Entrepreneurial Choice under Uncertainty

in Applied Econometrics

by

Bernt Petter Stigum

Department of Economics, the University of Oslo

Pb. 1095 Blindern, 0317 Oslo, Norway, +47 45851597

bstigum@econ.uio.no

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Abstract. The paper presents a theory of entrepreneurial choice under uncertainty. An entrepreneur is an individual who manages a firm that produces one commodity with labor, an intermediate good, and capital. He pays dividends to shareholders, invests in bonds and capital, and has an n-period planning horizon. His aim is to maximize the expected value of a utility function that varies with the dividends he pays each period and with his firm's balance sheet at the end of the planning horizon. The paper ends with an empirical formal-econometric test of the theory. The test demonstrates that the theory is empirically relevant.

Key Words. Entrepreneur, firm, uncertainty, neo-classical theory, marginal efficiency, empirical context, empirical relevance.

JEL A19, C01, C12, C21,C31, C49, C51, D21, D22, D84

1. Introduction

This paper presents a theory of entrepreneurial choice in a world in which the entrepreneur cannot foresee with certainty the behavior of prices during the periods of his planning horizon. I introduced the theory in (Stigum, 1969). Here, I develop the characteristics of entrepreneurial choice under uncertainty, and compare them with the way entrepreneurs act in the neo-classical theory of the firm. The characteristics which the two theories display differ in two important ways. Firstly, in the neo-classical theory the entrepreneur - with knowledge of prices and subject to the production constraints that he faces - maximizes the present value of the sum of profits which the firm earns each period during his planning horizon. In my theory, the entrepreneur - subject to the production and financial constraints that he faces - maximizes the expected value of a utility function that varies with the dividends he pays each period to his shareholders and with his firm's balance sheet at the end of his planning horizon. Secondly, in my theory the entrepreneur's reaction to a change in the price of a variable, say capital, can be analysed in terms of a substitution effect, an income effect, and an expectations effect. The income effect and expectations effect are missing in the neo-classical theory (cf., Chapter 7 in D. M. Kreps, 1990).

The paper ends with an empirical test of the empirical relevance of my theory. The test is formulated as an axiomatic system in which theory variables reside in a theory universe and data variables reside in a data universe. The universes are disjoint and connected by a bridge that relates theory variables to data variables in novel ways.

In this axiomatic system the data comprise four hundred observations of the components of a nineteendimensional random vector. The probabiliy distribution of this vector, the TPD, is taken to be a true rendition of the data generating process. In the TPD the data variables have finite means and finite positive variances.

Six of the axioms in the given axiomatic system present the economic theory that is at stake in the empirical analysis. The theory is a family of models of these axioms that describes salient characteristics of entrepreneurial choice under uncerainty. The variables of the theory comprise thirteen components of a random vector with a probability distribution, the RPD. In the RPD the theory variables have finite means and finite positive variances.

The RPD of the theory and the bridge principles induce a probability distribution of the data, the MPD, that may be very different from the TPD. Different models of the theory axioms and the bridge principles induce different models of the MPD. A model of the MPD that lies in a 95% confidence band of an estimate of the MPD is a data admissible MPD.

The theory is empirically relevanr in the given empirical context only if the bridge principles are empirically valid in that context. In Section 4.4 I show that all the data adissible MPD's are congruent models of the TPD. From this and the so-called Status of bridge principles in applied econometrics (cf., p. 7 in (Stigum, 2016)), it follows that the paper's bridge principles are empirically valid in the given empirical context.

The theory is empirically relevant if it contains an empirically relevant model. Looking for an empirically relevant model in the present case is not meaningful. Hence, I must estblish the empirical relevance of the theory in a different way. I begin by showing that the characteristics of entrepreneurial choice which the theory claims that the members of a sample population must share are shared in my sample when the data are MPD distributed. This I take to mean that the theory is empirically relevant when the data are MPD distributed. Thereafter, I show that the MPD and the TPD in a way mutually encompass each other. That and the fact that all the data admissible MPDs are congruent models of the TPD ensure that the theory is empirically relevant, also, in the present empirical context, where the data are TPD distributed.

2. A Model of Entrepreneurial Choice under Uncertainty

In this paper, the entrepreneur is an indvidual who operates a firm that is owned by many investors each one of which possesses a portion of the firm's outstanding shares. I assume that the entrepreneur owns one share himself, and that he under no circumstances will sell it. The shares and their price I denote by the letters, M and p_M.

The firm produces one output, y, with three inputs, L, x, and K, in accord with the prescriptions of a production function, $g(\cdot)$, as follows:

$$y = g(L, x, K), \text{ with } (y, L, x, K) \, \varepsilon \, R_+^3 \times R_{++}. \qquad (1)$$

Here L is short for labor, x for an intermediate good, and K for capital. The function, $g(\cdot)$, is an instantaneous point-input-point-output variety production function. I assume that $g(\cdot)$ is increasing, strictly concave, and twice differentiable with $\partial^2 g(L,x,K)\partial L\partial x > 0$. The prices of y, L, x, and K I denote by the letters, p_y, w, p_x, and p_K.

The entrepreneur is a price taker in all markets. He uses the firm's profit, $p_y y - wL - p_x x$, to pay the shareholders dividends, d, to invest in capital and in bonds that mature in one period, μ, and to adjust the number of outstanding shares. In a given period, i, the budget constraint for this activity is

$$p_{yi}y_i - w_iL_i - p_{xi}x_i - d_i - (p_{\mu i}\mu_i - \mu_{i-1}) - p_{Ki}(K_i - K_{i-1}) + p_{Mi}(M_i - M_{i-1}) \geq 0, \qquad (2)$$

where μ_{i-1}, K_{i-1}, and M_{i-1} record, respectively, the bonds and capital that the firm owns and the number of outstanding shares at the beginning of period i. I take bonds and shares to be continuous variables. Moreover, I take capital to be a fixed factor of production. Hence, the entrepreneur's investment in new capital in one period cannot be used in the production of y before the next period. Finally, I assume that there is no market for K_{i-1} in period i, and that there is no storage facility for commodities and intermediate goods.

A period is a week or a month. I assume that the entrepreneur has an n-period planning horizon, a utility function, V, and a subjective probability distribution, Q(dP), of the values which the respective prices assume in each period. The utility function is a function of the dividends that the entrepreneur pays the shareholders in each period and of the firm's balance sheet at the end of his planning horizon. Thus,

$$V = V(d_1, \ldots, d_n, \mu_n, K_n, M_n), \qquad (3)$$

where the function, $V(\cdot):R_+^n \times [-N_\mu,N_\mu] \times R_+ \times [1,N_M) \rightarrow R_+$, is taken to be twice differentiable, strictly concave, increasing in the d_is, μ_n, and K_n, and decreasing in M_n,. Moreover, a positive value of μ_n is an investment. A negative value of μ_n is a one-period loan. The interest rate in period n on such loans, r_n, equals $((1/p_{\mu n}) - 1)$. Finally, N_μ and N_M are finite positive constants with $N_M > 1$.

Let a circumstance be a vector of positive prices. I assume that the entrepreneur in the first period of his planning horizon chooses an optimal expenditure plan – that is a family of vectors,

$$(y_1, L_1, x_1, d_1, \mu_1, K_1, M_1, \ldots, y_n, L_n, x_n, d_n, \mu_n, K_n, M_n),$$

that, for i = 1, …, n, and for each and every circumstance that may occur, satisfies the conditions,

$$(y_i, L_i, x_i, d_i, K_i) \geq 0,\ N_\mu \geq \mu_i \geq -N_\mu,\ N_M \geq M_i \geq 1, \qquad (4)$$

$$y_i = g(L_i, x_i, K_{i-1}), \qquad (5)$$

$K_i \geq K_{i-1}$, with K_0 equal to a positive constant, and (6)

$$p_{yi}y_i - w_iL_i - p_{xi}x_i - d_i - (p_{\mu i}\mu_i - \mu_{i-1}) -$$

$$p_{Ki}(K_i - K_{i-1}) + p_{Mi}(M_i - M_{i-1}) \geq 0. \qquad (7)$$

and maxmizes the expected value of V(·) with respect to Q(dP) conditional upon the observed values of ($p_{y1}, w_1, p_{x1}, p_{\mu 1}, p_{K1}, p_{M1}$).

Formulating an optimal expenditure plan is a cumbersome way to determine what the entrepreneur's optimal first-period choice of variables is. However, under reasonable conditions on Q(dP), one can show - cf., Theorem T 30.5, p. 813 in (Stigum, 1990) - that there exists a function, U(·), such that the first-period part of an optimal expenditure plan, ($y_1, L_1, x_1, d_1, \mu_1, K_1, M_1$), is a vector that maximizes the value of U(·) subject to the first-period production and budget constraints. Specifically, there is a function,

$$U(\cdot): R_{++}^6 \times R_+ \times [-N_\mu, N_\mu] \times R_+ \times [1, N_M) \rightarrow R_+, \qquad (8)$$

of (($p_{y1}, w_1, p_{x1}, p_{\mu 1}, p_{K1}, p_{M1}$), d_1, μ_1, K_1, M_1), such that the

entrepreneur in the first period of his planning horizon chooses a vector, ($y_1, L_1, x_1, d_1, \mu_1, K_1, M_1$), that maximizes the value of U(·) subject to the conditions,

$$(y_1, L_1, x_1, d_1, K_1 - K_0) \geq 0,\ N\mu \geq \mu_1 \geq -N\mu,\ N_M \geq M_1 \geq 1, \qquad (9)$$

$$y_1 = g(L_1, x_1, K_0), \text{ and} \qquad (10)$$

$$p_{y1}y_1 - w_1L_1 - p_{x1}x_1 - d_1 - (p_{\mu 1}\mu_1 - \mu_0) - p_{K1}(K_1 - K_0)$$

$$+ p_{M1}(M_1 - M_0) \geq 0, \qquad (11)$$

where K_0 , μ_0 , M_0, N_μ , and N_M are suitable positive constants. In this paper I assume that U(·) is twice differentiable, strictly concave in (d_1, μ_1, K_1, M_1), increasing in (d_1, μ_1, K_1), and decreasng in M_1.

Here an example may be of help. The example describes a two-period model of the theory of entrepreneurial choice under uncertainty that I sketched above.

Example 1 In this example, $n = 2$, $\mu_0 = A$, $K_0 = 5$, $M_0 = 25$, and for $i = 1,2$, $(y_i,L_i,x_i) € R_+^3$. $(d_i,\mu_i,K_i) € R_+^3$, and $M_i €$ [1, 49]. The corresponding prices are

$$P_1 = (p_{y1},w_1,p_{x1},p_{\mu 1},p_{K1},p_{M1}),\ P_2 = (p_{y2},w_2,p_{x2},p_{\mu 2},p_{K2},p_{M2}),$$

with $P_i € R_{++}^6$, $i = 1,2$, $(p_{\mu 1},p_{K1}) < 1$, and $(p_{\mu 2},p_{K2}) < 1$.

For $i = 1,2$, the production and budget constraints are, respectively,

$$y_i = g(L_i,x_i,K_{i-1}) = L_i^{(1/4)} \cdot x_i^{(1/4)} + \gamma \log K_{i-1},$$

$$K_i \geq K_{i-1}, \text{ and} \qquad (12)$$

$$p_{yi}y_i - w_iL_i - p_{xi}x_i - d_i - (p_{\mu i}\mu_i - \mu_{i-1}) - p_{Ki}(K_i - K_{i-1})$$

$$+p_{Mi}(M_i - M_{i-1}) \geq 0, \qquad (13)$$

Finally, the two-period utility function, $V(\cdot)$, is

$$V(d_1,d_2,\mu_2,K_2,M_2) = d_1^{(1/3)} \cdot (d_2 \cdot \mu_2 \cdot K_2 \cdot (50-M_2))^{(1/6)} \qquad (14)$$

Subject to the pertinent production and budget constraints, one obtains the constrained maximum value of the second-period part of $V(\cdot)$ by maximizing the function,

$$(d_2 \cdot \mu_2 \cdot K_2 \cdot (50-M_2))^{(1/6)} + \lambda_1(y_2 - L_2^{(1/4)} \cdot x_2^{(1/4)} - \gamma \log K_1)$$

$$+ \lambda_2(p_{y2}y_2 - w_2L_2 - p_{x2}x_2 - d_2 - (p_{\mu 2}\mu_2 - \mu_1) - p_{K2}(K_2 - K_1)$$

$$+p_{M2}(M_2 - M_1)).$$

The necessary conditions for a maximum of this function are, first,

$$\lambda_1 = -\lambda_2 p_{y2};\ \lambda_1(1/4) \cdot L_2^{(-3/4)} \cdot x_2^{(1/4)} = -\lambda_2 w_2;$$

$$\lambda_1(1/4) \cdot L_2^{(1/4)} \cdot x_2^{-(3/4)} = -\lambda_2 p_{x2}; \text{ and}$$

$$y_2 = L_2^{(1/4)} \cdot x_2^{(1/4)} + \gamma \log K_1. \qquad (15)$$

for production. Then,

$$(1/6)\ d_2^{-(5/6)}(\mu_2 \cdot K_2 \cdot (50-M_2))^{(1/6)} = \lambda_2;$$

$$(1/6)\ \mu_2^{-(5/6)}\ (d_2 \cdot K_2 \cdot (50-M_2))^{(1/6)} = \lambda_2 p_{\mu 2};$$

$$(1/6)\ K_2^{-(5/6)} \cdot\ (d_2 \cdot \mu_2 \cdot (50-M_2))^{(1/6)} = \lambda_2 p_{K2};$$

$$(1/6)\ (50-M_2)^{-(5/6)} \cdot (d_2 \cdot \mu_2 \cdot K_2)^{(1/6)} = \lambda_2 p_{M2};\ \text{and}$$

$$p_{y2}y_2 - w_2L_2 - p_{x2}x_2 - d_2 - (p_{\mu 2}\mu_2 - \mu_1) - p_{K2}(K_2 - K_1) + p_{M2}(M_2 - M_1) = 0. \qquad (16)$$

for the budget.

In a solution of these conditions the entrepreneur chooses a y_2, L_2, and x_2 that maximizes the firm's profit. Their values are

$$L_2 = (p_{y2}/4)^2 w_2^{(-3/2)} \cdot p_{x2}^{(-1/2)};\ x_2 = (p_{y2}/4)^2 p_{x2}^{(-3/2)} w_2^{(-1/2)}:$$

$$y_2 = (p_{y2}/4) w_2^{(-1/2)} \cdot p_{x2}^{(-1/2)} + \gamma \log K_1.$$

In addition, with $\pi(p_{y2},w_2,p_{x2}) = p_{y2}y_2 - w_2L_2 - p_{x2}x_2$ being - for a given value of K_1 - the corresponding profit, and with

$$\pi^* = \pi(p_{y2},w_2,p_{x2}) + \mu_1 + p_{K2}K_1 - p_{M2}M_1,\ \text{and}$$

$$\pi^{**} = \pi^* + 50p_{M2}, \qquad (17)$$

the entrepreneur maximizes his utility by choosing λ_2, d_2, μ_2, K_2, and M_2 so that $\lambda_2 = (2/3) \cdot (d_2 \cdot \mu_2 \cdot K_2 \cdot (50-M_2))^{(1/6)} / \pi^{**}$; and

$$d_2 = \pi^{**}/4;\ \mu_2 = \pi^{**}/4p_{\mu 2};\ K_2 = \pi^{**}/4p_{K2},$$

$$(50-M_2) = \pi^{**}/4p_{M2}. \qquad (18)$$

From this it follows that the utility function, $U(\cdot)$, to be maximized in the first priod satisfies the equation,

$$U(p_{y1},w_1,p_{x1},p_{\mu 1},p_{K1},p_{M1},\ d_1,\ \mu_1,\ K_1,\ M_1) =$$

$$ed_1^{(1/3)} \cdot E\{(p_{\mu 2}.p_{K2} \cdot p_{M2})^{-(1/6)}[\pi(p_{y2},w_2,p_{x2}) + \mu_1 + p_{K2}K_1 + p_{M2}(50-M_1)]^{(2/3)} \mid P_1\}, \qquad (19)$$

where $e = (1/4)^{(2/3)}$, $E\{(\cdot) \mid P_1\}$ denotes the conditional

expected value of $(\cdot)$ given the observed value of P_1, the expectation is taken with respect to the values of the components of P_2, and the value of $\pi(p_{y2},w_2,p_{x2})$ depends on the value of K_1, at the end of period one.

2. Entrepreneurial Choice under Uncertainty

In the neo-classical theory of the firm, the firm's entrepreneur - subject to the production constraint he faces - maximizes the present value of the sum of profits which his firm earns in each period during his planning horizon. Moreover, he chooses his expenditure plan with knowledge of the values of the prices he will face in each and every period.

Since the neo-classical theory differs from my theory, it is interesting to see if fundamental theorems in the neo-classical theory are valid in my theory. I will do that by comparing important neo-classical theorems with related theorems in the one-period model that I described in equations (8) - (11). The two-period model in Example 1 shows that the one-priod theorems are valid in an n-period model.

I begin with the entrepreneur's choice of inputs and output. The entrepreneur in the neo-classical theory maximizes profit. Since the utility function is an increasing function of dividends, the entrepreneur in my theory, also, chooses his inputs and output to maximize profit. The two-period model in Example 1 suggests that he will plan to do so in each and every period of his planning horizon.

In the neo-classical theory the entrepreneur's choice of inputs and output maximizes the firm's profit only if his choice satisfies the following conditions: (1) The value of an input's marginal product equals its price; (2) the marginal cost of an output equals its price; (3) the change in profit, when an output's price or an input's price changes, equals, respectively, the equilibrium value of the output and the negative value of the equilibrium choice of the input (Hotelling's Lemma); and (4) the firm's supply of an output or use of an input is, respectively, an increasing function of the output's price and a decreasing function of the input's price. I will see if the entrepreneur's first-period choice of inputs and output in my theory satisfies these conditions.

For the given task, I must exhibit the necessary conditions for a constrained maximum of the utility function in (8) subject to the conditions in equations (9) - (11). They are displayed in the equations in (20) with two Lagrange multipliers, λ_1 and λ_2:

$$\lambda_1 = -\lambda_2 p_{y1};\ \lambda_1\partial g/\partial L_1 = -\lambda_2 w_1;\ \lambda_1\partial g/\partial x_1 = -\lambda_2 p_{x1};$$

$$y_1 = g(L_1,x_1,K_0);\ \partial U/\partial d_1 = \lambda_2;\ \partial U/\partial \mu_1 = \lambda_2 p_{\mu 1};$$

$$\partial U/\partial K_1 = \lambda_2 p_{K1};\ \partial U/\partial M_1 = -\lambda_2 p_{M1};\ p_{y1}y_1 - w_1L_1 -$$

$$p_{x1}x_1 - d_1 - (p_{\mu 1}\mu_1 - \mu_0) - p_{K1}(K_1 - K_0) + p_{M1}(M_1 - M_0) \geq 0. \quad (20)$$

From the equations in (20) I deduce, first, that

$$p_{y1}\cdot\partial g/\partial L_1 = w_1;\ \text{and}\ p_{y1}\cdot\partial g/\partial x_1 = p_{x1};$$

and, then, - with C short for the cost of producing y_1 - that,

$$\partial C/\partial y_1 = w_1(\partial L_1/\partial y_1) + p_{x1}(\partial x_1/\partial y_1) =$$

$$p_{y1}\cdot(\partial g/\partial L_1)(\partial L_1/\partial y_1) + p_{y1}\cdot(\partial g/\partial x_1)(\partial x_1/\partial y_1);\ \text{and}$$

$$p_{y1} = p_{y1}(\partial g/\partial L_1)(\partial L_1/\partial y_1) + p_{y1}(\partial g/\partial x_1)(\partial x_1/\partial y_1);$$

from which it follows that $\partial C/\partial y_1 = p_{y1}$. Hence, entrepreneur's first-period choice of inputs and output satisfies the first two conditions in the neo-classical theory.

To see if Hotelling's Lemma is valid in my theory, I use the equations in (20) to show that

$$p_{y1}\partial y_1/\partial w_1 = p_{y1}\partial g/\partial L_1\cdot\partial L_1/\partial w_1 + p_{y1}\partial g/\partial x_1\cdot\partial x_1/\partial w_1 =$$

$$w_1\partial L_1/\partial w_1 + p_{x1}\partial x_1/\partial w_1.$$

Similarly, in equilibrium $\partial\pi/\partial y_1 = 0$, and

$$\partial\pi/\partial p_{y1} = \partial\pi/\partial y_1\cdot\partial y_1/\partial p_{y1} + \partial\pi/\partial p_{y1} = y_1. \quad (21)$$

Consequently, $\partial\pi/\partial p_{y1} = y_1$, and

$$p_{y1}\partial y_1/\partial w_1 - w_1\partial L_1/\partial w_1 - L_1 - px_1\partial x_1/\partial w_1 = - L_1, \quad (22)$$

which shows that Hoteling's Lemma gives a valid description of my entrepreneur's first-period choice of inputs and output.

. To show that my theory satisfies the fourth condition, I deduce from the equations in (20) that

$$(\partial^2 g/\partial L_1\partial L_1)(\partial L_1/\partial w_1) + (\partial^2 g/\partial L_1\partial x_1)(\partial x_1/\partial w_1) = 1/p_{y1}. \quad (23)$$

$$(\partial^2 g/\partial x_1\partial L_1)(\partial L_1/\partial w_1) + (\partial^2 g/\partial x_1\partial x_1)(\partial x_1/\partial w_1) = 0, \text{ and} \quad (24)$$

$$\partial y_1/\partial w_1 - (\partial g/\partial L_1)(\partial L_1/\partial w_1) - (\partial g/\partial x_1)(\partial x_1/\partial w_1) = 0. \quad (25)$$

From these equations and the assumption that $g(\cdot)$ is strictly concave and twice differentiable with $\partial^2 g(L,x,K)\partial L\partial x > 0$, and from the assumption that the solutions to the equations in (20) constitute a maximum of the pertinent Lagrangian function, it is easy to show that $\partial L_1/\partial w_1 < 0$. Similar arguments show that $\partial x_1/\partial p_{x1} < 0$, and $\partial y_1/\partial p_{y1} > 0$. Hence, my theory satifies the fourth condition.

So much for inputs and output. Next, a few words about the marginal efficiency of capital and investments in μ and K. In the neo- classical theory, the marginal efficiency of capital is the rate of disount that will equate the price of fixed capital with the present value of the entrepreneur's income from the firm's fixed capital during his planning horizon (cf. Keynes 1936, p.135). My idea of the marginal efficiency of capital under conditions of uncertainty differs. It is like Irving Fisher's idea of a consumer's rate of time preference (Fisher 1961, p. 62). I describe it below for investments in μ and K.

Let $r = ((1/p_{\mu 1}) - 1)$ be the rate of interest on one-period loans; let m_{K1} be the entrepreneur's expected return during the planning horizon from a first-period additional unit of capital conditional on the observed values of first period prices; and let r_{K1} be defined by the equation, $(m_{K1}/1+r_{K1}) = p_{K1}$. It follows from the equations in (20) that

the entrepreneur invests in μ and K up to the point, where

$$((\partial U/\partial d_1 - \partial U/\partial \mu_1)/(\partial U/\partial \mu_1)) = r, \text{ and} \qquad (26)$$

$$((m_{K1}\cdot\partial U/\partial d_1 - \partial U/\partial K_1)/(\partial U/\partial K_1)) = r_{K1}. \qquad (27)$$

In these equations the term, $(\partial U/\partial d_1)$ records the expected value of the marginal utility of an extra unit of dividends in period one. In the same period, $(\partial U/\partial \mu_1)$ equals the expected value of the marginal utility to the entrepreneur of the income that would be forgone if a unit less is invested in μ. The two concepts combine to form what I in (Stigum, 1969) called the marginal efficiency of an extra unit of investment in μ. Similarly, $(m_{K1}\cdot\partial U/\partial d_1)$ and $(\partial U/\partial K_1)$ combine to mform a relation that I will call the marginal efficiency of capital. With these concepts in mind, equations (26) and (27) insist that in equilibrium the entrepreneur invests in μ and K up to the point, where the marginal efficiency of investments in μ and K equal, respectively, the interest rate on one-period loans and the conditionally expected rate of return from an additional unit of capital in period one.

To check whether the neo-classical symmetry conditions on factor demand functions are valid in the uncertainty theory, I must use the conditions in (20) to derive the derivatives of the entrepreneur's first-period demand and supply functions. Let D(1) and D(2) be, respectively, the four-by-nine and five-by-nine matrices below. In addition, let D and D_{ij}, i,j = 1, …, 9, be the determinant and the ijth co-factor of the nine-by-nine matrix whose first four lines are the lines in D(1) and the last five lines are the five

$$D(1) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & p_{y1} \\ 0 & \lambda_1\partial^2 g/\partial L_1\partial L_1 & \lambda_1\partial^2 g/\partial L_1\partial x_1 & \partial g/\partial L_1 & 0 & 0 & 0 & 0 & w \\ 0 & \lambda_1\partial^2 g/\partial\backslash x_1\partial L_1 & \lambda_1\partial^2 g/\partial_{x1}\partial x_1 & \partial g/\partial x_1 & 0 & 0 & 0 & 0 & p_{x1} \\ 1 & -\partial g/\partial L_1 & -\partial g/\partial x_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D(2) = \begin{pmatrix} 0 & 0 & 0 & 0 & \partial^2U/\partial d_1\partial d_1 & \partial^2U/\partial d_1\partial \mu_1 & \partial^2U/\partial d_1\partial K_1 & \partial^2U/\partial d_1\partial M_1 & -1 \\ 0 & 0 & 0 & 0 & \partial^2U/\partial \mu_1\partial d_1 & \partial^2U/\partial \mu_1\partial \mu_1 & \partial^2U/\partial \mu_1\partial K_1 & \partial^2U/\partial \mu_1\partial M_1 & -p_{\mu 1} \\ 0 & 0 & 0 & 0 & \partial^2U/\partial K_1\partial d_1 & \partial^2U/\partial K_1\partial \mu_1 & \partial^2U/\partial K_1\partial K_1 & \partial^2U/\partial K_1\partial M_1 & -p_{K1} \\ 0 & 0 & 0 & 0 & \partial^2U/\partial M_1\partial d_1 & \partial^2U/\partial M_1\partial \mu_1 & \partial^2U/\partial M_1\partial K_1 & \partial^2U/\partial M_1\partial M_1 & p_{M1} \\ p_{y1} & -w_1 & -p_{x1} & 0 & -1 & -p_{\mu 1} & -p_{K1} & p_{M1} & 0 \end{pmatrix}$$

lines in D(2). Then, by totally differentiating the necessary conditions in (20), I find that

$$\partial y_1/\partial p_{y1} = D^{-1}\{-\lambda_2 D_{11} - y_1 D_{91} - [D_{51}\partial^2U/\partial d_1\partial p_{y1} + D_{61}\partial^2U/\partial \mu_1\partial p_{y1} + D_{71}\partial^2U/\partial K_1\partial p_{y1} + D_{81}\partial^2U/\partial M_1\partial p_{y1}]\}; \text{ and} \qquad (28)$$

$$\partial K_1/\partial p_{K1} = D^{-1}\{\lambda_2 D_{77} + K_1 D_{97} - [D_{57}\partial^2U/\partial d_1\partial p_{K1} + D_{67}\partial^2U/\partial \mu_1\partial p_{K1} + D_{77}\partial^2U/\partial K_1\partial p_{K1} + D_{87}\partial^2U/\partial M_1\partial p_{K1}]\}. \qquad (29)$$

These equations and their analogues for $\partial L_1/\partial w_1$, $\partial x_1/\partial p_{x1}$, $\partial \mu_1/\partial p_{\mu 1}$, and $\partial M_1/\partial p_{M1}$ show that the rsponse of the entrepreneur's choice of y_1, L_1, x_1, d_1, K_1, μ_1, and M_1, to a change in its own price or the price of another variable can be analysed in terms of a substitution effect, an income effect, and an expectations effect. For example, in (29), $\partial K_1/\partial p_{K1}$ can be described as the sum of a substitution effect, $\lambda_2 D^{-1} D_{77}$, an income effect, $+ K_1 D^{-1} D_{97}$, and an expectations effect,

$$-D^{-1}[D_{57}\partial^2U/\partial d_1\partial p_K + D_{67}\partial^2U/\partial \mu_1\partial p_{K1} + D_{77}\partial^2U/\partial K_1\partial p_{K1} + D_{87}\partial^2U/\partial M_1\partial p_{K1}].$$

The income effect and the expectations effect are missing in the neo-classical theory of the firm.

The equations in (28) - (29) suggest that the symmetry condition, $\partial L_1/\partial p_{x1} = \partial x_1/\partial w_1$, might not be valid in my theory. However, a careful analysis of the cofactors of D reveals that

$$D_{ik} = 0 \text{ for } i = 5,\ldots,9 \text{ and } k = 1, 2, 3. \qquad (30)$$

Consequently, in the present uncertainty theory the entreprneur's choice of inputs and output can be analysed without an income effect and an expectations effect. In fact,

$$\partial y_1/\partial p_{y1} = -D^{-1}\lambda_2 D_{11};$$

$$\partial L_1/\partial w_1 = -D^{-1}\lambda_2 D_{22}: \ \partial x_1/\partial p_{x1} = -D^{-1}\lambda_2 D_{33};$$

$$\partial L_1/\partial p_{x1} = -D^{-1}\lambda_2 D_{32}; \text{ and } \partial x_1/\partial w_1 = -D^{-1}\lambda_2 D_{23}. \qquad (31)$$

The structural properties of the matrix of D ensure that $D_{23} = D_{32}$, and, hence, that $\partial L_1/\partial p_{x1} = \partial x_1/\partial w_1$.

The next example derives the first-period choices of the entrepreneur in Example 1.

Example 2. The necessary conditions for a constrained maximum of the utility function in (19) are - with F short for

$$(1/4)^{(2/3)}E\{(p_{\mu 2}p_{K2}p_{M2})^{-(1/6)}[\pi(py_2,w_2,px_2)+$$

$$\mu_1+p_{K2}K_1+p_{M2}(50-M_1)]^{(2/3)} \mid P_1\}-$$

$$\lambda_1 = -\lambda_2 p_{y1};\ \lambda_1\partial g/\partial L_1 = -\lambda_2 w_1;\ \lambda_1\partial g/\partial x_1 = -\lambda_2 p_{x1};$$

$$y_1 = g(L_1,x_1,K_0) = L_1^{(1/4)} \cdot x_1^{(1/4)} + \gamma \log K_0;$$

$$(1/3)d_1^{-(2/3)}F = \lambda_2;\ d_1^{(1/3)}\partial F/\partial\mu_1 = \lambda_2 p_{\mu 1};$$

$$d_1^{(1/3)}\partial F/\partial K_1 = \lambda_2 p_{K1};\ d_1^{(1/3)}\partial F/\partial M_1 = -\lambda_2 p_{M1};\text{ and}$$

$$p_{y1}y_1-w_1L_1-p_{x1}x_1-d_1-(p_{\mu 1}\mu_1-\mu_0)-p_{K1}(K_1-K_0)$$

$$+p_{M1}(M_1-M_0)\geq 0. \qquad (32)$$

From these conditions I deduce that the first-period equilibrium values of L_1, x_1, and y_1 are

$$L_1 = (p_{y1}/4)^2 w_1^{(-3/2)} \cdot p_{x1}^{(-1/2)};\ x_1 = (p_{y1}/4)^2 p_{x1}^{(-3/2)} w_1^{(-1/2)}\text{: and}$$

$$y_1 = (p_{y1}/4) w_1^{(-1/2)} \cdot p_{x1}^{(-1/2)} + \gamma \log K_0. \qquad (33)$$

The cross derivatives of the two demand functions are equal. To wit:

$$\partial L_1/\partial p_{x1} = -(1/2)(p_{y1}/4)^2 w_1^{(-3/2)} \cdot p_{x1}^{(-3/2)}\text{, and}$$

$$\partial x_1/\partial w_1 = -(1/2)(p_{y1}/4)^2 w_1^{(-3/2)} \cdot p_{x1}^{(-3/2)}.$$

From the necessary conditions in (32) I can, also, deduce that

$$p_{\mu 1} = 3d_1 \partial F/\partial \mu_1/F \qquad (34)$$

Consequently,

$$[F - (3d_1\partial F/\partial \mu_1)]/(3d_1\partial F/\partial \mu_1) =$$

$$[1 - (3d_1\partial F/\partial \mu_1/F)]/(3d_1\partial F/\partial \mu_1/F)$$

$$= (1 - p_{\mu 1})/p_{\mu 1} = [(1/p_{\mu 1}) - 1] = r. \qquad (35)$$

where r denotes the interest rate on one-period loans. But, if that is so, in equilibrium the entrepreneur invests in μ up to the point, where the marginal efficiency of such investments equals the interest rate.

A similar interpretation can be given to the equations,

$$p_{K1} = 3d_1\partial F/\partial K_1/F\text{, and} \qquad (36)$$

$$[m_{K1} \cdot F - (3d_1\partial F/\partial K_1)]/(3d_1\partial F/\partial K_1) =$$

$$[m_{K1} - (3d_1\partial F/\partial K_1/F)]/(3d_1\partial F/\partial K_1/F)$$

$$= (m_{K1} - p_{K1})/p_{K1} = [(m_{K1}/p_{K1}) - 1] = r_{K1}. \qquad (37)$$

In equilibrium the entrepreneur invests in K up to the point where the marginal efficiency of capital equals the conditionally expected rate of return from an additional first-period unit of capital, r_{K1}. Thus, the equilibrium conditions on investments in μ and K in the one-perio model in equations (8) - (11) are valid in Example 1's two-period model.

4. Entrepreneurial Choice in Applied Econometrics

The theory of entrepreneurial choice under uncertainty that I outlined in Section 2 is a family of models of Q(dP) and the equations in (1) - (3). The theory is not meant to describe entrepreneurial behavior under uncertainty. Instead it is a family of models that describe characteristic features of entrepreneurial choice in a world in which the entrepreneur cannot foresee the behavior of prices during his planning horizon.

Different families of models of Q(dP) and the equations in (1) - (3) constitute different theories of entrepreneurial choice under uncertainty. Moreover, members of such a family of models may be very different even though they describe characteristics of entrepreneurial choice in one and the same theory.

The way entrepreneurial choice varies with the models is interesting and of fundamental importance to the way theorym is used in the empirical analysis of entrepreneurial choice under uncertainty. For example, even though the solutions to the necessary conditions describe choice characteristics of many different entrepreneurs, the entrepreneurs share many characteristics. Their choice of y, L, and x satisfies Hotelling's Lemma, ensures that marginal cost equals the price of y, and maximizes the firm's profit. Similarly, their choice of d, K, μ, and M ensures that the marginal efficiency of the entrepreneur's investments in μ and K equals, respectively, the interest rate on one-period loans and the firm's conditionally expected rate of return from an additional unit of capital in period one.

A theory of entrepreneurial choice under uncertainty; i.e., a particular family of models of Q(dP) and the equations in (1) - (3), is empirically relevant if it contains a model that is empirically relevant. Looking for an empirically relevant model is not meaningful. To test the empirical relevance of the theory, one must look for choice characteristics which the solutions to the necessary conditions identify and the family's entrepreneurs share. The theory is empirically relevant only if the data do not reject the validity of one of them.

4.1 Axioms for Entrepreneurial Choice under Uncertainty in Applied Econometrics

I imagine that the variables in the family of models of Q(dP) and the equations in (1) - (3) belong in a theory universe. This theory universe is a triple, $(\Omega_T, \Gamma_T, (\Omega_T, \aleph_T, P_T(\cdot)))$, where Ω_T is a subset of a vector space, Γ_T is a finite set of assertions concerning properties of vectors in Ω_T, and $(\Omega_T, \aleph_T, P_T(\cdot))$ is a probability space. The latter comprises Ω_T, a σ field of subsets of Ω_T, $\aleph_T$, and a probability measure, $P_T(\cdot):\aleph_T \rightarrow [0,1]$.

The assertions in Γ_T consist of six axioms, A 1-A 6.

A 1 $\Omega_T \subset R^3 \times R^4 \times R^3 \times R^3 \times R \times R^7 \times R^6$. Thus $\omega_T \in \Omega_T$ only if $\omega_T = (y, L, x, d, \mu, K, M, p_y, w, p_x, p_\mu, p_K, p_M, \chi, u, z)$ for some $(y,L,x) \in R^3$, $(d,\mu,K,M) \in R^4$, $(p_y,w,p_x) \in R^3$, $(p_\mu,p_K,p_M) \varepsilon R^3$, $\chi \in R$, $u \in R^7$, $z \in R^6$, and $(y,L,x,d,\mu,K,M, p_y, w, p_x, p_\mu, p_K, p_M, \chi, u, z) \in R^{27}$.

A 2 For all $\omega_T \in \Omega_T$, (y, L, x) € R_+^3, and $(d, \mu, K, M) \in R_+ \times [-N_\mu,N_\mu] \times R_+ \times [1, N_M]$**.** Moreover, $(p_y, w, p_x, p_M) \in (0,50)^4$, and $(p_\mu,p_K) \in (0,1)^2$**.**

In the intended interpretation of y, L, x, d, μ, K, M, p_y, w, p_x, p_μ, p_K, and p_M, y denotes the firm's output, (L, x) denotes a pair of inputs. Moreover, d denotes dividends, a positive μ denotes a bond that matures in one period, and a negative μ denotes a one-period loan, K denotes the capital that is used in the production of y, and M denotes the firm's outstanding shares. Finally, the components of (p_y, w, p_x) denote the respective first-period prices of y, L, and x; and the components of (p_μ, p_K, p_M) denote the respective first-period prices of μ, K, and M. The χ and the components of u and z are error terms. The u and z are to be used to describe the relationship between theoretical variables and data variables.

The given theory variables also satisfy the conditions in axioms A 3 and A 4. In them, K_0 in A 3 and μ_0, K_0, and M_0 in A 4 denote initial quantities of μ, K, and M.

A 3 There is a function, $g(\cdot):R_+^3 \to R_+$, which is increasing, strictly concave, twice differentiable with $\partial^2 g(L,x,K)\partial L\partial x > 0$ such that, for all $(y,L,x,d,\mu,K,M,p_y,w,p_x,p_\mu,p_K,p_M,\chi,u,z) \in \Omega_T$,

$$y = g(L, x, K_0);\ p_y y - wL - p_x x \geq 0;$$

$$p_y\partial g(L,x,K_0)/\partial L = w;\ \text{ and } p_y\partial g(L,x,K_0)/\partial x = p_x.$$

A 4 Let $\pi = p_y y - wL - p_x x$, Let $\pi^* = \pi + \mu_0 + p_K K_0 - p_M M_0$, In addition, let P and $\mathbb{D}$, respectively, be short for $(p_y,w,p_x,p_\mu,p_K,p_M)$ and (d,μ,K,M). There exists a twice differentiable function,

$$U(\cdot):R_{++}^6 \times R_+ \times [-N_\mu,N_\mu] \times R_+ \times [1, N_M) \to R_+,$$

of $(p_y, w, p_x, p_\mu, p_K, p_M)$, d, μ, K, and M that is strictly concave in $\mathbb{D}$, increasing in (d, μ, K), and decreasing in M. Moreover, for all

$$(y, L, x, d, \mu, K, M, p_y, w, p_x, p_\mu, p_K, p_M, \chi, u, z) \in \Omega_T,$$

$$\partial U(P,\mathbb{D})/\partial d = A + \chi;\ \partial U(P,\mathbb{D})/\partial\mu = p_\mu\partial U(P,\mathbb{D})/\partial d;$$

$$\partial U(P,\mathbb{D})/\partial K = p_K\partial U(P,\mathbb{D})/\partial d;\ \partial U(P,\mathbb{D})/\partial M = -p_M\partial U(P,\mathbb{D})/\partial d;$$

and $\pi^* - d - p_\mu\mu - p_K K + p_M M \geq 0$.

In the intended interpretation of A 3 and A 4, the equations in A 3 record the necessary conditions on the entrepreneur's choice of y, L, and x that ensure that his choice maximizes the firm's profit. The equations in A 4 record the necessary conditions on the entrepreneur's choice of $\mathbb{D}$ that ensure that his choice maximizes his utility. The equations in both axioms concern them equilibrium values of g(·) and the partial derivatives of g(·) and U(·) and not properties of the functions themselves.

Relative to $P_T(\cdot)$, the components of (y, L, x, d, μ, K, M, p_y, w, p_x, p_μ, p_K, p_M, χ, u, z) are random variables. To wit:

A 5 Let $(y, L, x)(\cdot):\Omega_T \to R_+^3$, $(p_y, w, p_x)(\cdot):\Omega_T \to R_{++}^3$,

$(d, \mu, K, M)(\cdot):\Omega_T \to R_+ \times [-N_\mu, N_\mu] \times R_+ \times [1, N_M)$,

$(p_\mu, p_K, p_M)(\cdot): \Omega_T \to R_{++}^3$, and $(\chi, u, z)(\cdot):\Omega_T \to R^{14}$,

be defined by the equations,

$$((y,L,x)(\omega_T), (d,\mu,K,M)(\omega_T), (p_y,w,p_x)(\omega_T),$$

$$(p_\mu, p_K,p_M)(\omega_T), (\chi,u,z)(\omega_T)) = \omega_T, \text{ and } \omega_T \in \Omega_T.$$

The vector-valued functions, (y,L,x)(·), (d,μ,K,M)(·), $(p_y,w,p_x)(\cdot)$, $(p_\mu,p_K,p_M)(\cdot)$, and (χ,u,z)(·) are measurable with respect to $\aleph_T$. They have, subject to the conditions on which Γ_T insists, a well-defined joint probability distribution relative to $P_T(\cdot)$, the RPD, where R is short for researcher, P for probability, and D for distribution.

A 6 Relative to $P_T(\cdot)$, the components of

$$(y,L,x,d,\mu,K,M,p_y,w,p_x,p_\mu,p_K,p_M,\chi,u,z)(\cdot)$$

have finite means and finite positive variances. Moreover, the χ(·) and the components of u(·) and z(·) have means zero and are independently distributed of each other.

4.2 Axioms for the Data Generating Process

I imagine that the data that I will use to test the empirical relevance of my theory axioms belong in a data universe. This data universe is a triple, $(\Omega_P, \Gamma_P, (\Omega_P, \aleph_P, P_P(\cdot)))$, where Ω_P is a subset of a vector space, Γ_P is a finite set of assertions concerning properties of vectors in Ω_P, and $(\Omega_P, \aleph_P, P_P(\cdot))$ is a probability space. The latter comprises Ω_P, a σ field of subsets of Ω_P, $\aleph_P$, and a probability measure, $P_P(\cdot)$): $\aleph_P \to [0,1]$.

The assertions in Γ_P consist of four axioms, D 1-D 4.

D 1 $\Omega_P \subset R^7 \times R^6 \times R^2 \times R^4 \times R^6 \times R^6$. Thus, $\omega_P \in \Omega_P$ only if $\omega_P = (Y, V, mg, mu, \eta, \delta)$ for some $Y \in R^7$, $V \in R^6$, $mg \in R^2$, $mu \in R^4$, $\eta \varepsilon R^6$, $\delta \varepsilon R^6$, and $(Y, V, mg, mu, \eta, \delta) \in R^{31}$.

D 2 Suppose that $\omega_P \in \Omega_P$ and that $\omega_P = (Y, V, mg, mu, \eta, \delta)$ for some $(Y, V, mg, mu, \eta, \delta) \in R^{31}$. There exist constants, a_i, $i = 1,\ldots,6$, such that

$$mg_1 = a_1(V_2/V_1) + \delta_1, \text{ and } mg_2 = a_2(V_3/V_1) + \delta_2; \qquad (43)$$

$$mu_1 = a_3 + \delta_3,\ mu_2 = a_4 \cdot V_4 + \delta_4,\ mu_3 = a_5 \cdot V_5 + \delta_5,$$

$$\text{and } mu_4 = a_6 \cdot V_6 + \delta_6. \qquad (44)$$

In the intended interpretation of these axioms, the denotation of the components of Y are observations of the respective components of (y,L,x,d,μ,K,M), and the denotation of the components of V are observations of the respective components of $(p_y,w,p_x,p_\mu,p_K,p_M)$. Moreover, the components of mg are observations of the respective values of the partial derivatives $\partial g(L,x,K_0)/\partial L$ and $\partial g(L,x,K_0)/\partial x$; the components of mu are observations of the respective values of the partial derivatives, $\partial U(P,\mathbb{D})/\partial d$, $\partial U(P,\mathbb{D})/\partial \mu$, $\partial U(P,\mathbb{D})/\partial K$, and $\partial U(P,\mathbb{D})/\partial M$; and the components of η and δ are error terms.

Relative to $P_P(\cdot)$, the components of Y, V, mg, mu, η, and δ are random variables. To wit:

D 3 Let $Y(\cdot):\Omega_P \to R^7$, $V(\cdot):\Omega_P \to R^6$, $mg(\cdot):\Omega_P \to R^2$,

$$mu(\cdot):\Omega_P \rightarrow R^4,\ \eta(\cdot):\Omega_P \rightarrow R^6, \text{ and } \delta(\cdot):\Omega_P \rightarrow R^6$$

be defined by the equations,

$$(Y(\omega_P),V(\omega_P),mg(\omega_P),mu(\omega_P),\eta(\omega_P),\delta(\omega_P)) = \omega_P \text{ and } \omega_P \in \Omega_P.$$

The vector-valued functions, $Y(\cdot)$,$V(\cdot)$, $mg(\cdot)$, $mu(\cdot)$, $\eta(\cdot)$, $\delta(\cdot)$, are measurable with respect to $\aleph_P$ and have, subject to the conditions on which Γ_P insists, a well-defined joint probability distribution, the TPD, where T is short for true, P for probability, and D for distribution.

D 4 Relative to $P_P(\cdot)$, $Y(\cdot)$, $V(\cdot)$, $mg(\cdot)$, $mu(\cdot)$, $\eta(\cdot)$, and $\delta(\cdot)$ have finite means and finite positive variances. Moreover, (V_2/V_1) and (V_3/V_1) have finite means and variances. Finally, the components of δ are orthogonal to the components of V, and the components of η and δ have zero means and are independently distributed of each other.

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In the intended interpretation of D1- D 4, the TPD plays the role of the data generating process. Specifically, I assume that TPD has one model, and that this model is a true rendition of the data generating process. According to D 4, the variables in TPD have finite means and finite positive variances. Moreover, D 4 and a standard theorem in mathematical statistics imply that the equations in (43) and (44) have a TPD model. The researcher does not know the model of TPD.

For the empirical analysis I have a random sample of 400 observations of the components of Y, V, mg, and mu. If my assumptions about the TPD are valid, I can obtain good estimates of the variables' TPD means and variances and of the TPD values of the parameters in equations (43) and (44).

I begin with the six production variables, Y_1, Y_2, Y_3, V_1,V_2, and V_3. They must have finite means. Table 1 attests to that.

Table 1. TPD Means of Production Variables

	Mean	Std. err.	[95% conf. interval]	
Y_1	444.3416	1.7283	440.9438	447.7393
Y_2	125.3647	0.2608	124.8521	125.8774
Y_3	223.5203	2.3923	218.8171	228.2234
V_1	3.7201	0.0812	3.5605	3.8798
V_2	5.1477	0.1034	4.9445	5.3509
V_3	4.5191	0.0704	4.3808	4.6575

Table 2 records estimates of the TPD values of the parameters in (43) - with wc1 and wc2 short for (V_2/V_1) and (V_3/V_1). In the table, RMSE is short for square root of the mean square error of the residual, R-sq is short for R square, F designates F statistic, and P is short for Prob. > F

Table 2. Estimates of the TPD Values of the Parameters in (43)

Equation	Obs	Parms	RMSE	"R-sq"	F	P>F
mg_1	400	1	0.1031	0.9999	7028225	0.0000
mg_2	400	1	0.1420	0.9999	3102097	0.0000

Var.	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
wc1	1.0004	0.0004	2651.08	0.000	0.9997	1.0012
wc2	1.0005	0.0006	1761.28	0.000	0.9994	1.0016

So much for the production variables. Next I must consider Y_4, Y_5, Y_6, Y_7, V_4,V_5, and V_6. All of them except Y_5 must have finite positive means. Besides, the means of V_4 and V5 ought to be less than one. Table 3 attests to that. Table 4 records estimates of the TPD values of the parameters in (44).

Table 3. TPD Means of Dividends and Balancesheet Variables

Variabel	Mean	Std. err.	[95% conf. interval]	
Y4	16.1481	0.2066	15.7419	16.5543
Y5	21.8662	0.4076	21.0648	22.6676
Y6	70.8180	0.4958	69.8433	71.7927
Y7	59.8945	0.3401	59.2259	60.5632
V4	0.9089	0.0015	0.9060	0.9119
V5	0.9017	0.0013	0.8993	0.9042
V6	3.9878	0.0136	3.9610	4.0145

Table 4. Estimates of TPD Values of the Parameters in (44)

Equation	Obs	Parms	RMSE	"R-sq"	F	P>F
mu2	400	1	0.0581	0.9984	252750.7	0.0000
mu3	400	1	0.1463	0.9898	38813.54	0.0000
mu4	400	1	0.0099	1.0000	1.66e+08	0.0000

Variabel	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
Mean of mu1	1.5998	0.0064			1.5872	1.6123
V4	1.6065	0.0032	502.74	0.000	1.6003	1.6128
V5	1.5980	0.0081	197.01	0.000	1.5821	1.6140
V6	-1.6001	0.0001	-1.3e+04	0.000	-1.6003	-1.5998

It is important to observe that I have formulated D 1 - D4 without using the theory axioms. Hence, in the TPD there are no theory-based true values of the parameters in (43) and (44). I introduce the theory into the empirical analysis with the bridgeprinciples in B 1 - B 6. In reading them, note that I relate the entrepreneur's decision variables, y, L, x, d, μ, K, M, and the partial derivatives of g(·) and U(·) to the observed values of the corresponding components of Y, mg, and mu. In contrast and in the tradition of Trygve Haavelmo (cf. Haavelmo 1944, pp. 7,8), I relate the variables over which the entrepreneur has no control, p_y, w, p_x, p_μ, p_K, and p_M, to the true values in the data universe of the corresponding components of V.

4.3 Axioms for the Bridge

The Bridge is a pair, $(\Omega, \Gamma_{T,P})$, where Ω is a subset of $\Omega_T \times \Omega_P$, and $\Gamma_{T,P}$ is a set of six assertions about the vectors in Ω. It is understood that a researcher's observations consist of pairs,

(ω_T, ω_P), where $\omega_T \in \Omega_T$, $\omega_P \in \Omega_P$, and $(\omega_T, \omega_P) \in \Omega$.

The components of ω_T are unobservable, while the components of ω_P are observable. For example, in the present Bridge, one of the components of ω_T may record the entrepreneur's intended payment of dividends to shareholders, while the corresponding component of ω_P will record a sample entrepreneur's actual payment of dividends to his shareholders.

B 1 $\Omega \subset \Omega_T \times \Omega_P$. Thus $\omega \in \Omega$ only if $\omega = (\omega_T, \omega_P)$ for some $\omega_T \in \Omega_T$, $\omega_P \in \Omega_P$, and $(\omega_T, \omega_P) \in \Omega_T \times \Omega_P$; i.e., $\omega \in \Omega$ only if $\omega=((y,L,x,d,\mu,K,M,p_y,w,p_x,p_\mu,p_K,p_M,\chi,u,z),(Y,V,mg,mu,\eta,\delta))$ for some $(y,L,x,d,\mu,K,M,\underline{p_y,w,p_x,p_\mu,p_K,p_M, \chi,u,z) \in \Omega_T,}$ $\underline{(Y, V, mg, mu, \eta, \delta) \in \Omega_P, \text{and}}$ $((y, L, x, d, \mu, K, M, p_y, w, p_x, p_\mu, p_K, p_M, \chi, u, z), (Y, V, mg, mu, \eta, \delta)) \in \Omega_T \times \Omega_P$.

B 2 Ω_T and Ω_P are disjoint, and $\aleph_T$ and $\aleph_P$ are stochastically independent.

B 3 In the probability space, $(\Omega_T \times \Omega_P, \aleph, P(\cdot))$, which the probability spaces in the theory universe and the data universe generate, $\Omega \in \aleph$, and $P(\Omega) > 0$.

B 4 $\Omega_T \subset \{(y,L,x,d,\mu,K,M,p_y,w,p_x,p_\mu,p_K,p_M,\chi,u,z) \in \Omega_T$ for which there is a $(Y,V,mg,mu,\eta,\delta) \in \Omega_P$ such that $((y,L,x,d,\mu,K,M,p_y,w,p_x,p_\mu,p_K,p_M,\chi,u,z),(Y,V,mg,mu,\eta,\delta)) \in \Omega\}$.

B 5 For all $(\omega_T, \omega_P) \in \Omega$,

$$(y,L,x)(\omega_T) + (u_1,u_2,u_3(\omega_T) = (Y_1,Y_2,Y_3)(\omega_P)$$

$$(d,\mu,K,M)(\omega_T) + (u_4,u_5,u_6,u_7)(\omega_T) = (Y_4,Y_5,Y_6,Y_7)(\omega_P)$$

$$(p_y,w,p_x)(\omega_T) = (V_1,V_2,V_3)(\omega_P) - (\eta_1,\eta_2,\eta_3)(\omega_P);$$

$$(p_\mu,p_K,p_M)(\omega_T) = (V_4,V_5,V_6)(\omega_P) - (\eta_4,\eta_5,\eta_6)(\omega_P);$$

$$(\partial g(L,x,K_0)/\partial L,\partial g(L,x,K_0)/\partial x)(\omega_T)+(z_1,z_2)(\omega_T)=(mg_1,mg_2)(\omega_P);$$

$$(\partial U(P,\mathbb{D})/\partial d), \partial U(P,\mathbb{D})/\partial \mu, \partial U(P,\mathbb{D})/\partial K), \partial U(P,\mathbb{D})/\partial M)(\omega_T) + (z_3, z_4, z_5, z_6)(\omega_T) = (mu_1, mu_2, mu_3, mu_4)(\omega_P).$$

In the intended interpretation of these axioms, Axiom B 5 is not meant to establish an ordinary errors-in-variables relationship between theoretical variables and data variables. Instead, the first two equations and the last two equations delineate how the RPD of the left-hand variables is to be assigned to the corresponding data variables. This distribution, the MPD, may be very diffeent from their TPD. The third and fourth equation describe how the RPD of p_y, w, p_x, p_μ, p_K, and p_M is to be assigned to the true values of the corresponding components of V. This is the MPD of the true values of the components of V.

To obtain the MPD of the observed values of V, it is necessary to establish a theorem, and to add an axiom, B 6, about $\aleph_T$, the σ field of subsets of Ω_T. The theorem is an easy consequence of axioms A, D, and B. I will sketch a proof. it

Theorem 1. Suppose that the A, D, and B axioms are valid. For all $(\omega_T,\omega_P) \varepsilon \Omega$, let

$$u_{7+j}(\omega_T) = \eta_j(\omega_P),\ j = 1,\ldots,6.$$

The six $u_{7+j}(\cdot)$s are well defined on Ω, and the last two equations in B 5 can be rewritten as follows:

$$(p_y,w,p_x)(\omega_T) + (u_8,u_9,u_{10})(\omega_T) = (V_1,V_2,V_3)(\omega_P);\text{ and}$$

$$(p_\mu,p_K,p_M)(\omega_T) + (u_{11},u_{12},u_{13})(\omega_T) = (V_4,V_5,V_6)(\omega_P)$$

It suffices to consider one case in the proof of Theorem 1. Let j = 2 and consider the equation, $u_9(\omega_T) = \eta_2(\omega_P)$. Suppose that there are two pairs in Ω, (ω_T^0, ω_P^0) and (ω_T^1, ω_P^0), at which the two values of $u_9(\cdot)$ differ: i.e., where $u_9(\omega_T^0) \neq u_9(\omega_T^1)$. The two equations,

$$V_2(\omega_P^0) - \eta_2(\omega_P^0) = p_K(\omega_T^0)\text{ , and}$$

$$V_2(\omega_P^0) - \eta_2(\omega_P^0) = p_K(\omega_T^1)\text{ ,}$$

imply that $p_K(\omega_T^0) = p_K(\omega_T^1)$. But if that is so, the two equations,

$$V_2(\omega_P^0) = u_9(\omega_T^0) + p_K(\omega_T^0)\text{ , and}$$

$$V_2(\omega_P^0) = u_9(\omega_T^1) + p_K(\omega_T^1)\text{ ,}$$

imply that $u_9(\omega_T^1) = u_9(\omega_T^0)$.

Then the final axiom about the Bridge.

B 6 The vector valued function, $(u_8, \ldots, u_{13})(\cdot)$ is measurable with respect to $\aleph_T$. Relative to $P_T(\cdot)$, its components have zero means, finite positive variances, and are independently distributed of each other and of $\chi(\cdot)$, $z(\cdot)$, and $(u_1, \ldots, u_7)(\cdot)$.

4.4. The Empirical Analysis

My sample of 400 observations of the components of (Y,V, mg, mu) is a random sample. According to A 6 and B 5, the components of (Y,mg,mu) have finite means and positive variances in the MPD. According to A 6, B 2 - B 6, and Theorem1, the components of V have, also, finite means and positive variances in the MPD. For the empirical analysis, I add the assumption that (V2/V1) and (V3/V1) have finte means and variances in the MPD. From this it follows that Tables 1 - 4 TPD estimates of the means of Y and V and of the parameters in equations (43) and (44) are, also, MPD estimates of MPD means, variances, and parameters. In the MPD there are theory-based true values of a_1, a_2, and a_3. They are, respectively, 1, 1, and A. The first two lie in the confidence intervals of the estimates of a_1 and a_2. A 3 does not insist on a true value of A, but the MPD estimate of A in Table 3 suggests that the true value of A with 95% certainty lies in the interval, (1.5872, 1.61238).

In the intended interpretation of Axiom A 3, the axiom describes characteristics of an entrepreneur's choice of production variables to maximize his firm's profit. With that interpretation in mind, I can deduce from A 3, B 2 - B 6, and Theorem 1 all the characteristics of such choice that I deduced from the equations in (20). They are characteristics that the entrepreneurs in my sample must share if my theory is empirically relevant.

To see if my sample entrepreneurs' choices have the required charaeristics, I begin by recording in Table 5 the correlation matrix of the production variables. The table shows that the entrepreneurs' supply of y varies positively with its price, and that their demand for an input varies negatively with its price.

Table 5. MPD Correlation Matrix of Production Variables

	Y_1	Y_2	Y_3	V_1	V_2	V_3
Y_1	1.0000					
Y_2	-0.0129	1.0000				
Y_3	-0.1837	-0.2179	1.0000			
V_1	0.1158	0.1330	-0.3485	1.0000		
V_2	0.0331	-0.0319	-0.0462	-0.0173	1.0000	
V_3	0.0150	-0.0603	-0.1308	0.0547	0.1287	1.0000

Next, I will obtain estimates of the data version of the relations which the last two equations in A 3 depict. I do that by regressing $V_1 \cdot mg_1$ on V_2 and $V_1 \cdot mg_2$ on V_3. In Table 6, mv1 is short for $V_1 \cdot mg_1$ and mv2 is short for $V_1 \cdot mg_2$. If my interpretation of the variables is empirically relevant, the confidence intervals of both parameters must contain the number 1, which they do.

Table 6. MPD Estimates of the Parameters in Axiom A 3

Equation	Obs	Parms	RMSE	"R-sq"	F	P>F
mv1	400	1	0.4182	0.9944	70383.36	0.0000
mv2	400	1	0.5233	0.9881	33137.81	0.0000

Variables	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
mv1 on V_2	1.0001	0.0038	265.30	0.000	0.9927	1.0075
mv2 on V_3	1.0065	0.0055	182.04	0.000	0.9956	1.0174

It will be interesting to see if my observations, also, accord with Hotelling's Lemma. I do that by regressing mπ1 – the data version of π - on V_1, V_2, and V_3.

Table 7. An MPD Test of Hotelling's Lemma

Equation	Obs	Parms	RMSE	"R-sq"	F	P>F
mπ1	400	3	259.7794	0.9140	1406.773	0.0000

Variables	Coefficient	Std. err.	t	P>|t|	[95conf. interval]	
V_1	452.2555	7.0351	64.29	0.000	438.4248	466.0862
V_2	-130.2235	5.5835	-23.32	0.000	-141.2006	-119.2465
V_3	-222.8214	7.3486	-30.32	0.000	-237.2684	-208.3744

The data accord with the Lemma only if the the respective confidence intervals of the estimated coefficients of V_1, V_2 and V_3 contain the mean value of Y_1 and the negative values of the means of Y_2 and Y_3. Tables 1 and 7 show they do.

It remains to check if the observations agree that the firm's marginal cost of producing Y_1 equals its price. I do that by regressing the cost of producing Y_1 on Y_1 and mπ1. The result is displayed in Table 8, where wmc = $V_2 \cdot Y_2 + V_3 \cdot Y_3$ and mπ1 = $V_1 \cdot Y_1$ – wmc. According to Tables 1 and 8, the confidence interval of the estimated coefficient of Y_1 contains the observed mean value of the price of Y_1. Hence, I cannot reject the hypothesis that the sample-entrepreneurs' choices satisfy the marginal cost condition.

Table 8. An MPD Estimate of the Marginal Cost of Y_1

Equation	Obs	Parms	RMSE	"R-sq"	F	P>F
wmc	400	2	409.5867	0.9440	3354.542	0.0000

Variabel	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
Y1	3.7216	0.0460	80.97	0.000	3.6312	3.8119
mπ1	-0.3259	0.0232	-14.04	0.000	-0.3715	-0.2795

So much for the production variables. Next I must consider the interpretation of Y4, Y5, Y6, Y7, V4, V5, and V6. In the intended interpretation of Axiom A 4, the axiom describes characteristics of an entrepreneur's choice of dividends and balance-sheet variables that maximize the value of his utility in (8) subject to the conditions in (9) - (11). With that interpretation in mind, I can deduce from A 4, B 2 - B 6, and Theorem 1 all the characteristics of such choices that I deduced from the equations in (20). The solutions of the equations in (20) depict characteristics that the entrepreneurs in my sample must share if my theory is empirically relevant.

I begin wih the marginal efficiency condition on investments in bonds. There are six variables involved in the analysis of the entrepreneur's investment in bonds, dividends -Y_4, bonds - Y_5, price of bonds -V_4, two of the marginal-utility variables in the equations in (44) - mu_1 and mu_2, the interest rate on one-period loans - ccr1, and the marginal efficiency of the investment in Y_5 - mefmu1. The definition of the last two variables are as follows:

$$mefmu1 = ((mu_1 – mu_2)/mu_2) \text{ and } ccr1 = (1/V_4) – 1.$$

Table 9. MPD Means of Variables Involved in Bond Investment

Variables	Mean	Std. err.	[95% conf. interval]	
V4	0.9089	0.0015	0.9060	0.9119
ccr1	0.1014	0.0018	0.0979	0.1049
mu1	1.5998	0.0064	1.5872	1.6124
mu2	1.4520	0.0037	1.4447	1.4593
mefmu1	0.1042	0.0050	0.0944	0.1141

The mean values of the two mus, mefmu1, and ccr1 are listed in Table 9. My interpretation of the variables is empirically relevant in the present empirical context only if the mean value of ccr1 lies in the confidence interval of the mean value of mefmu1. It does.

Next, the marginal eficiency condition on investment in capital. There are six variables involved in the empirical analysis of the entrepreneur's investment in capital, capital - Y_6, price of capital -V_5, two of the marginal-utility variables in the equations in (44) - mu_1 and mu_3, the rate of return to capital - ccr3, and the marginal efficiency of the investment in Y_6 -mefmu3. With the $m_{K1} = 1$ in (27), the definitions of the last two variables are as follows:

mefmu3 = ((mu_1 – mu_3)/mu_3) and ccr3 = (1/V_5) – 1.

The mean values of the two mus and mefmu3 and ccr3 are listed in Table 10. My interpretation of the variables is empirically relevant in the present empirical context omly if the mean value of ccr3 lies in the confidence interval of the mean of mefmu3 – as it does. For the present test the value of m_{K1} is irrelevant sinc (m_{K1}·mu_1/mu_3) – 1 = (m_{K1}/V_5) – 1, and the 1 and the m_{K1} cancel.

Table 10. MPD Means of Variables Involved in Capital Investment

Variables	Mean	Std. err	[95% conf. interval]	
V5	0.9017	0.0013	0.8993	0.9042
ccr3	0.1098	0.0015	0.1068	0.1129
mu1	1.5998	0.0064	1.5872	1.6124
mu3	1.4410	0.0076	1.4261	1.4559
mefmu3	0.1222	0.0074	0.1078	0.1367

It remains to see if the last three equations in A 4 are empirically relevant. For that I use Stata 17's nonlimear regression program to regress mu_2 on V_4·mu_1, mu_3 on V_5·mu_1, and mu_4 on V_6·mu_1. Tables 11, 12, and 13 record the results. My theory is empirically relevant only if the 95% confidence intervals of the parameter estimates in Tables 11, 12, and 13 contain, respectively, the numbers 1, 1, and -1. They do.

Table 11. An MPD nl-Estimate of a Parameter in Axiom A 4

```
nl(mu2 = ({b0 = 1}*V4)*mu1)
Iteration 0:  residual SS = 6.2503;
Iteration 1:  residual SS =   6.2503;
```

Source	SS	df	MS	
				Number of obs = 400
Model	839.2776	1	839.2776	R-squared = 0.9926
Residual	6.2503	399	0.0157	Adj R-squared = 0.9926
				Root MSE = 0.1252
Total	845.5280	400	2.1138	Res. dev. = -528.38

mu2	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
/b0	0.9921	0.0043	231.47	0.000	0.9836	1.0005

Table 12 An MPD nl-Estimate of a Parameter in Axiom A

nl(mu3 = ({b0 = 1}*V5)*mu1)

Iteration 0: residual SS = 13.9836;

Iteration 1: residual SS = 13.9836;

Source	SS	df	MS	
				Number of obs = 400
Model	825.8249	1	825.8249	R-squared = 0.9833
Residual	13.9836	399	0.0350	Adj R-squared = 0.9833
				Root MSE = 0.1872
Total	839.8086	400	2.0995	Res. dev. = -206.2803

mu3	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
/b0	0.9920	0.0065	153.50	0.000	0.9793	1.0047

Table 13. An MPD nl-Estimate of a Parameter in Axiom A 4

nl(mu4 = ({b0 = 1}*V6)*mu1)

Iteration 0: residual SS = 105.5411:

Iteration 1: residual SS = 105.5411;

Source	SS	df	MS	
				Number of obs = 400
Model	16256.1373	1	16256.1373	R-squared = 0.9935
Residual	105.5411	399	0.2645	Adj R-squared = 0.9935
				Root MSE = 0.5143
Total	16361.6784	400	40.9042	Res. dev. = 602.205

mu4	Coefficient	Std. err.	t	P>|t|	[95% conf. interval]	
/b0	-0.9943	0.0040	-247.90	0.000	-1.0022	-0.9864

I have, now, checked the empirical relevance of all the characteristics that my sample entrepreneurs must share if the theory is empirically relevant. The checks were carried out with MPD distributed data variables. They did not give me reason to reject the empirical relevance of the theory in an empirical context in which the data are MPD distributed. It remains to see if the theory is empirically relevant in an empirical context in which the TPD is the data generating process – i.e., in the present empirical context.

To demonstrate that my theory is empirically relevant in the present empirical context, I must show that the MPD in some sense encompasses the TPD. Let M_T be an econometric model whose variables are listed in D 1 and satisfy the conditions imposed on them in D 1 and D 2. Assume that they have the TPD distribution described in D 3 and D 4, and let Δ be a vector whose components are the parameters whose estimated values are listed in Tables 1-13. Moreover, let s_n denote a sample of n observations of the data variables, and let $m_T{}^0(\cdot)$ and $m_P{}^0(\cdot)$ be, respectively, the Stata 17 estimators of the components of Δ in the TPD and the MPD distributions. Finally, let $TP(\cdot):\aleph_P \rightarrow [0,1]$ be the probability11 measure on $(\Omega_P, \aleph_P)$ corresponding to TPD, and let MP(·):$\aleph_P \rightarrow$[0,1] be the probability measure on (Ω_P, $\aleph_P$) which – in accord with Kolmogorv's Consistency Theorem (cf. theorem T 15.23 on p. 347 in (Stigum, 1990)) - is induced by a given MPD. This measure varies with the MPD in question. Since the two estimators are identical, it is the case, both in TP(·) measure and in MP(·) measure, that $m_T^0(s_n) = m_P^0(s_n)$ a.e.. The estimates in Tables 1-4 are MPD estimates as well as TPD estimates. Similarly, the estimates in Tables 5-13 are TPD estimates as well as MPD estimates. Consequently, the two pairs, (M1,$m_P^0(s_n)$) and (M2,$m_T^0(s_n)$), in fact, mutually encompass each other (cf. in this context, Bontemps and Mizon 2008, pp. 727-728).

The preceding observation provides the missing link in the proof that my theory is empirically relevant. The given MPD is coherent with the a priori theory in D 1 and D 2 - in the sense that the equations in (43) and (44) have an MPD model – and it encompasses the TPD. Hence (cf. Definition 2 on p. 6 in (Stigum, 2016)), it is a congruent model of the TPD. Since all the data admissible MPDs have the coherence-with-a priori-theory and mutual-encompassing properties of the given MPD, all the data admissible MPDs are congruent models of the TPD. From this and the so-called Status of bridge principles (cf. p. 7 in (Stigum, 2016)), it follows that the bridge principles in B 1 – B 6 are empirically valid in the present empirical context. But if that is so, in the present case the fact that my theory is empirically relevant in an empirical context with MPD distributed data implies that the theory is empirically relevant in an empirical context in which the data are TPD distributed.

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