})$ for all $t \in \mathbb{N}$ and all $\boldsymbol{w} \in \Omega^{\mathbb{N}}$. Since R is an extension of $\succsim$, $[\boldsymbol{u}^1, \dots, \boldsymbol{u}^T] \succ [\boldsymbol{u}^1, \dots, \boldsymbol{u}^T]$ for all $T \in \mathbb{N}$. By (9a), $\boldsymbol{u}P^O\boldsymbol{v}$. Since R^O is a subrelation of R, we obtain $\boldsymbol{u}P\boldsymbol{v}$. By using (9b) instead of (9a), we can analogously show that R satisfies **IC**, and we omit its proof.

'Only if.' From Theorem 1, there exists an ordering $\succsim$ on Ω satisfying **A***, **ECCL***, and **EI*** such that R is an extension of $\succsim$. We show that R^O associated with $\succsim$ is a subrelation of R. To show that $P^O \subseteq P$, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose $\boldsymbol{u}P^O\boldsymbol{v}$. By

(9a), there exists $T^* \in \mathbb{N}$ such that, for all $T \geq T^*$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. By Lemma 2, there exists $\alpha \in \mathbb{R}$ satisfying (1). Define $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by

$$\begin{cases} \tilde{\boldsymbol{u}}^1 = \boldsymbol{u}^1, \ \tilde{\boldsymbol{u}}^t = [\boldsymbol{u}^t, \alpha] \text{ for all } t \in \{2, \ldots, T^*\}, \ \text{ and } \tilde{\boldsymbol{u}}^{+T^*} = \boldsymbol{u}^{+T^*}; \\ \tilde{\boldsymbol{v}}^1 = \boldsymbol{v}^1, \ \tilde{\boldsymbol{v}}^t = [\boldsymbol{v}^t, \alpha] \text{ for all } t \in \{2, \ldots, T^*\}, \ \text{ and } \tilde{\boldsymbol{v}}^{+T^*} = \boldsymbol{v}^{+T^*}. \end{cases}$$

Since R is transitive, we obtain by (1) that $\tilde{\boldsymbol{u}} I \boldsymbol{u}$ and $\tilde{\boldsymbol{v}} I \boldsymbol{v}$. We next define $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by

$$\begin{cases} \bar{\boldsymbol{u}}^1 = [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T^*}], \ \bar{\boldsymbol{u}}^t = \alpha \text{ for all } t = 2, \ldots, T^*, \ \text{ and } \bar{\boldsymbol{u}}^{+T^*} = \boldsymbol{u}^{+T^*}; \\ \bar{\boldsymbol{v}}^1 = [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T^*}], \ \bar{\boldsymbol{v}}^t = \alpha \text{ for all } t = 2, \ldots, T^*, \ \text{ and } \bar{\boldsymbol{v}}^{+T^*} = \boldsymbol{v}^{+T^*}. \end{cases}$$

Since $\succsim$ satisfies $\mathbf{A}^*$, $[\tilde{\boldsymbol{u}}^1, \ldots, \tilde{\boldsymbol{u}}^{T^*}] \sim [\bar{\boldsymbol{u}}^1, \ldots, \bar{\boldsymbol{u}}^{T^*}]$ and $[\tilde{\boldsymbol{v}}^1, \ldots, \tilde{\boldsymbol{v}}^{T^*}] \sim [\bar{\boldsymbol{v}}^1, \ldots, \bar{\boldsymbol{v}}^{T^*}]$. Since R is an extension of $\succsim$, we obtain $\tilde{\boldsymbol{u}} I \bar{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}} I \bar{\boldsymbol{v}}$. Since R is transitive, we obtain

$$\boldsymbol{u} R \boldsymbol{v} \Leftrightarrow \bar{\boldsymbol{u}} R \bar{\boldsymbol{v}}.$$

We show $\bar{\boldsymbol{u}} P \bar{\boldsymbol{v}}$ to prove $P^O \subseteq P$. Recall that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ for all $T \geq T^*$. Thus, $\bar{\boldsymbol{u}}^1 \succ \bar{\boldsymbol{v}}^1$. Since $\succsim$ satisfies $\mathbf{EI}^*$, we obtain $[\bar{\boldsymbol{u}}^1, \ldots, \bar{\boldsymbol{u}}^T] \succ [\bar{\boldsymbol{v}}^1, \ldots, \bar{\boldsymbol{v}}^T]$ for all $T = 2, \ldots, T^*$. Further, since $\succsim$ satisfies $\mathbf{A}^*$ and $\mathbf{EI}^*$ and it is transitive, we obtain $[\bar{\boldsymbol{u}}^1, \ldots, \bar{\boldsymbol{u}}^T] \succ [\bar{\boldsymbol{v}}^1, \ldots, \bar{\boldsymbol{v}}^T]$ for all $T > T^*$. Since R is an extension of $\succsim$, we obtain $(\bar{\boldsymbol{u}}^{-T}, \boldsymbol{w}^{+T}) P (\bar{\boldsymbol{v}}^{-T}, \boldsymbol{w}^{+T})$ for all $T \in \mathbb{N}$ and all $\boldsymbol{w} \in \Omega^{\mathbb{N}}$. By **WPC**, $\bar{\boldsymbol{u}} P \bar{\boldsymbol{v}}$. Using (9b) and **IC** instead of (9a) and **WPC**, the proof that $I^O \subseteq I$ is analogous.

(ii) 'If.' Since R^O associated with $\succsim$ is a subrelation of R^C associated with $\succsim$, R satisfies **FA**, **WECL**, **EI**, and **IC**. Applying the same argument as the proof of (i) using (11a) instead of (9a), the proof that R satisfies **SPC** is analogous.

'Only if.' By the same argument as the proof of (i), we can analogously show that R^C associated with $\succsim$ is a subrelation of R. Specifically, since $I^O = I^C$, it suffices to show $P^C \subseteq P$. This can be shown by using (11a) and **SPC** instead of (9a) and **WPC**. ■

If we add strong Pareto and minimal inequality aversion in Theorem 4, $\succsim$ must satisfy strong Pareto* and minimal inequality aversion*. We state this result as the following theorem. The proof is analogous to Theorem 4.

Theorem 5. *(i) An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, minimal inequality aversion, weak preference consistency, and indifference consistency if and only if* *there exists an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of constant critical levels*, existence independence*, and minimal inequality aversion* such that R^O associated with $\succsim$ is a subrelation of R.*

(ii) An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, minimal inequality aversion, strong preference consistency, and indifference consistency if and only if there exists an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of constant critical levels*, existence independence*, and minimal inequality aversion* such that R^C associated with $\succsim$ is a subrelation of R.

5 Avoidance of the Repugnant Conclusion

In this section, we evaluate the generalized overtaking and generalized catching-up criteria, R^O and R^C, using the axiom of avoidance of the repugnant conclusion in Kamaga (2016). We will show that if we add the axiom of avoidance of the repugnant conclusion to the axioms in Theorem 5, we obtain characterizations of R^O and R^C associated with an ordering $\succsim$ on Ω that also satisfies existence of positive constant critical levels*.

One of the most fundamental issues is how we can avoid the repugnant conclusion that was pointed out by Parfit (1976, 1982, 1984) against the classical utilitarianism. In the context of evaluating social states with finite and variable population size, the repugnant conclusion is defined as the following ethically unacceptable evaluation, namely, some social state where every member of the population has a high positive utility level is declared to be worse than another social state with a much larger population where each member has a utility level which is positive but barely above zero. Applying Parfit's argument generation by generation, Kamaga (2016) reformulated the repugnant conclusion in the current framework as follows. An SWR R on $\Omega^{\mathbb{N}}$ is said to imply the *repugnant conclusion* if and only if, for any stream of population sizes $(n_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and for any stream of positive utility levels of generations $(\xi_t)_{t\in\mathbb{N}}, (\varepsilon_t)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$ satisfying $(\xi_t)_{t\in\mathbb{N}} \gg (\varepsilon_t)_{t\in\mathbb{N}}$, there exists a stream of population sizes $(m_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $(m_t)_{t\in\mathbb{N}} \gg (n_t)_{t\in\mathbb{N}}$ such that $(\varepsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}} P (\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}}$. The following axiom requires the repugnant conclusion to be avoided.

Avoidance of the Repugnant Conclusion (ARC): There exist $(n_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(\xi_t)_{t\in\mathbb{N}}, (\varepsilon_t)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$ with $(\xi_t)_{t\in\mathbb{N}} \gg (\varepsilon_t)_{t\in\mathbb{N}}$ such that for all $(m_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with

$(m_t)_{t\in\mathbb{N}} \gg (n_t)_{t\in\mathbb{N}}$, $(\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}} R (\varepsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}}$.

Note that this axiom implies the negation of the repugnant conclusion.

The following theorem shows that if we add avoidance of the repugnant conclusion to the axioms in Theorem 5, R^O and R^C associated with an ordering $\succsim$ that has a positive constant critical level are characterized.

Theorem 6. (i) *An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, minimal inequality aversion, weak preference consistency, indifference consistency, and avoidance of the repugnant conclusion if and only if there exists an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of positive constant critical levels*, existence independence*, and minimal inequality aversion* such that R^O associated with $\succsim$ is a subrelation of R.*

(ii) *An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, minimal inequality aversion, strong preference consistency, indifference consistency, and avoidance of the repugnant conclusion if and only if there exists an ordering $\succsim$ on Ω satisfying strong Pareto*, anonymity*, existence of positive constant critical levels*, existence independence*, and minimal inequality aversion* such that R^C associated with $\succsim$ is a subrelation of R.*

Proof. (i) 'If.' Since **EPCCL*** implies **ECCL***, it follows from Theorem 5 (i) that we only need to show that R satisfies **ARC**. Since $\succsim$ satisfies **EPCCL***, there exists $\alpha \in \mathbb{R}_{++}$ such that for all $\boldsymbol{u}^t \in \Omega$, $[\boldsymbol{u}^t, \alpha] \sim \boldsymbol{u}^t$. Let $\xi_t = \alpha$ and $\varepsilon_t \in (0, \alpha)$ for all $t \in \mathbb{N}$. Then, for any $(m_t)_{t\in\mathbb{N}}, (n_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $(m_t)_{t\in\mathbb{N}} \gg (n_t)_{t\in\mathbb{N}}$, we obtain that for all $t \in \mathbb{N}$,

$$[\xi_1 \mathbf{1}_{n_1}, \ldots, \xi_t \mathbf{1}_{n_t}] \sim [\xi_1 \mathbf{1}_{m_1}, \ldots, \xi_t \mathbf{1}_{m_t}].$$

Since $\succsim$ satisfies strong Pareto*, it follows that for all $t \in \mathbb{N}$,

$$[\xi_1 \mathbf{1}_{m_1}, \ldots, \xi_t \mathbf{1}_{m_t}] \succ [\varepsilon_1 \mathbf{1}_{m_1}, \ldots, \varepsilon_t \mathbf{1}_{m_t}].$$

By the transitivity of $\succsim$, we obtain that for all $t \in \mathbb{N}$,

$$[\xi_1 \mathbf{1}_{n_1}, \ldots, \xi_t \mathbf{1}_{n_t}] \succ [\varepsilon_1 \mathbf{1}_{m_1}, \ldots, \varepsilon_t \mathbf{1}_{m_t}].$$

From (9a), it follows that

$$(\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}} P^O (\varepsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}}.$$

Since R^O is a subrelation of R, we obtain $(\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}} P (\varepsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}}$. Thus, R satisfies **ARC**.

'Only if.' From Theorem 5 (i), there exists an ordering $\succsim$ on Ω satisfying **SP**$*$, **A***, **ECCL***, **EI***, and **MIA*** such that R^O associated with $\succsim$ is a subrelation of R. From **ECCL***, there exists $\alpha \in \mathbb{R}$ such that for all $\boldsymbol{u}^t \in \Omega$, $[\boldsymbol{u}^t, \alpha] \sim \boldsymbol{u}^t$. We show that $\alpha > 0$. By way of contradiction, suppose $\alpha \leq 0$. Let $(n_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(\xi_t)_{t\in\mathbb{N}}, (\varepsilon_t)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$ with $(\xi_t)_{t\in\mathbb{N}} \gg (\varepsilon_t)_{t\in\mathbb{N}}$. Consider $(m_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ that satisfies that for all $t \in \mathbb{N}$, $m_t > n_t$ and

$$m_t \varepsilon_t > n_t \xi_t + (m_t - n_t)\alpha.$$

Define $\boldsymbol{u}, \bar{\boldsymbol{u}} \in \Omega^{\mathbb{N}}$ by, for all $t \in \mathbb{N}$, $\boldsymbol{u}^t = [\xi_t \mathbf{1}_{n_t}, \alpha \mathbf{1}_{m_t - n_t}]$ and $\bar{\boldsymbol{u}}^t = \delta_t \mathbf{1}_{m_t}$, where

$$\delta_t = \frac{n_t \xi_t + (m_t - n_t)\alpha}{m_t}.$$

Then, we obtain that for all $t \in \mathbb{N}$

$$[\xi_1 \mathbf{1}_{n_1}, \ldots, \xi_t \mathbf{1}_{n_t}] \sim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t].$$

Since $\succsim$ is transitive and satisfies **EI*** and **MIA***, it follows that for all $t \in \mathbb{N}$,

$$[\bar{\boldsymbol{u}}^1, \ldots, \bar{\boldsymbol{u}}^t] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t]$$

Since $\delta_t < \varepsilon_t$ for all $t \in \mathbb{N}$ and $\succsim$ satisfies **SP***, we obtain that for all $t \in \mathbb{N}$

$$[\varepsilon_1 \mathbf{1}_{m_1}, \ldots, \varepsilon_t \mathbf{1}_{m_t}] \succ [\bar{\boldsymbol{u}}^1, \ldots, \bar{\boldsymbol{u}}^t].$$

Thus, by the transitivity of $\succsim$, we obtain that for all $t \in \mathbb{N}$,

$$[\varepsilon_1 \mathbf{1}_{m_1}, \ldots, \varepsilon_t \mathbf{1}_{m_t}] \succ [\xi_1 \mathbf{1}_{n_1}, \ldots, \xi_t \mathbf{1}_{n_t}].$$

From (9a), it follows that $(\varepsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}} P^O_L (\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}}$, which implies

$$(\varepsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}} P (\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}}.$$

This means that R implies the repugnant conclusion. However, this is a contradiction since R satisfies **ARC**. Thus, $\alpha > 0$ and $\succsim$ satisfies **EPCCL***.

(ii) The proof of the if-part is straightforward from Theorems 5 (ii) and 6 (i). Also, the proof of the only-if-part is straightforward from Theorem 6 (i) since, if R^C associated with $\succsim$ is a subrelation of R, then R^O associated with $\succsim$ is also a subrelation of R. ■

Now, we consider two more issues in population ethics: the weak repugnant conclusion due to Broome (1992) and the mere addition principle in Parfit (1984). We consider the infinite-horizon reformulations of the weak repugnant conclusion and the mere addition principle in Kamaga (2016). We say that an SWR R on $\Omega^{\mathbb{N}}$ implies the *weak repugnant conclusion* if and only if, for any $(n_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and any $(\xi_t)_{t\in\mathbb{N}}, (\epsilon_t)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$ with $(\xi_t)_{t\in\mathbb{N}} \gg (\epsilon_t)_{t\in\mathbb{N}} \gg (\alpha, \alpha, \ldots)$, there exists $(m_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $(m_t)_{t\in\mathbb{N}} \gg (n_t)_{t\in\mathbb{N}}$ such that $(\epsilon_t \mathbf{1}_{m_t})_{t\in\mathbb{N}} P (\xi_t \mathbf{1}_{n_t})_{t\in\mathbb{N}}$, where $\alpha \in \mathbb{R}_{++}$ is a critical level for all $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ at any $t \in \mathbb{N}$. An SWR R on $\Omega^{\mathbb{N}}$ satisfies the *mere addition principle* if and only if, for all $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ and all $(\xi_t)_{t\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_{++}$, $([\boldsymbol{u}^t, \xi_t])_{t\in\mathbb{N}} R \boldsymbol{u}$.

It is easy to check that R^O_L associated with a positive critical level implies the weak repugnant conclusion, Further, it can be checked that R^O_L associated with a positive critical level α violates the mere addition principle since, for any $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ and any $\xi \in (0, \alpha)$, we obtain $\boldsymbol{u} P ([\boldsymbol{u}^t, \xi])_{t\in\mathbb{N}}$. This observation applies to any SWR that includes R^O_L associated with $\alpha > 0$. Thus, we state the following remark.

Proposition 1. *Every SWR in the classes characterized in Theorem 6 implies the weak repugnant conclusion and violates the mere addition principle.*

Proof. For any SWR R on $\Omega^{\mathbb{N}}$, there exists an ordering $\succsim$ on Ω satisfying **SP**$*$, **A***, **EPCCL***, **EI***, and **MIA*** such that R^O associated with $\succsim$ is a subrelation of R. Since $\succsim$ satisfies **EPCCL***, there exists $\alpha \in \mathbb{R}_{++}$ such that for all $\boldsymbol{u}^t \in \Omega$, $[\boldsymbol{u}^t, \alpha] \sim \boldsymbol{u}^t$. Note that α is a critical level of utility for all $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ at any $t \in \mathbb{N}$. Applying the same argument as the proof of the only-if-part of Theorem 6, we can show that R implies the weak repugnant conclusion and we do not explicitly state it.

To show show that R violates the mere addition principle, define $\boldsymbol{u} \in \Omega^{\mathbb{N}}$ by $\boldsymbol{u}^t = \alpha$ for all $t \in \mathbb{N}$ and let $\xi \in (0, \alpha)$. Then, we obtain that for all $t \in \mathbb{N}$,

$$[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t] \sim [\underbrace{\alpha\mathbf{1}_2, \ldots, \alpha\mathbf{1}_2}_{t \text{ times}}].$$

Define $\boldsymbol{v}, \bar{\boldsymbol{v}} \in \Omega^{\mathbb{N}}$ by, for all $t \in \mathbb{N}$. $\boldsymbol{v}^t = (\alpha, \xi)$ and $\bar{\boldsymbol{v}}^t = ((\alpha + \xi)/2, (\alpha + \xi)/2)$. Since

$\succsim$ satisfies $\mathbf{SP}^*$ and $\mathbf{MIA}^*$, it follows that for all $t \in \mathbb{N}$,

$$[\underbrace{\alpha\mathbf{1}_2, \ldots, \alpha\mathbf{1}_2}_{t \text{ times}}] \succ [\bar{\boldsymbol{v}}^1, \ldots, \bar{\boldsymbol{v}}^t]$$

and

$$[\bar{\boldsymbol{v}}^1, \ldots, \bar{\boldsymbol{v}}^t] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^t].$$

Since $\succsim$ is transitive, we obtain that for all $t \in \mathbb{N}$,

$$[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^t] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^t].$$

From (9a), it follows that $\boldsymbol{u}P^O\boldsymbol{v}$. Since R^O is a subrelation of R, we obtain $\boldsymbol{u}P\boldsymbol{v}$. This means that $\succsim$ violates the mere addition principle. ■

6 Application to the critical-level leximin principle

In this section, we apply the results of the generalized criteria in Sect. 4 and 5 to axiomatizing infinite-horizon extensions of the critical-level leximin principle, each of which is a specific representation of the generalized criteria associated with a critical-level leximin ordering $\succsim_{L,\alpha}$ on Ω.

We will call the dominance-in-tails criterion associated with a critical-level leximin ordering on Ω *critical-level leximin SWR* and denote it by $R^D_{L,\alpha}$. Formally, the critical-level leximin SWR associated with a given α is defined as follows: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{u}R^D_{L,\alpha}\boldsymbol{v} \Leftrightarrow \text{ there exists } T \in \mathbb{N} \text{ such that } \boldsymbol{u}^t \succsim_S \boldsymbol{v}^t \text{ for all } t > T \text{ and} \\ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim_{L,\alpha} [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]. \end{aligned} \tag{12}$$

For any $\alpha \in \mathbb{R}$, the restriction of the associated critical-level leximin SWR to $\mathbb{R}^{\mathbb{N}}$ coincides with the leximin quasi-ordering for infinite utility streams introduced by Bossert, Sprumont, and Suzumura (2007).

To axiomatize the critical-level leximin SWR, we introduce the equity axiom called Hammond equity, which is a reformulation of the axiom introduced by Hammond (1976) in the fixed population social choice.[4] It formalizes the equity property that an order-preserving change that diminishes the inequality of utilities between two conflicting individuals in some generation is socially preferable. Our

[4]See Blackorby, Bossert, and Donaldson (1996, 2002, 2005) for a version of the axiom formalized in the finite-horizon framework of variable population social choice.

version of the axiom requires that this property hold for at least one generation.

Hammond Equity (HE): There exists $t \in \mathbb{N}$ such that for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{t'} = \boldsymbol{v}^{t'}$ for all $t' \neq t$, if $\mathbf{n}(\boldsymbol{u}^t) = \mathbf{n}(\boldsymbol{v}^t)$ and there exist $i, j \in \{1, \ldots, \mathbf{n}(\boldsymbol{u}^t)\}$ such that $v_i^t < u_i^t \leq u_j^t < v_j^t$ and $u_k^t = v_k^t$ for all $k \neq i, j$, then $\boldsymbol{u}R\boldsymbol{v}$.

Note that Hammond equity implies minimal inequality aversion.

In the following theorem, we present an axiomatization of a critical-level leximin SWR by strengthening minimal inequality aversion to Hammond equity in the set of axioms in Theorem 3.

Theorem 7. *An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, and Hammond equity if and only if there exists $\alpha \in \mathbb{R}$ such that $R^D_{L,\alpha}$ associated with α is a subrelation of R.*

Proof. 'If.' From Theorem 3, it follows that R satisfies **SP**, **FA**, **WECL**, and **EI**. It is easy to show that R satisfies **HE** and we do not explicitly state it.

'Only if.' From Theorem 3, there exists an ordering $\succsim$ on Ω satisfying **SP***, **A***, **ECCL***, and **EI*** such that R^D associated with $\succsim$ is a subrelation of R. Further, since R satisfies **HE** and **FA** and it is transitive, $\succsim$ satisfies the following property corresponding to **HE**.

Hammond Equity* **(HE*****)**: For all $n \in \mathbb{N}$ and all $\boldsymbol{u}^t, \boldsymbol{v}^t \in \mathbb{R}^n$, if there exists $i, j \in \{1, \ldots, n\}$ such that $v_i^t < u_i^t < u_j^t < v_j^t$ and $u_k^t = v_k^t$ for all $k \neq i, j$, then $\boldsymbol{u}^t \succsim \boldsymbol{v}^t$.

From Theorem 6. 13 in Blackorby, Bossert, and Donaldson (2005), if an ordering $\succsim$ on Ω satisfies **SP***, **A***, **ECCL***, **EI***, and **HE***, then there exists $\alpha \in \mathbb{R}$ such that $\succsim = \succsim_{L,\alpha}$. ■

The generalized overtaking and catching-up criteria associated with a critical-level leximin ordering, which we will call *critical-level leximin overtaking* and *critical-level leximin catching-up* SWRs respectively, consecutively applies a critical-level leximin ordering associated with a given $\alpha \in \mathbb{R}$ to the heads of streams of utility vectors. Given $\alpha \in \mathbb{R}$, the critical-level leximin overtaking SWR $R^O_{L,\alpha}$ associated with α is defined as follows: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\begin{gathered}\boldsymbol{u}P^O_{L,\alpha}\boldsymbol{v} \Leftrightarrow \text{there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ_{L,\alpha} [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T],\end{gathered} \tag{13a}$$

$$\begin{gathered}\boldsymbol{u}I^O_{L,\alpha}\boldsymbol{v} \Leftrightarrow \text{there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim_{L,\alpha} [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T].\end{gathered} \tag{13b}$$

For any $\alpha \in \mathbb{R}$, the restriction of the associated critical-level leximin overtaking SWR to $\mathbb{R}^{\mathbb{N}}$ coincides with the leximin version of the overtaking criterion, called W-leximin quasi-ordering, for infinite utility streams introduced by Asheim and Tungodden (2004).

Given $\alpha \in \mathbb{R}$, the critical-level leximin catching-up SWR $R^C_{L,\alpha}$ associated with α is defined as follows: for all $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$,

$$\begin{aligned} \boldsymbol{u} R^C_{L,\alpha} \boldsymbol{v} \Leftrightarrow \text{ there exists } T^* \in \mathbb{N} \text{ such that, for all } T \geq T^* \\ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim_{L,\alpha} [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]. \end{aligned} \tag{14}$$

For any $\alpha \in \mathbb{R}$, the restriction of the associated critical-level leximin catching-up SWR to $\mathbb{R}^{\mathbb{N}}$ coincides with the leximin version of the catching-up criterion, called S-leximin quasi-ordering, for infinite utility streams in Asheim and Tungodden (2004).

In the following theorem, we axiomatize a critical-level leximin overtaking SWR and a critical-level leximin catching-up SWR strengthening minimal inequality aversion to Hammond equity in Theorem 5.

Theorem 8. *(i) An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, Hammond equity, weak preference consistency, and indifference consistency if and only if there exists $\alpha \in \mathbb{R}$ such that $R^O_{L,\alpha}$ associated with α is a subrelation of R.*

(ii) An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, Hammond equity, strong preference consistency, and indifference consistency if and only if there exists $\alpha \in \mathbb{R}$ such that $R^C_{L,\alpha}$ associated with α is a subrelation of R.

Proof of Theorem 8. We only prove (i). The proof of (ii) is analogous. 'If.' From Theorem 5 that R satisfies **SP**, **FA**, **WECL**, **EI**, **WPC**, and **IC**. Since $R^D_{L,\alpha}$ is a subrelation of $R^O_{L,\alpha}$, it follows from Theorem 7, R also satisfies **HE**.

'Only if.' From Theorem 5 (i), there exists an $\succsim$ ordering on Ω satisfying **SP***, **A***, **ECCL***, and **EI*** such that R^O associated with $\succsim$ is a subrelation of R. From Lemmas 3 and 4, R^D associated with $\succsim$ is a subrelation of R^O associated with $\succsim$. From Theorem 7, there exists $\alpha \in \mathbb{R}$ such that $\succsim = \succsim_{L,\alpha}$. ■

Applying Theorem 6, if we add avoidance of the repugnant conclusion to the axioms in Theorem 4, we obtain axiomatizations of critical-level leximin overtaking and catching-up SWRs associated with a positive critical-level.

Theorem 9. *(i) An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, Hammond equity, weak preference consistency, indifference consistency, and avoidance of the repugnant conclusion if and only if there exists $\alpha \in \mathbb{R}_{++}$ such that $R^O_{L,\alpha}$ associated with α is a subrelation of R.*

(ii) An SWR R on $\Omega^{\mathbb{N}}$ satisfies strong Pareto, finite anonymity, weak existence of critical levels, existence independence, Hammond equity, strong preference consistency, indifference consistency, and avoidance of the repugnant conclusion if and only if there exists $\alpha \in \mathbb{R}_{++}$ such that $R^C_{L,\alpha}$ associated with α is a subrelation of R.

Proof. We only prove (i). The proof of (ii) is analogous. 'If.' From Theorems 6 and 8, it follows that R satisfies all the axioms in the theorem statement.

'Only if.' From Theorem 8, there exists $\alpha \in \mathbb{R}$ such that $R^O_{L,\alpha}$ associated with α is a subrelation of R. From Theorem 6, $\succsim_{L,\alpha}$ satisfies **EPCCL***. Thus, $\alpha > 0$. ■

In what follows, we examine other population ethics properties of the critical-level leximin overtaking and catching-up SWR associated with a positive critical level α. To this end, we consider an infinite-horizon variant of the very sadistic conclusion that was introduced by Arrhenius (2000, forthcoming) in the finite-horizon context of population ethics. Let $\Omega_{++} = \cup_{n \in \mathbb{N}} \mathbb{R}^n_{++}$ and $\Omega_{--} = \cup_{n \in \mathbb{N}} \mathbb{R}^n_{--}$. Following Kamaga (2016), we say that an SWR R on $\Omega^{\mathbb{N}}$ implies the *very sadistic conclusion* if and only if, for any stream of negative utility vectors $\boldsymbol{u} \in \Omega^{\mathbb{N}}_{--}$, there exists a stream of positive utility vectors $\boldsymbol{v} \in \Omega^{\mathbb{N}}_{++}$ such that $\boldsymbol{u}P\boldsymbol{v}$.

To examine whether $R^O_{L,\alpha}$ and $R^C_{L,\alpha}$ associated with $\alpha > 0$ avoid the very sadistic conclusion, we define the infinite-horizon extension of the axiom of priority for lives worth living in Blackorby, Bossert, and Donaldson (2005) as follows.

Priority for Lives Worth Living (PLWL): For all $\boldsymbol{u} \in \Omega^{\mathbb{N}}_{--}$ and all $\boldsymbol{v} \in \Omega^{\mathbb{N}}_{++}$, $\boldsymbol{v}P\boldsymbol{u}$.

Note that priority for lives worth living implies the negation of the very sadistic conclusion.

The following proposition shows that every SWR that includes R^O_L associated with $\alpha > 0$ as a subrelation avoids the very sadistic conclusion.

Proposition 2. *Suppose that an SWR R on $\Omega^{\mathbb{N}}$ includes $R^O_{L,\alpha}$ associated with $\alpha \in \mathbb{R}$ as a subrelation. R satisfies priority for lives worth living if and only if $\alpha \geq 0$.*

Proof. 'If.' Let $\boldsymbol{u} \in \Omega_{--}^{\mathbb{N}}$, $\boldsymbol{v} \in \Omega_{++}^{\mathbb{N}}$, and $\alpha \geq 0$. Then, for all $T \in \mathbb{N}$, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succ_{L,\alpha} [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. Since $R^O_{L,\alpha}$ associated with α, we obtain by (9a), $\boldsymbol{v}P\boldsymbol{u}$.

'Only if.' To prove the contraposition, suppose $R^O_{L,\alpha}$ associated with $\alpha < 0$ is a subrelation of R. Consider $\boldsymbol{u} \in \Omega_{--}^{\mathbb{N}}$ and $\boldsymbol{v} \in \Omega_{++}^{\mathbb{N}}$ such that for all $t \in \mathbb{N}$, $\boldsymbol{u}^t = (\varepsilon, \varepsilon)$ with $\varepsilon \in (\alpha, 0)$ and $\boldsymbol{v}^t = -\alpha$. Then, for all $T \in \mathbb{N}$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ_{L,\alpha} [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. Since R^O_L associated with α, we obtain by (9a), $\boldsymbol{u}P\boldsymbol{v}$. Thus, R violates **PLWL**. ■

From Proposition 2, the critical-level leximin overtaking and catching-up SWRs associated with a non-negative critical level α are characterized by replacing avoidance of the repugnant conclusion with priority for lives worth living in Theorem 9.

Combining Propositions 1 and 2 with Theorem 9, the critical-level leximin overtaking and catching-up SWRs associated with a positive critical level α avoid the repugnant conclusion and very sadistic conclusion and satisfy priority for lives worth living, whereas they imply the weak repugnant conclusion and violate the mere addition principle.

7 Conclusion

In this paper, we examined generalized infinite-horizon extensions of an ordering of variable dimensional utility vectors. We have shown that the axioms of finite anonymity, weak existence of critical levels, and existence independence jointly imply that a social welfare relation for streams utility vectors must apply an ordering of variable dimensional utility vectors satisfying the corresponding properties to utility profiles of finite generations as long as streams have a common tail. The three generalized evaluation criteria we axiomatized have specific representations employing, for example, a critical-level generalized utilitarian ordering, a critical-level leximin ordering, and their lexicographic composition, as an ordering used to evaluate utility profiles of finite generations. As we demonstrated using a critical-level leximin ordering, the results of generalized criteria are useful useful stepping stones to analyzing possible infinite-horizon extensions of an ordering of variable dimensional utility vectors with the use of existing results in variable population social choice.

Our results of the generalized criteria, however, have a limitation in exploring infinite-horizon extensions of some well-established orderings of variable dimensional utility vectors. Specifically, as we showed in Lemmas 3 and 4, the existence independence property of an ordering of variable dimensional utility vectors is included in a sufficient condition for the ordering to be extended as an SWR in the forms of the generalized criteria we presented. In the literature of variable population social choice, there have been proposed many orderings that violate the existence independence property, e.g., average generalized utilitarianism in Blackorby, Bossert, and Donaldson (1999, 2005), number-dampened (generalized) utilitarianism in Blackorby, Bossert, and Donaldson (2005) and Ng (1986), rank-discounted critical-level generalized utilitarianism in Asheim and Zuber (2014), and a version of the critical-level leximin principle in Arrhenius (forthcoming). Our results suggest that we may need to explore other forms of infinite-horizon extension of an ordering of variable dimensional utility vectors to construct infinite-horizon reformulations of those orderings. We should address this issue in future research.

Appendix

Proof of Lemma 3. To prove that R^D is an SWR, we first show that R^D is reflexive. Let $\boldsymbol{u} \in \Omega^{\mathbb{N}}$. Since $\succsim$ and $\succsim_S$ are reflexive, we obtain $\boldsymbol{u}^1 \succsim \boldsymbol{u}^1$ and $\boldsymbol{u}^t \succsim_S \boldsymbol{u}^t$ for all $t > 1$. By (7), $\boldsymbol{u}R^D\boldsymbol{u}$. Next, to show that R^D is transitive, let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Omega^{\mathbb{N}}$ and suppose that $\boldsymbol{u}R^D\boldsymbol{v}$ and $\boldsymbol{v}R^D\boldsymbol{w}$. By (7), there exists $T \in \mathbb{N}$ such that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ and $\boldsymbol{u}^t \succsim_S \boldsymbol{v}^t$ for all $t > T$, and there exists $T' \in \mathbb{N}$ such that $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^{T'}]$ and $\boldsymbol{v}^t \succsim_S \boldsymbol{w}^t$ for all $t > T'$. If $T = T'$, since $\succsim$ and $\succsim_S$ are transitive, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ and $\boldsymbol{u}^t \succsim_S \boldsymbol{w}^t$ for all $t > T$. Thus, by (7), $\boldsymbol{u}R^D\boldsymbol{w}$. Now, consider the case that $T \neq T'$. Without loss of generality, we assume $T > T'$. Since $\succsim$ satisfies $\mathbf{EI}^*$, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^{T'}]$ implies $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}, \boldsymbol{w}^{T'+1}, \ldots, \boldsymbol{w}^T] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$. Since $\succsim$ satisfies $\mathbf{SP}^*$ and $\mathbf{A}^*$ and it is transitive, we obtain $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}, \boldsymbol{w}^{T'+1}, \ldots, \boldsymbol{w}^T]$. By transitivity of $\succsim$, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$. Further, since $\succsim_S$ is transitive, $\boldsymbol{u}^t \succsim_S \boldsymbol{w}^t$ for all $t > T$. Thus, by (7), $\boldsymbol{u}R^D\boldsymbol{w}$. Next, to show that R^D is finitely complete, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^{+T} = \boldsymbol{v}^{+T}$. Since $\succsim$ is complete, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ or $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. Since $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ for all $t > T$, we obtain, by (7), $\boldsymbol{u}R^D\boldsymbol{v}$ or $\boldsymbol{v}R^D\boldsymbol{u}$. Finally, to show that R^D is intratemporally anonymous, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that, for all $t \in \mathbb{N}$, there exists a bijection $\pi^t : \{1, \ldots, \mathbf{n}(\boldsymbol{u}^t)\} \to \{1, \ldots, \mathbf{n}(\boldsymbol{v}^t)\}$ such that $\boldsymbol{u}^t = \left(v^t_{\pi^t(1)}, \ldots, v^t_{\pi^t(\mathbf{n}(\boldsymbol{u}^t))}\right)$. Since $\succsim$ satisfies $\mathbf{A}^*$ and it is transitive, we obtain $\boldsymbol{u}^1 \sim \boldsymbol{v}^1$. Further, we obtain $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ for all $t > 1$. Thus, by (7), $\boldsymbol{u}I^D\boldsymbol{v}$.

We next prove (8a) and (8b). First, we prove the if-part of (8a). Let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t \succsim_S \boldsymbol{v}^t$ for all $t > T$ and

$[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. By (7), $\boldsymbol{u} R^D \boldsymbol{v}$. We show $\neg \boldsymbol{v} R^D \boldsymbol{u}$ by contradiction. Suppose $\boldsymbol{v} R^D \boldsymbol{u}$. By (7), there exists $T' \in \mathbb{N}$ such that $\boldsymbol{v}^t \succsim_S \boldsymbol{u}^t$ for all $t > T'$ and $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T'}]$. Since we obtain a contradiction to $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ if $T = T'$, we consider the case that $T \neq T'$. Without loss of generality, we assume $T > T'$. Since $\succsim$ satisfies $\mathbf{EI}^*$, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T'}]$ implies $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}, \boldsymbol{u}^{T'+1}, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. Since $\succsim$ satisfies $\mathbf{SP}^*$ and $\mathbf{A}^*$ and it is transitive, we obtain $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}, \boldsymbol{u}^{T'+1}, \ldots, \boldsymbol{u}^T]$. Thus, by transitivity, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. This is a contradiction to $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. Thus, $\neg \boldsymbol{v} R^D \boldsymbol{u}$.

Next, we prove the only-if-part of (8a). Let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose $\boldsymbol{u} P^D \boldsymbol{v}$. By (7), there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t \succsim_S \boldsymbol{v}^t$ for all $t > T$ and $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. We distinguish two cases: (a) $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ for all $t > T$ and (b) $\boldsymbol{u}^{T^*} \succ_S \boldsymbol{v}^{T^*}$ for some $T^* > T$. First, consider case (a). We show $\neg[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$ by contradiction. Suppose $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. By (7), $\boldsymbol{v} R^D \boldsymbol{u}$. This is a contradiction to $\boldsymbol{u} P^D \boldsymbol{v}$. Thus, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. Next, consider case (b). Since $\succsim$ satisfies $\mathbf{EI}^*$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ implies $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T, \boldsymbol{v}^{T+1}, \ldots, \boldsymbol{v}^{T^*}] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T^*}]$. Since $\succsim$ satisfies $\mathbf{SP}^*$ and $\mathbf{A}^*$ and it is transitive, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T^*}] \succ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T, \boldsymbol{v}^{T+1}, \ldots, \boldsymbol{v}^{T^*}]$. By transitivity, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T^*}] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T^*}]$.

To prove the if-part of (8b), let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ for all $t > T$ and $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. By (7), $\boldsymbol{u} R^D \boldsymbol{v}$ and $\boldsymbol{v} R^D \boldsymbol{u}$, or equivalently, $\boldsymbol{u} I^D \boldsymbol{v}$. Next, to prove the only-if-part of (8b), let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose $\boldsymbol{u} I^D \boldsymbol{v}$. By (7), there exists $T \in \mathbb{N}$ such that $\boldsymbol{u}^t \succsim_S \boldsymbol{v}^t$ for all $t > T$ and $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$, and there exists $T' \in \mathbb{N}$ such that $\boldsymbol{v}^t \succsim_S \boldsymbol{u}^t$ for all $t > T'$ and $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T'}]$. Without loss of generality, we assume $T \geq T'$. Then, we obtain $\boldsymbol{u}^t \sim_S \boldsymbol{v}^t$ for all $t > T$. If $\neg[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$, then by (8a), we obtain $\boldsymbol{u} P^D \boldsymbol{v}$, a contradiction to $\boldsymbol{u} I^D \boldsymbol{v}$. Thus, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. ■

Proof of Lemma 4. (i) First, we prove that R^O is well defined as a binary relation on $\Omega^{\mathbb{N}}$. To this end, we show that $P^O \cap I^O \neq \emptyset$ and that P^O and I^O are, respectively, asymmetric and symmetric. We show $P^O \cap I^O \neq \emptyset$ by contradiction. Suppose $\boldsymbol{u} P^O \boldsymbol{v}$ and $\boldsymbol{u} I^O \boldsymbol{v}$. By (9a) and (9b), there exists $T \in \mathbb{N}$ such that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ and $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. This is a contradiction to that $\succsim$ is a binary relation on Ω. Next, to show that P^O is asymmetric, suppose $\boldsymbol{u} P^O \boldsymbol{v}$. By (9a), there is no $T^* \in \mathbb{N}$ such that, for all $T \geq T^*$, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succ [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. Thus, $\neg \boldsymbol{v} P^O \boldsymbol{u}$. Now, to show that I^O is symmetric, suppose $\boldsymbol{u} I^O \boldsymbol{v}$. By (9b), there exists $T^* \in \mathbb{N}$ such that, for all $T \geq T^*$, $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \sim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. Thus, $\boldsymbol{v} I^O \boldsymbol{u}$.

Next, we prove that R^O is an SWR. First, to show that R^O is reflexive, let

$\boldsymbol{u} \in \Omega^{\mathbb{N}}$. Then, for all $T \in \mathbb{N}$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$. By (9a), $\boldsymbol{u} R^O \boldsymbol{u}$. Next, to show that R^O is transitive, let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Omega^{\mathbb{N}}$ and suppose that $\boldsymbol{u} R^O \boldsymbol{v}$ and $\boldsymbol{v} R^O \boldsymbol{w}$. By (9a) and (9b), there exist $T^* \in \mathbb{N}$ such that (a) $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ for all $T \geq T^*$ or $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ for all $T \geq T^*$ and (b) $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succ [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ for all $T \geq T^*$ or $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \sim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ for all $T \geq T^*$. Since $\succsim$ is transitive, we obtain that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ for all $T \geq T^*$ or $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ for all $T \geq T^*$. By (9a) and (9b), $\boldsymbol{u} R^O \boldsymbol{v}$. Now, to show that R^O is finitely complete, let $T \in \mathbb{N}$ and $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ with $\boldsymbol{u}^{+T} = \boldsymbol{v}^{+T}$. Since $\succsim$ is complete, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ or $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$ Since $\succsim$ satisfies **EI***, we obtain that, for all $T' > T$

$$[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \Leftrightarrow [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T'}] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}]$$

and

$$[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \Leftrightarrow [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{T'}] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{T'}].$$

Thus, by (9a) and (9b), $\boldsymbol{u} R^O \boldsymbol{v}$ or $\boldsymbol{v} R^O \boldsymbol{u}$. Finally, to show that R^O is intratemporally anonymous, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that, for all $t \in \mathbb{N}$, there exists a bijection $\pi^t : \{1, \ldots, \mathbf{n}(\boldsymbol{u}^t)\} \to \{1, \ldots, \mathbf{n}(\boldsymbol{v}^t)\}$ such that $\boldsymbol{u}^t = \left(v^t_{\pi^t(1)}, \ldots, v^t_{\pi^t(\mathbf{n}(\boldsymbol{u}^t))}\right)$. Since $\succsim$ satisfies $\mathbf{A}^*$ and it is transitive, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ for all $T \in \mathbb{N}$. By (9b), $\boldsymbol{u} I^O \boldsymbol{v}$.

(ii) First, we prove that R^C is an SWR. By (9a), (9b), and (10), $R^O \subseteq R^C$. Thus, by Lemma 4 (i), R^C is finitely complete. To show that R^C is reflexive, let $\boldsymbol{u} \in \Omega^{\mathbb{N}}$. Since $\succsim$ is reflexive, we obtain that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T]$ for all $T \in \mathbb{N}$. By (10), $\boldsymbol{u} R^C \boldsymbol{v}$. Next, to show that R^C is transitive, let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \Omega^{\mathbb{N}}$ and suppose that $\boldsymbol{u} R^C \boldsymbol{v}$ and $\boldsymbol{v} R^C \boldsymbol{w}$. By (10), there exists T^* such that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ and $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ for all $T \geq T^*$. Since $\succsim$ is transitive, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{w}^1, \ldots, \boldsymbol{w}^T]$ for all $T \geq T^*$. By (10), $\boldsymbol{u} R^C \boldsymbol{v}$. Finally, to show that R^C is intratemporally anonymous, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose that, for all $t \in \mathbb{N}$, there exists a bijection $\pi^t : \{1, \ldots, \mathbf{n}(\boldsymbol{u}^t)\} \to \{1, \ldots, \mathbf{n}(\boldsymbol{v}^t)\}$ such that $\boldsymbol{u}^t = \left(v^t_{\pi^t(1)}, \ldots, v^t_{\pi^t(\mathbf{n}(\boldsymbol{u}^t))}\right)$. Since $\succsim$ satisfies $\mathbf{A}^*$ and it is transitive, we obtain $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$ for all $T \in \mathbb{N}$. By (10), $\boldsymbol{u} I^C \boldsymbol{v}$.

Next, we prove (11a) and (11b). Let R_A and R_B be the binary relations on $\Omega^{\mathbb{N}}$ defined by (11a) and (11b), respectively. We show that $R_A \cup R_B = R^C$ and R_A and R_B are asymmetric and symmetric. By (11a) and (11b), it is straightforward that R_A is asymmetric and R_B is symmetric. To show that $R_A \cup R_B \subseteq R^C$, let $\boldsymbol{u}, \boldsymbol{v} \in \Omega^{\mathbb{N}}$ and suppose $(\boldsymbol{u}, \boldsymbol{v}) \in R_A \cup R_B$. By (11a) and (11b), there exists $T^* \in \mathbb{N}$ such that, for all $T \geq T^*$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. By (10), $\boldsymbol{u}R^C\boldsymbol{v}$. Next, to show that $R^C \subseteq R_A \cup R_B$, suppose $\boldsymbol{u}R^C\boldsymbol{v}$. By (10), there exists $T^* \in \mathbb{N}$ such that, for all $T \geq T^*$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succsim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$. If there exists $T' \geq T^*$ such that, for all $T \geq T'$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$, then we obtain $\boldsymbol{u}R_B\boldsymbol{v}$ by (11b). If there is no $T' \geq T^*$ such that, for all $T \geq T'$, $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \sim [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$, then, for all $T' \geq T^*$, there exists $T \geq T'$ such that $[\boldsymbol{u}^1, \ldots, \boldsymbol{u}^T] \succ [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^T]$, and we obtain $\boldsymbol{u}R_A\boldsymbol{v}$ by (11a). Thus, $(\boldsymbol{u}, \boldsymbol{v}) \in R_A \cup R_B$. ■

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