A Game is "essentially" a mechanism
The outcome depends on the (rules of the) game
Comparative static: How outcome varies with the rules
"Reverse game theory": How the rules should vary with the outcome

1. Given a particular outcome/goal: What is the appropriate game?
2. Is it possible to find a game implementing the desired outcome?
3. Given an objective/welfare function: What is the best game?
Mechanism $\iff$ Bayesian Game

- Take the external environment as given: The players $\{1, \ldots, I\}$, the set of alternatives $X$, the payoffs $u_i(x, \theta_i)$, the type space $\theta_i \in \Theta_i$, and the priors (pdf over type space) $\phi : \Theta_1 \times \cdots \Theta_I \rightarrow [0, 1]$.

- A mechanism is a set of strategies and an outcome-function: $\Gamma = (S_1, \ldots, S_I, g(\cdot))$, $g : S_1 \times \cdots S_I \rightarrow X$.

- So, the mechanism specifies who can do what and the consequences of these actions.

- Interpretation: the institutions, the rules of the game, the electoral system, bargaining game, ownership, the economic system...

- The game and the external environment is together the strategic normal form Bayesian game:

$$G = [I, \{S_i\}, u_i(g(s), \theta_i), \Theta_1 \times \cdots \Theta_I, \phi]$$
Game theory: For a given game, find the outcomes

\[ G \Rightarrow x(\theta) \]

Mechanism design: For a given desirable outcome, find the games

\[ x(\theta) \Rightarrow \Gamma \Rightarrow G \]

The "engineering" part of economics
The Implementation Problem

- What is a desirable outcome?
- **Def23B1:** A *scf* is a function $f : \Theta_1 \times ... \Theta_I \rightarrow X$
- **Def23B2:** $f$ is ex post efficient if $\exists (\theta, x)$ s.t. $u_i (x, \theta_i) \geq u_i (f (\theta), \theta_i) \forall i$ with strict inequality for some.
- **Def23B4:** $\Gamma$ implements $f$ if there is one equilibrium of $\Gamma$, $s^* (.)$, s.t. $g (s^* (\theta)) = f (\theta) \forall \theta$.

**Ambiguities:**
- Is one equilibrium enough? ($\Gamma$ can have multiple equilibria: require "full implementation"?).
- What is an "equilibrium"? Bayesian? Nash? Dominant? (Harder)

1. When is $f$ implementable?
2. What if $f$ is not implementable?
What is an equilibrium?

- **Def23D1**: $s^*$ is a *Bayesian Nash equilibrium* of $\Gamma$ if

\[
E_{\theta_{-i}} \left[ u_i \left( g \left( s^*_i (\theta_i) , s^*_{-i} (\theta_{-i}) \right) , \theta_i \right) \right] \geq E_{\theta_{-i}} \left[ u_i \left( g \left( s'_i , s^*_{-i} (\theta_{-i}) \right) , \theta_i \right) \right] \]

for all $(s'_i , i , \theta_i)$. 

- **Def23C1**: $s^*$ is a *dominant strategy equilibrium* of $\Gamma$ if

\[
u_i \left( g \left( s^*_i (\theta_i) , s'_{-i} \right) , \theta_i \right) \geq u_i \left( g \left( s'_i , s'_{-i} \right) , \theta_i \right) \]

for all $(s'_i , s'_{-i} , i , \theta_i)$. 

- $s^*$ is a dominant strategy equilibrium $\Rightarrow$ $s^*$ is a Bayesian Nash equilibrium. 

- $f$ implementable in dominant strategies $\Rightarrow f$ implementable in Bayesian Nash equilibrium.
Direct Revelation Mechanism

- Problem: There is a large number of mechanisms...
- **Def23B5:** \( \Gamma \) is a direct revelation mechanism if \( S_i = \Theta_i \forall i \) and \( g(\theta) = f(\theta) \forall \theta. \)
- **Def23B6:** \( f \) is truthfully implementable (=incentive compatible) if the direct revelation mechanism \( \Gamma \) has one equilibrium \( s^* \) where \( s^*_i(\theta_i) = \theta_i \forall (i, \theta_i) \).

- A very simple mechanism: Just ask for the preferences, and implement \( f \) as if reports=messages were true.
- A very naïve mechanism!
- Seems very limited and hard to satisfy!?
The Revelation Principle

**Proposition**

\( \exists \Gamma \) implementing \( f \) \( \iff \) \( f \) is truthfully implementable

**Proof** (Dominant strategies): If \( \Gamma \) implements \( f \)

\[
\begin{align*}
    u_i \left( g \left( s^*_i (\theta_i) , s'_{-i} \right) , \theta_i \right) & \geq u_i \left( g \left( s'_i , s'_{-i} \right) , \theta_i \right) \Rightarrow \\
    u_i \left( g \left( s^*_i (\theta_i) , s^*_{-i} (\theta_{-i}) \right) , \theta_i \right) & \geq u_i \left( g \left( s'_i , s^*_{-i} (\theta_{-i}) \right) , \theta_i \right) \Rightarrow \\
    u_i \left( g \left( s^*_i (\theta_i) , s^*_{-i} (\theta_{-i}) \right) , \theta_i \right) & \geq u_i \left( g \left( s^*_i \left( \hat{\theta}_i \right) , s^*_{-i} (\theta_{-i}) \right) , \theta_i \right) \Rightarrow \\
    u_i \left( f (\theta) , \theta_i \right) & \geq u_i \left( f \left( \hat{\theta}_i , \theta_{-i} \right) , \theta_i \right).
\end{align*}
\]

- Good: Not difficult to find \( \Gamma \)
- Bad: Difficult to find implementable \( f \)!
The Gibbard-Satterthwaite Theorem

Suppose $X$ is finite and $|X| \geq 3$, preferences are strict and $f(\Theta) = X$.

**Proposition**

- The scf $f$ is implementable in dominant strategies iff $f$ is dictatorial.

- The scf $f$ is **dictatorial** if there exists an $i$ such that for all $\theta \in \Theta$

  $$f(\theta) \in \{x \in X : u_i(x, \theta_i) \geq u_i(y, \theta_i) \ \forall y \in X\}$$

- The simplest proof builds on Arrow’s impossibility theorem.
- Proven independently by Gibbard ’73 and Satterthwaite ’75.
**Vickrey-Clarke-Groves Mechanism**

- **Quasilinear case / transferable utility**
  
  \[ u_i(x, \theta_i) = v_i(k, \theta_i) + t_i \]

- **Project** \( x = (k, t_1, ..t_I) \), \( X = \{k \in K, \sum t_i \leq 0\} \)

- **Note**: \( f \) specifies \( k \) \textit{and} transfers as a function of \( \theta \)

- **Def23B2 implies**: \( f \) is ex post efficient only if \( k(.) \) is efficient, i.e.
  
  \[ \sum v_i(k(\theta), \theta_i) \geq \sum v_i(k', \theta_i) \forall (\theta, k'). \]

- **Prop23C4**: Suppose \( k \) is efficient. Then \( f \) is implementable in dominant strategies if

  \[ t_i(\theta) = \left[ \sum_{j \neq i} v_j(k(\theta), \theta_j) \right] + h_i(\theta_{-i}), \]

  no matter the functions \( h_i(\theta_{-i}). \)

- **Example**: \( I = 2, h_i(\theta_j) = -v_j(k(\theta_j, \theta), \theta_j); \) second-price auction.
VCG: Proof

Proof by contradiction. If truth is *not* a dominant strategy, then it is possible that

\[
v_i \left( k \left( \hat{\theta}_i, \theta_{-i} \right), \theta_i \right) + t_i \left( \hat{\theta}_i, \theta_{-i} \right) > v_i \left( k (\theta), \theta_i \right) + t_i (\theta)
\]

\[
\Leftrightarrow
\]

\[
v_i \left( k \left( \hat{\theta}_i, \theta_{-i} \right), \theta_i \right) + \sum_{j \neq i} v_j \left( k \left( \hat{\theta}_i, \theta_{-i} \right), \theta_j \right) > v_i \left( k (\theta), \theta_i \right) + \sum_{j \neq i} v_j \left( k (\theta), \theta_j \right)
\]

\[
\Rightarrow
\]

\[
\sum v_j \left( k', \theta_j \right) > \sum v_j \left( k (\theta), \theta_j \right),
\]

so \( k (\theta) \) is *not* ex post efficient.
Assume A: If $V = \{v : K \rightarrow \mathbb{R}\}$ then $\{v_i(., \theta_i) : \theta_i \in \Theta_i\} = V$.

Prop23C5: If $k(.)$ is efficient and A, it is implementable if and only if, for some $h_i(\theta_{-i})$:

$$t_i(\theta) = \left[ \sum_{j \neq i} v_j(k(\theta), \theta_j) \right] + h_i(\theta_{-i}),$$

Def23B2 also implies: $f$ is ex post efficient iff $k(.)$ is efficient and $\sum t_i(\theta) = 0 \forall \theta$

Prop23C6: If $f$ is ex post efficient and A, it is not implementable in dominant strategies.

Relax assumption: If one $v_i$ is constant, then ex post efficient $f$ is implementable in dominant strategies
The Expected Externality Mechanism / AGV

- Relax equilibrium concept: If $f$ is ex post efficient (and types independent), then $f$ is implementable in Bayesian Nash equilibrium.

- ...with the Expected Externality Mechanism (d’Aspremont and Gerard-Varet 79, Arrow 79):

  $$ t_i(\theta) = \mathbb{E}_{\theta_{-i}} \left[ \sum_{j \neq i} v_j(k(\theta), \theta_j) \right] + h_i(\theta_{-i}), $$

- Trick is to carefully select $h_i(\theta_{-i})$

  $$ h_i(\theta_{-i}) = \frac{-1}{l-1} \sum_{j \neq i} \left[ \mathbb{E}_{\theta_{-j}} \left[ \sum_{l \neq j} v_l(k(\theta), \theta_l) \right] \right] $$

- Problem in all above mechanism: Participation constraints
Suppose seller’s and buyer’s valuation of the good has a nonempty intersection.

There exists a function $f$ which is ex post efficient, implementable in Bayesian Nash equilibrium, and which gives every type nonnegative expected gains from participation.

Easy to check in example with $\{c, \overline{c}\}$ and $\{v, \overline{v}\}$

**OPTIMAL MECHANISMS:**

- Often, the ideal $f$ is not implementable
- The second-best problem: Maximize a given objective, s.t. (IC), (IR) and (BB)
- This is often easier than to detect implementable $f$
Incomplete Contracts - What is that?

- Some acts are contractible, others are not
- Are non-contractible actions suboptimal?
  - What do they depend on?
  - How do they depend on the contractible acts?
- How can we influence non-contractible acts by carefully selecting the contractible ones?
Incomplete Contracts - Examples

- Firms: Ownership is contractible, effort is not
- Trade: Quantity contractible, investments not
- Finance: Repayment is partly contractible, risk-taking not
- Climate: Emission is contractible, R&D is not
- EU: Voting rules are contractible, local policies are not
Incomplete Contracts - General setting

1. Agents $I$ may contract on $q$
2. Actions $x_i$ may also be feasible, $i \in I$
3. Agents negotiate a decision $k \in K$ and $t_i$

- Payoff $V_i (x; q, k) + t_i - \psi_i (x_i)$
- What is the equilibrium $x_i$?
- How does $x_i$ depend on $q$?
- What is the optimal $q$?
- Can we find $q$ s.t. $x$ is first-best?
Incomplete Contracts - Bilateral example

- Let $K = \{yes, no\}$
- Let $\Phi_i(x_1, x_2; q) = V_i(x_1, x_2; q, no)$
- Let $\alpha_i \in [0, 1]$ be $i$'s "bargaining power"
- Suppose $k = yes$ will be ex post optimal
- Bargaining surplus is
  \[ \sum_l V_i(.) - \sum_l \Phi_i(.) . \]
- Agent 1 receives $\Phi_1(.) + \alpha_i \cdot \text{Bargaining-surplus}$
When choosing $x_i$, the foc is

$$ \alpha_i \left( \frac{\partial V(.)}{\partial x_i} - \frac{\partial \Phi(.)}{\partial x_i} \right) + \frac{\partial \Phi_i(.)}{\partial x_i} = \psi'_i(x_i) $$

First-best $x_i$ is

$$ \frac{\partial V(.)}{\partial x_i} \equiv \sum_{j \in I} \frac{\partial V_j(.)}{\partial x_i} = \psi'_i(x_i^*) . $$

Why is $x_i \neq x_i^*$? When is $x_i = x_i^*$?
Incomplete Contracts - Solutions

- Consider assets \( \{a_1, a_2\} \) but \( \Phi_i(.) = 0 \) unless \( i \) owns both.
- Suppose \( \Phi'_1(x_1, x_2, q) = \Phi'_1(x_1, \{a_1, a_2\}) > 0, \ V_{12} > 0 \)
- Then... non-integration is dominated
  - If \( \Phi'_i(x_1, x_2, \{a_1, a_2\}) < 0 \), non-integration better
- Complementary assets should be owned together
- If only one invests, that agent should own both assets
- If only one invests, that agent should have all bargaining power
- The agent having an outside option/market should own