Learning in a medium-scale DSGE model with expectations based on small forecasting models

By Sergey Slobodyan and Raf Wouters*

Draft: April 1, 2010

This paper evaluates the empirical performance of a medium–scale DSGE model (Smets and Wouters 2007) when agents form expectations about forward variables by using small forecasting models. Agents learn about these simple forecasting models through Kalman filter updating. A model with this type of adaptive learning fits the data better than the rational expectations model. Agents’ beliefs about the persistence of inflation explain the observed decline in both the mean and the volatility of inflation as well as the flattening of the Phillips curve. Learning about inflation dynamics also results in lower estimates for the persistence of the exogenous processes that drive price and wage dynamics in the rational expectation version of the model. We also find that expectations based on small forecasting models are closely related to the survey evidence on inflation expectations. (JEL: C11, C52, D84, E30)

*Slobodyan: CERGE-EI, Politickych veznu 7, 111 21 Prague 1, Czech Republic (e-mail: sergey.slobodyan@cerge-ei.cz); Wouters: National Bank of Belgium/Université catholique de Louvain, Boulevard de Berlaimont, B-1000 Brussels, Belgium (e-mail: rafael.wouters@nbb.be). First author expresses sincere gratitude to NBB for hospitality and support which made this project possible. We thank participants in the San Francisco Fed Conference on "Macroeconomic Models for Monetary Policy", the EABCN Workshop on "Uncertainty over the Business Cycle", the Magyar Nemzeti Bank/CEPR Macroeconomic Policy Research Workshop on "DSGE Models: A Closer Look at the Workhorse of Macroeconomics", 2009 SED Conference, “Adaptive Learning and Macroeconomic Policy” conference at Cambridge University, Joint Lunchtime Seminar of European Central Bank (ECB), Center for Financial Studies (CFS), and Deutsche Bundesbank, and 2008 “Learning Week” conference at St. Louis Fed. We particularly thank our discussants George Evans, Marco Del Negro, Kaushik Mitra and Krisztina Molnár for very useful comments. The views expressed are solely our own and do not necessarily reflect those of the National Bank of Belgium.
In modern macroeconomics, the behavior of economic agents is defined by a series of intertemporal optimization problems. The solution to these problems depends crucially on how agents form their expectations about future events. From a theoretical perspective, rational expectations (RE) appear as a logical assumption: expectations are formed consistently with the underlying model and the policy environment, and all available information is used efficiently. This hypothesis is extremely useful for a macroeconomist as it tightens the link between theory and estimation, and it allows for an efficient estimation of the deep parameters of the model by exploiting all the cross-equation restrictions that are imposed through the model-consistent expectations hypothesis. However, RE do not provide a description of the information problem that agents have to solve to discover these systematic relations. Therefore, from an empirical point of view, RE appear as an implausibly strong assumption. Households and firms do not have complete knowledge about the correct form of the underlying economic model, about the exact value of the model parameters, or about the complete state vector of variables, and especially about the exogenous and latent disturbances that hit the economy. Instead, agents, like econometricians, need to find out the dynamic structure of the economy using the data available in real time. As processing information is costly, it is more realistic to assume that they will concentrate on a limited set of information and that they update their beliefs about the underlying economic relations as new data become available, in order to capture possible changes in the stochastic structure or in the policy environment. If expectations are allowed to deviate from the RE solution, the model dynamics will change as well and expectations become, potentially, an important additional source of business cycle fluctuations.

Therefore, in this paper, we assume that expectations are based on a limited information set, meaning that agents use small forecasting models in forming their beliefs about future realizations, i.e. by using simple autoregressive models, and that they adapt the coefficients of these forecasting models by a simple Kalman filter updating procedure. This approach implies only a minimal departure from the RE model as we leave the decision problems of the agents intact and we only replace the model-consistent expectations in their optimizing behavioral rules by a more realistic adaptive expectations mechanism. By allowing agents to use misspecified and under-parameterized expectation models, the macroeconomist receives some additional degree of freedom as he has to make a decision on the relevant forecasting model or, in other words,  

---

1 These limitations of the RE approach are widely recognized in the macroeconomic literature and generally acknowledged as a serious weakness and challenge for the new generation of DSGE models that are now popular in both academic and applied research. Over the last decades, alternative approaches to model expectations have been suggested in the literature: the Bounded Rationality approach of Sargent (1993), the Rational Inattention approach of Sims (2003), the Epidemiology of Macroeconomic Expectations by Carroll (2001), the Sticky Information model of Mankiw and Reis (2002), the Partial Information model of Pearlman et al. (1986) and Svensson and Woodford (2003), and the learning approach of Evans and Honkapohja (1999).
on the information set that agents are using in their expectations model. However, this does not appear as an important empirical issue for several reasons. First, it turns out that the data are informative on this decision, so that the additional flexibility can be exploited optimally to fit the overall model on the data without any danger of ending up in an unidentified system. Second, the estimates of structural parameters and the identification of the shocks do not seem to be very sensitive to the specific choice of the information set included in the forecasting model. Finally, the use of explicit data on private sector expectations available through surveys or forward looking variables, like asset prices, might be very helpful in the selection of the relevant forecasting model.

Recent empirical DSGE models systematically retain the hypothesis of rational expectations. Smets and Wouters (2003, 2007) have shown that these models, when equipped with a rich set of frictions and a general stochastic structure, explain the data relatively well. However it is still somewhat problematic that these models require highly persistent exogenous shocks to explain the observed persistence in the data (Chari et al. 2009). Milani (2006, 2008) and Orphanides and Williams (2005a) claim that learning can significantly influence the macroeconomic dynamics and increase the persistence in the responses to shocks. For instance, Milani estimates a small scale model both under RE and learning and shows that the learning reduces the scale of structural frictions and results in an improved marginal likelihood relative to the RE model. Orphanides and Williams (2005a) illustrate how adaptive learning can lead to inflation persistence. Slobodyan and Wouters (2007) analyze the learning dynamics in the SW model and find that learning hardly influences model dynamics if the information set used in the learning process is the same as under rational expectations. Restricting information available to the agents improves the model fit and produces IRFs that match those from the best-fitting DSGE-VAR models.

In this paper, we show that the empirical fit of the standard medium-scaled DSGE model is significantly improved when private agents are assumed to form their expectations on the basis of updated small forecasting models. Expectations and learning turn out to play a significant role in accounting for business cycle and inflation fluctuations. This empirical success depends crucially on the fact that private sector expectations about future inflation are allowed to adjust to the changing persistence in the observed inflation dynamics over the last forty years. With this type of learning dynamics in the expectations, the exogenous disturbances to price and wage dynamics reduce to i.i.d. innovations. In other words, the learning process substitutes for the exogenous dynamics in the shocks that are typically required in the RE model. Furthermore, the implied time variation in the propagation mechanism of these shocks can explain the upward and downward trend in inflation volatility during the great inflation and subsequent great moderation,
and is also consistent with the observed flattening of the Phillips curve over the last twenty years. The important role of the inflation expectations dynamics in our model, and in particular the possible divergence between private sector expectations and the complete information-based rational expectation forecast, also justifies why central bankers are so concerned about these expectations.

In the next section, we explain how we change the expectation assumption in the medium-scale DSGE model of Smets and Wouters (SW, 2007) and how the learning mechanism is specified. Section 3 presents the estimation results under learning with limited information and compares these results with the RE version. In Section 4, we interpret the learning dynamics and provide additional evidence in favor of our time-varying expectations mechanism. Section 5 discusses how the learning dynamics affect the propagation mechanism of the shocks in the model and explain the changes in volatility and the slope of the Phillips curve in the US data over the last four decades. Finally, in Section 6, an extensive sensitivity analysis is performed to show the robustness of the results.

I. Model and learning dynamics

In this paper, we evaluate the potential role of adaptive learning (AL) in an estimated medium-scale DSGE model. The model that we consider in this application is the one estimated in SW (2007) for the US economy, updated with the most recent data covering 1966-2008. This DSGE model, following the work of Christiano et al. (2005), contains many frictions that affect both nominal and real decisions of households and firms. Households maximize expected utility over an infinite horizon. Consumption appears in this utility function relative to a time-varying external habit variable. Their labor services are further differentiated by a union that sets the nominal wage according to a Calvo model. Households decide on how much capital to accumulate, given the investment adjustment cost function, and rent this capital to firms. Depending on the rental rate, the capital stock will be used more or less intensively. Firms produce differentiated goods, decide on labor and capital inputs, and set prices again according to the Calvo model. Their marginal cost depends on wages, the rental rate of capital and the exogenous productivity process. The Calvo models for price- and wage-setting assume partial indexation to lagged inflation, so that the inflation dynamics have both a forward- and a backward-looking component. The standard Dixit-Stiglitz aggregator in both the goods and labour market is replaced by a more general aggregator which allows for time-varying demand elasticity.

\(^2\)We refer to Smets and Wouters (2007) for the formal presentation of the model. The equations and corresponding coefficients are repeated in the appendix.
As in SW, monetary policy is described by a generalized Taylor rule with inertia in the policy reaction to inflation and the output gap. We deviate, however, from SW by defining the output gap simply as the deviation of output from its underlying neutral productivity process and not as the natural output gap. In doing so, we avoid the modelling of the flexible economy, which considerably reduces the number of forward variables on which agents have to form expectations.

The model contains 14 endogenous variables summarized by the vector $y_t$. In addition, the stochastic structure of the model is determined by seven exogenous disturbances and their innovations. Neutral and investment-specific technological progress, risk premiums, exogenous spending and non-systematic monetary policy actions are represented by a first-order autoregressive process, while the price and wage markup disturbances are modelled as an ARMA(1,1) process. The vector $w_t$ represents both the 7 exogenous variables and the lagged innovations $\epsilon_{t-1}$ for the markup shocks. After linearization around the deterministic steady state, the model can be represented as follows:\footnote{We implemented this AL estimation approach in the Dynare program, and therefore we follow the notation used by Julliard (1996). The code is available upon request from the authors.}:

$$
A_0 \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + A_1 \begin{bmatrix} y_t \\ w_t \end{bmatrix} + A_2 E_t y_{t+1} + B_0 \epsilon_t = \text{const.}
$$

Under rational expectations, the solution of the model is provided by

$$
\begin{bmatrix} y_t \\ w_t \end{bmatrix} = \mu + T \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + R \epsilon_t.
$$

where the matrices $T$ and $R$ are non-linear functions of the structural parameters of the model, $\Theta$; the intercept, $\mu$, is a zero vector under RE. The vector $y$ can be further decomposed into state variables $y^s$ (those appearing with a lag), forward variables $y^f$ (appearing with a lead in the model) and the so-called static variables. More specifically, in the SW model, agents have to form expectations on 7 forward variables: consumption, investment, hours worked, wages, inflation, price and return of existing capital.

In this paper, we relax the RE assumption and, following Marcet and Sargent (1989) and Evans and Honkapohja (2001), we assume that the agents forecast the values of the forward variables as a reduced-form linear function of the state variables\footnote{This type of learning, promoted by Evans and Honkapohja (2001), is called Euler equation learning: the agents forecast only immediate future variables which are typically present in Euler equations for firms and/or consumers. An alternative description of learning — long–horizon learning — has been suggested by Marcet and Sargent (1989) and promoted by Preston (2005): he considers agents forecasting economic variables (present in their budget constraint and}. Furthermore, we assume that agents use
only a limited information set, $X_j$, so that the forecasting model takes the form: $y_j^f = X_j^T \beta_j$. More in particular, in the baseline specification, we assume that agents use a simple extrapolative AR(2) model to form expectations, where $X_j$ contains a constant and two lags of the forward variable $y_j^f$. This specification of the PLM model delivers results which are representative for the outcomes under a broad set of small forecasting models (see section 6). Note that this set-up deviates from RE in three substantial ways: firstly, the coefficients in the forecasting models are not restricted to be consistent with the decision rules of the agents; secondly, the information set that is used in the forecasting models is much smaller than the state vector that would be used under RE, and finally, the coefficients of the forecasting model are updated based on new observations. The precise learning procedure is defined as follows.

Agents estimate the forecasting model at each point in time given the information set available at that time. We assume that they use an efficient Kalman filter updating mechanism. They believe that the coefficients $\beta$ (a vector obtained by stacking all $\beta_j$) follow a vector autoregressive process around $\bar{\beta}$ (which will be specified later):

\begin{equation}
\text{vec}(\beta_t - \bar{\beta}) = F \cdot \text{vec}(\beta_{t-1} - \bar{\beta}) + v_t,
\end{equation}

where $F$ is a diagonal matrix with $\rho \leq 1$ on the main diagonal. Errors $v_t$ are assumed to be i.i.d. with variance-covariance matrix $V$.

We can write the forecasting model in the following SURE format:

\begin{equation}
\begin{bmatrix}
y_{1t}^f \\
y_{2t}^f \\
\vdots \\
y_{mt}^f
\end{bmatrix}
= \begin{bmatrix}
X_{1,t-1} & 0 & \ldots & 0 \\
0 & X_{2,t-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & X_{m,t-1}
\end{bmatrix}
\begin{bmatrix}
\beta_{1,t-1} \\
\beta_{2,t-1} \\
\vdots \\
\beta_{m,t-1}
\end{bmatrix}
+ \begin{bmatrix}
u_{1,t} \\
u_{2,t} \\
\vdots \\
u_{m,t}
\end{bmatrix},
\end{equation}

The errors $u_{j,t}$ depend on a linear combination of the true model innovations $\epsilon_t$ and therefore

exogenous to their decision–making) infinitely many periods ahead.

For a theoretical discussion on these two approaches to adaptive learning, see Preston (2005) and Honkapohja et al. (2002). For a discussion of effects of the learning type on the behavior of estimated DSGE models, see Milani (2006) and references therein.

5In the adaptive learning literature, this equation is called the Perceived Law of Motion (PLM).

6Sargent and Williams (2005) showed that even if Kalman filter and constant gain learning are asymptotically equivalent on average, their transitory behavior may differ a lot. In particular, Kalman filter tends to result in much faster adjustment of agents’ beliefs. With faster adjustment of beliefs, we are able to better understand whether the initial beliefs or time–varying coefficients matter more for the improved model fit.

7The SURE format and the corresponding GLS estimator are necessary to get an efficient estimator of the complete forecasting model because the variables appearing on the RHS in each equation are not identical.
they are likely to be correlated, making the variance-covariance matrix non-diagonal:

\[(5) \quad \Sigma = E \left[ u_t \cdot u_t^T \right].\]

With the above notation, the Kalman filter updating and transition equations for the belief coefficients and the corresponding covariance matrix are given by

\[(6a) \quad \beta_{it} = \beta_{it-1} + P_{it-1} X_{it-1} \left[ \Sigma + X_{it-1}^T P_{it-1} X_{it-1} \right]^{-1} \times \left( y_{it} - X_{it-1} \beta_{it-1} \right),\]

\[\text{with } (\beta_{i+1|t} - \beta) = F \cdot (\beta_{i|t} - \beta).\]

\[(6b) \quad P_{it} = P_{it-1} - P_{it-1} X_{it-1} \left[ \Sigma + X_{it-1}^T P_{it-1} X_{it-1} \right]^{-1} \times X_{it-1}^T P_{it-1},\]

\[\text{with } P_{i+1|t} = F \cdot P_{i|t} \cdot F^T + V.\]

These best estimates for the beliefs \((\beta_{it-1})\) are then substituted for \(\beta_t\) in (4) to generate expectations of forward-looking variables, \(E_t y_{t+1}^f\). Plugging these expectations into (1), we obtain a purely backward-looking representation of the model\(^8\):

\[(7) \quad \begin{bmatrix} y_t \\ w_t \end{bmatrix} = \mu_t + T_t \begin{bmatrix} y_{t-1} \\ w_{t-1} \end{bmatrix} + R_t \epsilon_t.\]

The resultant time-dependent matrices \(\mu_t\), \(T_t\), and \(R_t\) replace the constant equivalents in the RE solution (2). These matrices depend now on both the parameters of the decision problem (\(\Theta\)) and on the best estimates of the forecasting model (\(\beta_{it-1}\)), and contain all necessary information to describe the dynamics and the propagation of the shocks in the model under learning.

In order to initialize this Kalman filter for the belief coefficients, we need to specify \(\beta_{i0} = \beta\), \(P_{i0}\), \(\Sigma\), and \(V\). In our baseline approach, all these expressions are derived from the correlations between the model variables implied by the RE Equilibrium evaluated for the corresponding structural parameter vector \(\Theta\). In other words, the initial beliefs are assumed to be model consistent.

Using the fact that \(\widehat{\beta}_{OLS} = (X^T X)^{-1} X^T y\) is unbiased, we use the theoretical moment matrices \(E \left[ X^T X \right]\) and \(E \left[ X^T y \right]\) from the RE solution and set

\[(8) \quad \beta_{i0} = (E \left[ X^T X \right])^{-1} \cdot E \left[ X^T y \right].\]

\(^8\text{Note that we expand the state vector } y \text{ in this representation with additional lags that occur in the forecasting models.}\)
Given $\beta_{1|0}$, we calculate $\Sigma$ as

$$
\Sigma = E \left[ \left( y_t^j - X_{t-1}^T \beta_{1|0} \right) \left( y_t^j - X_{t-1}^T \beta_{1|0} \right)^T \right],
$$

again using the RE theoretical moments. Finally, $P_{1|0}$, the initial guess about the mean square forecast error of the belief coefficients, and $V$, the variance–covariance matrix of shocks $v_t$ to these coefficients, are both taken to be proportional to $(X^T \Sigma^{-1} X)^{-1}$:

$$
P_{1|0} = \sigma_0 \cdot \left( X^T \Sigma^{-1} X \right)^{-1},
$$

$$
V = \sigma_v \cdot \left( X^T \Sigma^{-1} X \right)^{-1}.
$$

This initialization leaves just three parameters, $\sigma_0$, $\sigma_v$, and $\rho$, to fully describe the learning dynamics.

II. Estimation Results

In this section, we document the estimation approach and we present the estimation results in terms of the posterior distribution of the estimated parameters and the marginal likelihood of the model. We compare the results under RE and under our baseline learning model that assumes a simple extrapolative AR(2) forecasting model.

A. Estimation approach

As in SW 2007, the model is estimated using seven key macroeconomic US time series as observable variables: the log difference of real GDP, real consumption, real investment and the real wage, log hours worked, the log difference of the GDP deflator and the federal funds rate.

$^9 (X^T \Sigma^{-1} X)^{-1}$ is equal to $\nu \omega [\hat{\beta}_{GLS}]$ where $\hat{\beta}_{GLS} = \left( X^T \Sigma^{-1} X \right)^{-1} X^T \Sigma^{-1} y$, which gives an efficient estimator for the SURE model. Given knowledge of theoretical moments and of $\Sigma$, the matrix $\left( X^T \Sigma^{-1} X \right)^{-1}$ could be readily calculated.
The corresponding measurement equation is:

\[
O_t = \begin{bmatrix}
    d\ln GDP_t \\
d\ln Cons_t \\
d\ln INV_t \\
d\ln Wages_t \\
\ln Hours_t \\
d\ln P_t \\
FEDFUNDS_t
\end{bmatrix} = \begin{bmatrix}
    \bar{y} \\
    \bar{y} \\
    \bar{y} \\
    \bar{y} \\
    \bar{\pi} \\
    \bar{\pi} \\
\bar{\pi}
\end{bmatrix} + \begin{bmatrix}
    \bar{y}_t - \bar{y}_{t-1} \\
    \bar{c}_t - \bar{c}_{t-1} \\
    \bar{\iota}_t - \bar{\iota}_{t-1} \\
    \bar{\omega}_t - \bar{\omega}_{t-1} \\
    \bar{\pi}_t \\
    \bar{\pi}_t \\
\bar{\pi}_t
\end{bmatrix},
\]

where \(\bar{y}\) and \(d\bar{y}\) stand for log and log difference respectively, \(\bar{y} = 100(\gamma - 1)\) is the common quarterly trend growth rate to real GDP, consumption, investment and wages, \(\bar{\pi} = 100(\Pi - 1)\) is the quarterly steady-state inflation rate and \(\bar{\pi} = 100(\gamma \cdot \Pi_s / \beta - 1)\) is the steady-state nominal interest rate. Given the estimates of the trend growth rate and the steady-state inflation rate, the latter will be determined by the estimated discount rate. Finally, \(\bar{\iota}\) is steady-state hours worked.

The model is estimated over the sample period from 1966:1 until 2008:4.

The Bayesian estimation method proceeds in two steps. First, we estimate the mode of the posterior distribution by maximizing the log posterior function, which combines the prior information on the parameters with the likelihood of the data. In a second step, the Metropolis-Hastings algorithm is used to get a complete picture of the posterior distribution and to evaluate the marginal likelihood of the model. The prior assumptions on the structural parameters are exactly the same as in SW 2007. For the learning parameters, it turned out that the three parameters are not simultaneously identified by the estimation procedure. In the baseline version, we estimate \(\rho\) using a uniform prior over the interval \([0,1]\), and we fix the scale of the initial variance-covariance matrix of the belief coefficients to some large number \((\sigma_0 = 0.03)\) relative to the variance–covariance of the shocks to the beliefs \((\sigma_v = 0.003)\).

The likelihood of the model is evaluated with a Kalman filter that uses equation (7) as the state equation and (12) as the measurement equation. At each moment in time, the beliefs of the agents on their forecasting model are updated using (6), which delivers the necessary input to solve for next period \(\mu_t, T_t\) and \(R_t\). The forecasting model is evaluated and updated in terms of the filtered states of the overall model\(^{10}\).

\(^{10}\)Note that not all forward variables are observed, while others are observed in first differences but appear in the forecasting models in levels, or more precisely, in log deviations from the steady state growth path.
B. Posterior estimates

Table (1) summarizes the marginal likelihood and the posterior estimates for a selection of the parameters under RE and under AL with small forecasting models. The first observation is the noticeable improvement in the marginal likelihood for the model under AL (-982.23) versus RE (-1005.22), which implies that the posterior odds ratio is definitely in favor of the learning approach.

**Table 1—Posterior Estimates for RE and AL Models**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>RE</th>
<th>AL</th>
<th>AL (Baseline)</th>
<th>AL no updating</th>
</tr>
</thead>
<tbody>
<tr>
<td>wage markup AR(1)</td>
<td>0.96</td>
<td>0.53</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[0.93-0.99]</td>
<td>[0.28-0.79]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>wage markup MA(1)</td>
<td>0.88</td>
<td>0.43</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[0.81-0.95]</td>
<td>[0.13-0.73]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>price markup AR(1)</td>
<td>0.85</td>
<td>0.28</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[0.75-0.94]</td>
<td>[0.06-0.49]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>price markup MA(1)</td>
<td>0.70</td>
<td>0.48</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[0.55-0.86]</td>
<td>[0.30-0.66]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>wage indexation</td>
<td>0.51</td>
<td>0.18</td>
<td>0.21</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>[0.30-0.71]</td>
<td>[0.07-0.29]</td>
<td>[0.09-0.32]</td>
<td>[0.19-0.54]</td>
</tr>
<tr>
<td>price indexation</td>
<td>0.24</td>
<td>0.29</td>
<td>0.19</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>[0.10-0.38]</td>
<td>[0.11-0.45]</td>
<td>[0.08-0.29]</td>
<td>[0.15-0.53]</td>
</tr>
<tr>
<td>wage stickiness</td>
<td>0.77</td>
<td>0.83</td>
<td>0.84</td>
<td>0.84</td>
</tr>
<tr>
<td></td>
<td>[0.68-0.85]</td>
<td>[0.78-0.88]</td>
<td>[0.80-0.88]</td>
<td>[0.79-0.89]</td>
</tr>
<tr>
<td>price stickiness</td>
<td>0.72</td>
<td>0.66</td>
<td>0.65</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td>[0.65-0.80]</td>
<td>[0.61-0.72]</td>
<td>[0.59-0.71]</td>
<td>[0.61-0.75]</td>
</tr>
<tr>
<td>marg.lik.</td>
<td>-1005.22</td>
<td>-982.23</td>
<td>-983.86</td>
<td>-999.76</td>
</tr>
</tbody>
</table>

Secondly, there are some important changes in the estimated stochastic structure depending on the retained assumption about the expectations. Especially the process of the price and wage markup shocks changes quite dramatically. While these shocks were estimated as an ARMA(1,1) process with a very persistent component under RE, these shocks follow basically an i.i.d. process under learning: with both the autoregressive and the moving average coefficient close to the mean of the prior distribution centered around 0.5, the implied dynamics for these processes are equivalent to a white noise process. This interpretation is confirmed by re-estimating a version of the learning model in which the two markup shocks are explicitly defined as i.i.d. processes. The marginal likelihood of the model hardly changes (-983.86). It appears that the propagation of the markup shocks under learning is completely captured by the expectations mechanism and by the internal dynamics of the decision rules, while it was dependent on the persistence in the
exogenous dynamics of the markup process under RE. Some of the other exogenous disturbances are also influenced by the learning dynamics, but these changes go in various directions: the persistence in the TFP and risk premium increases, while it decreases for the investment shock. The learning model with *i.i.d.* price and wage markup shocks is considered as the baseline AL model in the rest of the paper.

Thirdly, the structural parameters that govern the decision rules of the agents remain relatively robust under alternative hypotheses on the expectations. The estimated mode of several parameters changes, but the posterior intervals under RE and under AL do overlap considerably in most cases. The most significant change is observed for the wage indexation parameter which drops from 0.51 under RE to 0.21 under learning. Note that the indexation for price-setting is estimated consistently low at a level of 0.20 in both cases. This result points in the same direction as the change in the markup shocks, namely that the learning dynamics provide a strong propagation mechanism for the markup shocks so that alternative frictions such as the indexation mechanism are less crucial to explain the observed persistence in the inflation process. The estimated price and wage stickiness, however, are not significantly different across models.

These results can be considered as an interesting achievement because empirical DSGE models are often criticized for depending too much on the dynamics incorporated in the exogenous disturbances, which leaves only a minor role for the endogenous propagation mechanism. Learning dynamics seem to substitute for the exogenous persistence but leave the structural parameters of the model unaffected. It is also important to note that the identified historical innovations to the exogenous processes are very similar under RE and under AL: the correlation between the estimates of these *i.i.d.* innovation series varies from 0.83 for the price markup shock to 0.98 for the productivity shock.

Finally, note that the estimated persistence ($\rho$) in the belief coefficients is estimated to be quite high (0.96–0.97). In the sensitivity analysis, we will consider alternative assumptions about the parameters driving the learning dynamics.

As mentioned before, our learning model deviates from RE by considering a smaller information set and by allowing for time-variation in the beliefs based on the latest observations. To identify which of these two deviations is more important for explaining the improved fit of the model, we consider a model in which agents use the same small forecasting model but with constant belief coefficients fixed at the RE–implied initial coefficients ($\beta_{1(0)}$). This model produces a marginal likelihood of -999.76, which is still an improvement on the RE-model: the model with constant expectations based on a limited information set improves the model fit compared to the fully model consistent expectations case. But updating of the expectations through the KF-
learning process is more important for improving the fit: learning explains roughly three-quarters of the overall improvement in the marginal likelihood.

C. Improved forecasting performance of inflation dynamics

The good fit of the AL model is also reflected in the out-of-sample prediction performance. Table (2) compares the root mean squared forecast error at different forecast horizons under RE and under the baseline AL model. The overall forecast performance of the AL model is better at a one-quarter forecast horizon, which is consistent with the higher marginal likelihood. The biggest gain is realized in the forecast of inflation which improves by 9% relative to the RE model. Given that the learning dynamics are most active for the inflation expectations, this is a promising result for the AL approach. On the other hand, the forecasts for consumption and real wages are slightly worse under AL even for one-period-ahead forecasts. The forecast performance of the AL model generally deteriorates at longer horizons for all variables but wages. This result applies independently of the exact procedure that is used to construct the long term forecasts\textsuperscript{11}. Longer-horizon forecasts for consumption and the interest rate deteriorate the most. For these variables, the restrictions imposed by the RE assumption appear to be very useful in longer-horizon forecasts. This suggests that there is a trade-off between the flexibility of the learning dynamics in capturing the short-run behavior on the one hand, and the RE restrictions that are useful for the long-run forecasts on the other hand. In this context, it would be very interesting to test whether Preston’s infinite-horizon learning approach (2005) can resolve this trade-off more efficiently by using the restrictions from the budget constraints in the forecasts.

III. Expectations implied by the learning model

In this section, we discuss in detail the behavior of the expectations implied by the time-varying small forecasting or PLM model. First, the time variation in the belief coefficients for the various forward variables is illustrated graphically. This time variation or updating of the coefficients is driven by the forecasting errors and has a simple intuitive interpretation. In particular, we illustrate how the updating of the inflation belief coefficients responds to the innovations in the exogenous disturbances. Secondly, we evaluate the quality of the forecasts implied by these small forecasting models by comparing the RMSE with alternative forecasting models, and by

\textsuperscript{11}The results reported in Table 2 are produced under the hypothesis that the ALM process, exemplified by (7), remains constant over the forecast horizon. Minor improvements are obtained if the ALM is updated over the forecast horizon in line with the perceived mean reversion in the belief coefficients as captured by the estimated $\rho$ parameter.
TABLE 2—OUT-OF-SAMPLE PREDICTION PERFORMANCE OF RE AND AL MODELS

<table>
<thead>
<tr>
<th>RMSE of the RE-model at different forecast horizons</th>
<th>GDP</th>
<th>π</th>
<th>R</th>
<th>Hours</th>
<th>Wage</th>
<th>Cons.</th>
<th>Inv.</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1q</td>
<td>0.65</td>
<td>0.27</td>
<td>0.13</td>
<td>0.52</td>
<td>0.63</td>
<td>0.60</td>
<td>1.66</td>
<td>-12.13</td>
</tr>
<tr>
<td>2q</td>
<td>1.01</td>
<td>0.27</td>
<td>0.23</td>
<td>0.85</td>
<td>0.99</td>
<td>0.88</td>
<td>2.85</td>
<td>-7.15</td>
</tr>
<tr>
<td>4q</td>
<td>1.63</td>
<td>0.23</td>
<td>0.40</td>
<td>1.35</td>
<td>1.51</td>
<td>1.15</td>
<td>5.19</td>
<td>-3.06</td>
</tr>
<tr>
<td>8q</td>
<td>2.62</td>
<td>0.23</td>
<td>0.54</td>
<td>2.00</td>
<td>2.63</td>
<td>1.73</td>
<td>8.53</td>
<td>1.13</td>
</tr>
<tr>
<td>12q</td>
<td>3.11</td>
<td>0.23</td>
<td>0.56</td>
<td>2.26</td>
<td>3.73</td>
<td>2.06</td>
<td>9.49</td>
<td>2.21</td>
</tr>
</tbody>
</table>

Percentage gains (+) or losses (-) of AL relative to RE model

| 1q | 5.08 | 8.93 | 3.96 | 2.65 | -2.13 | -2.60 | 6.20 | 1.56   |
| 2q | 2.08 | 1.79 | -3.24| -3.51| -2.19 | -11.44| 5.68 | -1.17  |
| 4q | 3.95 | -14.62| -8.09| -4.17| -19.08| 4.64  | -3.42|       |
| 8q | 6.83 | -4.38| -23.45| -8.46| -25.00| 0.27  | -6.45|       |
| 12q| 0.25 | -5.14| -38.53| -12.03| 4.95  | -42.22| -8.09| -10.9  |

The forecast period is 1990:1-2008:4. Models are re-estimated each year, starting in 1966:1 and ending in the fourth quarter of the year before the forecast starts. The overall measure of forecast performance is the log determinant of the uncentered forecast error covariance matrix. Gains and losses in the overall measure are expressed as the difference in the overall measure divided by the number of variables and divided by two to convert the variance to standard errors (times 100).

Comparison with empirical evidence on these expectations as measured by the Survey of Professional Forecasters. This evidence confirms that the small forecasting models provide a plausible description of the way expectations are formed by agents in the real world.

A. Time variation in the beliefs

KF learning leads to important time variation in the coefficients of the forecasting model. Figure (1) illustrates the time variation in the coefficients of the AR(2) forecasting model for four of the seven forward variables in the model. To facilitate the interpretation of the time variation, we can rewrite the AR(2) forecasting equation in the following format:

\[ y_{j,t}^f = \mu_{j,t} + (\beta_{j,t}^1 + \beta_{j,t}^2) * y_{j,t-1}^f - \beta_{j,t}^2 * \Delta y_{j,t-1}^f. \]

Figure (1) provides the evidence on the constant (\(\mu_{j,t}\)), the persistence in the expectations (\(\beta_{j,t}^1 + \beta_{j,t}^2\)), and (\(-\beta_{j,t}^2\)) which we refer to as the persistence in the growth rate of the expectations.

First, it is clear that the constants vary a lot for all four forward variables. Under RE, these constants would be zero, reflecting the fact that all real variables are modelled as deviations from a constant deterministic growth rate and that inflation and the nominal rate fluctuate around a constant inflation objective and a corresponding nominal rate as determined by the monetary policy reaction function. By allowing for a time-varying constant in the belief models, these
restrictions are relaxed. The perceived trend growth rate and inflation objective can vary now over time and across variables as a function of the observed trends in the recent history. The constants in the real variables fluctuate over the cycle reflecting the past growth rates observed in each of the individual variables. Clearly, the constant for the expected investment rate is the most cyclical, while the constants for consumption and real wages more closely reflect the long-term growth rates in these variables which deviate quite persistently from the imposed common productivity growth rate in the model. For inflation, the constant also reflects the trend in the past observed inflation rate. The constant term in the inflation beliefs rose during the seventies and started to decline only slowly after the disinflation of the early eighties. The coefficient has stabilized around zero since the mid nineties, meaning that the expected mean inflation of the private agents has converged to the constant inflation objective of the central bank since then. Note that the perceived long-run inflation rate that is implied by these belief equations does not only depend on the constant term but also on the perceived persistence.

Second, the perceived persistence, as measured by the sum of the AR(2) coefficients, is stable
and close to one for the real variables. These coefficients suggest that the true data-generating processes for these expectations are close to an AR(1) in first differences. The more interesting coefficient is therefore the persistence in the growth rate (measured here by $-\beta^2_{j,t}$) which is clearly positive for the real variables, and in the case of consumption it was slightly higher during the seventies but declining later on.

The most important updating dynamics seem to be taking place in the perceived inflation persistence. This process followed a clear upward trend in the 1960s and 1970s with a peak around the mid '70s and again around 1980, followed by a quick decline towards a level of 0.6 since the mid eighties, and a further downward shift in the most recent period$^{12}$. Movements in the inflation rate were perceived as much more persistent in the seventies then they were during the sixties or the more recent period. These updating dynamics for the perceived inflation persistence correspond with the statistical properties of the observed inflation process over this period. For instance, Cogley, Primiceri and Sargent (2007) obtain a very similar pattern for the persistence in the inflation gap. In the following sections, we discuss in detail how this perceived persistence affects the impulse responses of various shocks and how they can be helpful to understand the great inflation in the seventies and the moderation afterwards. These estimates confirm the general observation that monetary policy and the inflation target of the central bank have become much more credible over the last decades.

This time variation in the beliefs of private agents has a relatively simple and intuitive interpretation. The updating expression in (6) states that the updating in the beliefs is determined by the forecast errors multiplied by a Kalman gain matrix. These expressions are very general and complex because the forecasting model takes the form of a SURE model that treats all forward variables jointly, and because the Kalman gain matrix itself depends on updates of the second-moment matrices. The outcome of this mechanism nevertheless has a straightforward interpretation. For the constant terms, this simply means that higher (lower) than expected realizations of a forward variable results in upward (downward) revisions in the constant of the forecasting equation. The updating of the persistence, which is especially relevant for the inflation beliefs, follows a slightly more complicated logic because of its state-dependent nature. When inflation is high relative to its long-run mean value, agents will expect inflation to decline in the future, with the speed of convergence depending on the perceived persistence. With a new realization of inflation that is higher than expected, agents will tend to revise their perceived persistence upwards in

$^{12}$The extremely high perceived inflation persistence in the mid- and late seventies also explains why the updating in the beliefs during these years sometimes leads to explosive AR(2) beliefs. As is standard in the learning literature, the projection facility in our estimation process eliminates updates in the beliefs that would result in unstable dynamics for the model’s ALM.
order to avoid underestimation in the future. With a new realization that is lower than expected, the perceived persistence will be adjusted downwards. So, in periods when the inflation rate is high, revisions in persistence are positively correlated with the inflation innovations. This interpretation explains why the perceived inflation persistence was rising in the seventies and quickly declining in the early eighties. But the opposite relation applies when inflation is at a relatively low level: when a positive inflation innovation is realized in a low inflation state, the perceived inflation persistence will decline as inflation seems to rise towards its long-run target faster than previously expected. This negative relationship between inflation realizations and revisions in the persistence is relevant in the more recent periods with low inflation rates.

To further illustrate this updating process, we show in Figure (2) the reaction of the perceived inflation persistence to various structural innovations. The figure shows how the perceived inflation persistence reacts to four types of shocks (one standard error innovation to productivity, monetary policy, price markup and wage markup shock) during four different moments in our sample (68q3, 79q3, 93q1 and 01q4). For each observation, we plot the IRF of the perceived inflation persistence during five years following the shock. Two types of IRFs are calculated: the dotted line represents the updates in the belief coefficients that are implied by the new realizations of the forward variables, but disregarding the feedback effect of these belief updates on the actual law of motion of the economy, while the full line takes into account this feedback effects of the beliefs on $\mu_t$, $T_t$ and $R_t$. Several observations can be noted. First, the sign of the belief updates changes over time in line with the above discussion. Second, beliefs are very sensitive to innovations in the shocks especially in the first part of our sample, but this sensitivity decreases during the more recent period when actual inflation is more in line with long term expectations. Third, perceived inflation persistence is mainly driven by innovations to the price markup, which generates mainly short-run volatility in the beliefs, and by innovations to the wage markup and productivity process which have a more gradual but also more persistent effect on the beliefs. Finally, monetary policy shocks have almost no direct impact on the perceived inflation persistence, represented by the dotted line, but they had some indirect effects through the feedback effects at least during the first sub-period. Monetary policy affects inflation only through the marginal costs and aggregate economic activity, and expectations and beliefs underlying these variables contribute to the overall reaction of inflation and inflation expectations.

A complete decomposition of the belief dynamics in terms of historical shocks is not possible because of the highly non–linear nature of the updating process. Nevertheless, the impulse response exercise, as presented in Figure (2), suggests that the upward trend in perceived inflation in the 1970s was mostly driven by positive price and wage markup and negative productivity
shocks. The series of accommodating monetary policy shocks during the great inflation period also contributed significantly to the destabilization of inflation beliefs, but it is not possible to quantify this contribution exactly. The role of the restrictive monetary policy innovations on the inflation beliefs during the Volker disinflation process is even harder to assess. Our learning model implies a relatively quick decline in the perceived inflation persistence during this period, following the sharp decline in actual inflation realizations. Monetary policy shocks alone are not able to generate such a quick response in our updating mechanism, which suggests that other shocks, and price markup shocks in particular, were active during this period as well.

B. More evidence on the expectations model

In order to further justify the choice of the small forecasting models as an approximation for the expectations of the private sector, it is useful to analyze in detail the quality and properties of the implied forecasts. Two types of evidence are presented in this section. First, we evaluate the forecasts performance of the small PLM model relative to RE model and to the actual law of
motion (ALM) of the learning model. It would be difficult to maintain the hypothesis that our small forecasting model provides a reasonable approximation for actual historical expectations if the forecasting performance of this model is much worse compared to alternative forecasting models. Secondly, we compare the forecasts implied by the small PLM models and the RE model with survey evidence on actual historical forecasts of economic agents. In principle, this type of evidence enables the historical relevance of alternative expectation models to be tested directly.

Table (3) summarizes the RMSE statistics from the PLM forecasts for the forward variables. We report results for the five forward variables, out of seven, that are also included in the list of observed variables. To assess this performance, similar statistics are provided for the model under RE, in which expectations are model-consistent and the PLM and the ALM yield the same forecast. We also report the RMSE for the ALM under learning, and for the PLM of the AR(2) model without updating: the same model that was considered in Table 1, with the AR(2) coefficients fixed at the initial beliefs. The statistics are all based on in-sample forecasts, in contrast to Table (2) where out-of-sample forecasts were considered, because here we want to evaluate the historical forecasts of the PLM model that are implicit in the learning model when estimated over the full sample. Only one-quarter-ahead forecasts are considered because these are the relevant ones in the Euler equation learning approach.

<table>
<thead>
<tr>
<th>Table 3—RMSE Comparison of the PLM Forecasts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>PLM</td>
</tr>
<tr>
<td>RE</td>
</tr>
<tr>
<td>ALM</td>
</tr>
<tr>
<td>PLM-no update</td>
</tr>
</tbody>
</table>

The first figure stands for the RMSE over the complete estimation sample 66q1-08q4, the second figure applies for the period 90q1-08q4, which is the same period considered in the out-of-sample forecasts in Table (2).

The RE and ALM in-sample forecast errors are minimized in the estimation procedure, and therefore they provide a minimum bound against which the PLM performance can be evaluated. Overall, the AR(2)-based PLM does seem to perform reasonably well compared to these optimized predictors. The maximum loss for the small model forecasts relative to RE model is realized for consumption, with a deterioration of 15% in the RMSE statistic. For inflation, the small-model forecasts are only slightly worse than the benchmark. Note also that the PLM forecasts perform almost as well as the RE forecasts for the more recent sample. Updating of the belief coefficients in the AR(2) forecasting model has only a minor impact on the actual forecast performance.
A reasonable performance of the expectation models in terms of RMSE is preferable for obtaining a good overall fit of the model, but provides only indirect evidence on the empirical validity of these expectations. To test this last objective, it is more useful to compare the forecast errors in our expectation models with direct evidence on these forecast errors as collected in empirical surveys. The Survey of Professional Forecasters (SPF) contains information about private sector expectations on future outcomes for inflation (GDP deflator), consumption and investment, which are three variables that are also present in our list of forward variables. This survey evidence is therefore directly relevant to evaluate our expectation model.

A problem that complicates any direct comparison of the forecasts is related to the fact that the surveys are based on real-time data, while our PLM forecasts are based on *ex-post* revised data. To overcome this problem, we concentrate on the forecast errors rather than on the forecast levels, and we focus on systematic bias during prolonged periods rather than on the period-by-period forecast performance. To construct the forecast errors, we use the survey forecast at time $t$ for changes in the variable between $t + 1$ and $t$, and calculate the prediction error as the difference between these expected changes and changes realized in the next quarter (based on the first data release as reported in the SPF database). Table (4) compares the correlation between these forecast errors in the surveys and the forecast errors realized by our PLM and RE model. Clearly, the correlation of the PLM forecast errors with the survey errors is higher than for the RE forecast errors. The difference in correlation is most pronounced for inflation and especially during the 1990s.13

| Table 4—Correlation of Forecast Errors with SPF Data |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                                | Inflation (PLM) | Inflation (RE)  | Consumption (PLM) | Consumption (RE) | Investment (PLM) | Investment (RE) |
| Complete sample                | 0.53            | 0.48            | 0.56            | 0.52            | 0.77            | 0.71            |
| 69q1-80q4                      | 0.47            | 0.54            | -               | -               | -               | -               |
| 81q4-90q4                      | 0.44            | 0.34            | 0.66            | 0.59            | 0.71            | 0.62            |
| 91q1-00q4                      | 0.33            | 0.04            | 0.38            | 0.41            | 0.79            | 0.75            |
| 01q4-08q4                      | 0.71            | 0.61            | 0.56            | 0.52            | 0.81            | 0.76            |

13 There is a number of other papers which considered survey expectations to be an outcome of some adaptive learning process. In Orphanides and Williams (2005b) agents use a three–variable VAR with constant gain Recursive Least Squares to form expectations of inflation, the unemployment rate, and the federal funds rate. They fit the model to the SPF expectations and conclude that geometrically discounting past data with a rate of one to two percent per quarter produces forecasts close to those found in SPF. Branch and Evans (2006) consider both constant gain and Kalman filter adaptive learning and find that constant gain learning with gain 0.0345 fits the SPF expectations on GDP growth and GDP deflator inflation the best. Nunes (2009) considers inflation expectations to be a weighted average of rational expectations and SPF forecasts that could be represented as an outcome of a simple constant gain learning process.
To further illustrate the relevance of this difference, Figure (3) plots the cumulative forecast errors for inflation observed in the SPF data and in the PLM and RE model. All three forecasts display similar systematic bias over different sub-periods in our sample: forecasters were surprised by the general increase in inflation in the 1970s and the disinflation in the early 1980s resulting in cumulated negative forecast errors first and a reversion in the trend later on. However, it appears very clearly from this picture that expectations in the survey and in the small PLM models tend to overestimate inflation systematically during the 1990s, whereas the RE model underestimates inflation systematically during this period. This evidence suggests that the expectations in the PLM are more in line with the survey evidence during the 1990s, which is exactly the period during which the learning model outperforms the RE model in predicting inflation (see Table (2)) and in terms of overall likelihood. We consider these results as favorable evidence for the superiority of the small model PLM hypothesis relative to the RE-based expectations. This result also suggests that the overall improvement in the marginal likelihood of the learning model is based on a more realistic modelling of private sector expectations.

![Figure 3. Cumulative Forecast Errors for Inflation](image-url)
IV. Macrodynamics implied by the learning process

In this section, we discuss the implications of the time-varying beliefs for the overall dynamics of the model. All results presented in this section take the time-varying PLM ($\beta_t$) and ALM processes ($\mu_t$, $\tau_t$ and $R_t$) as given and their consequences for various properties of the overall model are assessed.

A. Time variation in the impulse response functions

The mechanism for the transmission of structural shocks depends crucially on the way private agents form their expectations. Therefore, it is interesting to illustrate how the IR functions depend and vary over time depending on the belief coefficients. In Figure (4), we plot the time-varying IRF for the productivity shock, the risk premium shock, the monetary policy shock and the wage markup shock for the baseline learning model with beliefs based on the AR(2) model. Only the effects on output and inflation are shown. The IRFs reported here are calculated for fixed belief coefficients (and corresponding ALM matrices) at each point in time and disregard the updating of these beliefs that might be caused by the shock, as discussed in the section 4.1. In doing so, these pseudo-IRFs might underestimate the persistence and the magnitude of the actual responses. We add the corresponding IRF in the RE model at the end of the sample for comparison.

For all shocks, the reaction of inflation depends crucially on the perceived persistence of inflation by the private agents. Inflation reacted much more strongly and persistently to the shocks in the 1970s when inflation was perceived as very persistent. During the periods when perceived inflation persistence was more moderate, the reaction of inflation was also much smaller and more gradual. This profile of the inflation response under learning with small PLM models contrasts sharply with the typical response under RE. The impact of the shocks on output displays less time variation. This is especially the case for the impact effects, while the transmission of the shocks in the subsequent quarters tends to vary somewhat more with larger and more persistent effects in the 1970s as well. The profile of the output reaction is more in line with the reaction under RE for most shocks but not so for the markup shocks which have a much smaller impact on output under learning.

B. Time variation in the volatility

Given the time variation in the way agents formed expectations in our PLM model and the effect of this on the transmission mechanism of the different shocks, it is interesting to evaluate
FIGURE 4. IRF FUNCTIONS IN THE AL AND RE MODEL
how this contributed to the overall volatility in the economy. The results are most outspoken for inflation. The model produces both a higher mean inflation and a higher inflation volatility in the seventies than in the period since 1984, see Table (5)\textsuperscript{14}. The outcome for the mean inflation is mainly related to the time varying constant in the belief equations. The higher volatility is explained by the higher perceived inflation persistence and the stronger and more persistent reaction of inflation to all the shocks in the seventies. Averaging over the sub–periods before and after 1984, the model explains a drop in inflation volatility from 0.55 to 0.32, a 42% drop in volatility ($\frac{0.55-0.32}{0.55} \times 100\%$) while in the historical data the observed decline is 58%. Thus, the model explains 0.67 of the historical moderation in inflation volatility. These results clearly illustrate the crucial role of inflation expectations to explain the great inflation experience of the 1970s. The series of upward inflation shocks that arose in the mid-seventies led to an upward revision in the mean expected inflation rate by private agents and at the same time they also revised their perceived inflation persistence which reinforced the impact of the unfavorable shocks on inflation even further. This revision in the inflation expectations of the private sector happened independently of the systematic monetary policy behavior, as the policy rule in our model is assumed to be constant over the complete estimation period. At the beginning of the 1980s, a combination of restrictive monetary policy and negative inflation shocks caused agents to revise downward their expectations about future mean inflation and the perceived inflation persistence, so that inflation gradually converged towards the inflation objective of the central bank. The crucial mechanism in this explanation of the great inflation is the interaction between the way inflation expectations are formed and the specific series of historical shocks that appear over time. This interpretation suggests that monetary policy-makers should continuously be careful about inflation expectations and how these anticipations react to positive inflation shocks.

For output, the model is able to replicate the increase in the average growth rate over the two sub–periods, but it does only explain a small fraction of the great moderation in the volatility of the real variables. The baseline AL model explains on average 18% of the decline in volatility in the growth rate of output, or 15% in terms of HP–filtered output gap volatility. Most of the real moderation generated by our model seems to be related to consumption behavior.

Note that the data reported in Table (5) are averages. The exercise also makes it possible to calculate the probability of the observed decline according to our posterior distribution. For inflation, we find that the volatility declines by 58% or more in 10% of the simulations, while

\textsuperscript{14}To produce these numbers, 500 draws from the MCMC were randomly selected. At every parameter draw, the time–varying $\mu$, $T$, and $R$ implied by the changing beliefs, were saved. Then this time–varying ALM model was simulated 500 times to produce 500 hypothetical alternative histories for our sample period. Before– and after–84 means and standard deviations were then averaged over all histories, and then over all parameter draws.
less than 1% of simulations reproduce an observed decline of 47% for output growth volatility. Despite the relatively large uncertainty surrounding these statistics, the baseline model is hard to reconcile with the real moderation. For a successful explanation of the output moderation by the learning approach, it is necessary that the model captures a sufficiently strong decline in the perceived persistence in the growth of the real variables. The baseline model does take up some decline for consumption growth persistence, but this effect is not strong enough. Some of the alternative specifications that are discussed in the sensitivity analysis are more successful on this dimension. This suggests that the learning approach is potentially able to explain a larger fraction of the real moderation with an appropriate selection of the belief processes.

<table>
<thead>
<tr>
<th>TABLE 5—MEAN AND VOLATILITY OF INFLATION AND OUTPUT OVER SUB-SAMPLES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DATA</strong></td>
</tr>
<tr>
<td>Before 1984</td>
</tr>
<tr>
<td>Inflation</td>
</tr>
<tr>
<td>mean</td>
</tr>
<tr>
<td>st.dev.</td>
</tr>
<tr>
<td>Output growth</td>
</tr>
<tr>
<td>mean</td>
</tr>
<tr>
<td>st.dev.</td>
</tr>
<tr>
<td>HP-output</td>
</tr>
<tr>
<td>st.dev.</td>
</tr>
</tbody>
</table>

C. **Time variation in the inflation-output relation**

The time variation in the learning model has also interesting implications for the relation between inflation and output. In the literature, the flattening of the Phillips curve has been observed in various contexts, cf. Atkeson (2001), Stock and Watson (2006), Borio and Filardo (2007), Kuttner and Robinson (2008), and others. Using the time-varying representation of the learning model, we can calculate for each point in time the implied coefficients for a projection of current inflation on the current measure of economic conditions (marginal cost or output gap) and past inflation. The resulting coefficients are given in Figure (5). The first panel shows that the coefficient of the real marginal cost in a simple backward-looking Phillips curve equation has indeed declined in our learning model in line with the evidence on a flattening of the Phillips curve after 1985. For the regressions using the output gap instead, the conclusion is less clear, as there are two periods of significant flattening separated by a marked increase in the slope of the Phillips curve in early 1980s. However, if we calculate the effect of a sustained four-quarter change in economic conditions on the one-year-ahead inflation rate, the conclusion is clear: over
time, the one-year-ahead impact on inflation drops significantly. In the case where the marginal cost is used as a proxy for economic conditions, the impact closely follows the perceived inflation persistence, while in the output gap case, the impact clearly depends on both the persistence and the slope coefficient.

\[ \pi_t = \beta \pi_{t-1} + \gamma MC_t \]

\[ \pi_t = \beta \pi_{t-1} + \gamma ygap_t \]

**Figure 5. Time Variation in the Phillips Curve Coefficients**

V. Sensitivity and robustness analysis

In this section, we provide evidence on the robustness of the learning dynamics. First of all, we illustrate that the results are not dependent on the choice of one specific model for the PLM. Secondly, we show that results are not sensitive to the sample period and that the learning model
still outperforms the RE model when estimated over the more stable period since 1984. Finally, we also document the robustness of the estimation outcomes across alternative specifications and initialization of the learning dynamics.

A. Alternative assumptions on the PLM–model

In the baseline learning model, the PLM model was specified as a simple AR(2) model. There are clearly no obvious arguments for choosing this particular forecasting model, and it is crucial to show that the results of the baseline model are robust across alternative choices for the PLM model. Therefore, we also considered an extended set-up of the learning dynamics in which we allow agents to consider a set of alternative small forecasting models. These are five small forecasting models in which the expectations about the forward variables are modelled either as a simple AR(1) process, an AR(2) process as in the baseline model, an AR(2) plus the lagged inflation rate, an AR(2) plus the lagged inflation and interest rate, and an AR(2) plus lagged inflation, interest rate and output. Agents will combine these forecasts either by simple equal weights (EW version) or by a Bayesian Model Averaging method in which the weight attached to each model varies with the past forecasting performance of each of the small forecasting models (BIC version)\textsuperscript{15}. The exercise confirms the robustness of the results in various dimensions: the AL model with a set of small forecasting models produces a marginal likelihood that is very similar to that of our baseline AR(2) specification, and also outperforms the RE model on this criterion. The persistence of inflation in the average forecasting model, measured by the sum of the lagged inflation coefficients, follows a profile over the sample that is close to the one in the baseline model. Structural parameters and identified shock series do not change significantly. Some IRFs do change mainly in their short-run response. For instance, with the interest rate included in the PLM specification, the model predicts a positive inflation response to a restrictive monetary policy shock in the 1970s: raising the policy rate created higher future inflation expectations. This response resembles the price puzzle in the SVAR literature.

We also considered alternative PLM models in which we allow agents to augment the AR(2) specification with additional RHS variables specific for each of the seven forward variables. These variables were selected based on the direct relevance of the variables suggested by the structural model: marginal cost for inflation, interest rates for the demand components etc. Such

\textsuperscript{15}It is interesting to note that in our estimations, a simple fixed-weight forecast combination works better than sophisticated time-varying re-weighting of the forecasts. Timmerman (2006) surveys a broad literature which reaches a similar conclusion, namely that simple forecast combinations “often dominate more refined combination schemes aimed at estimating the theoretically optimal combination weights”. For demonstration of dominance of simple averaging of forecasts of quarterly GDP growth, see Watson and Stock (2004).
extensions of the PLM model do not improve the fit of the model, and in some cases they lead
to significantly worse (but still better than under RE) marginal likelihoods. This finding may be
related to the fact that while the model suggests a contemporaneous relationship between these
variables and the forward variables, in adaptive learning we are restricting to variables’ lags in-
stead.

Moving in the other direction, however, produced better results: using the baseline beliefs,
we allowed two of the forward-looking variables (inflation and price of capital) to be AR(1)
processes, leaving the remaining five forwards as AR(2). This specification produced a marginal
likelihood rather close to that of the baseline model, and was more successful in matching the
post-1984 moderation experience.

B. Alternative sample periods

We estimated the model under RE and under learning for the baseline AR(2)-PLM specification
for two sub-periods: 1966q1-1979q2 and 1984q1-2008q4. In both cases, the model with AL
improves on the RE model in terms of marginal likelihood, see Table (6). Perceived inflation
persistence is the main difference between the two sub-sample models. As we have already
illustrated with the out-of-sample prediction exercise over the period since 1990, the model with
learning dynamics based on small PLM models improves upon the RE model during the more
stable environment of the great moderation period too.
C. Alternative specification and initialization of the learning dynamics

The estimation results for the learning model are also robust across alternative initializations and specifications of the learning set-up. To illustrate this, we re-estimated the model over the same sample, but now taking into account pre-sample data ranging from 1955q1 until 1965q4 to initialize the Kalman filter for the belief coefficients. The marginal likelihood of this model is very similar to the baseline version, illustrating that our results are not particularly sensitive to the initialization based on the RE-implied belief coefficients. It is interesting to note that the beliefs about inflation persistence during this pre-sample period follow an interesting pattern, with first a strong decline in the beginning of the 1960s, followed by an increase so that the beliefs at the start of our maximization period (66q1) are again close to the initialization under the baseline approach. After a few more years, the beliefs have completely converged to their counterparts in the baseline estimation.

Secondly, we experimented with alternative initial beliefs to test their impact on the results. Ideally, one would like to estimate the initial beliefs, but this is quite complicated in our application as the number of coefficients would increase considerably. One step in that direction is to estimate a model with learning where the initial beliefs are kept constant and equal to the initial beliefs in the baseline model. This resembles a first step of an iterative optimization procedure, where initial beliefs and model parameters are optimized during alternate steps. Compared to the baseline, the marginal likelihood of this model improves by another 10 units. The learning dynamics, in particular beliefs about inflation, are very similar to the baseline model.

We also tested the robustness of the learning dynamics with respect to the three learning-specific coefficients $\rho$, $\sigma_v$, and $\sigma_0$. The role of $\sigma_0$ is limited to the first few observations, and given our four-period initialization of the Kalman filter, this parameter is basically neutral for the results. For simple univariate cases, it can be shown that $\sigma_v$ and $\rho$ affect the Kalman gain in a similar direction, so that an exact identification of these two parameters simultaneously is most unlikely. Therefore, we estimate $\sigma_v$ and $\sigma_0$ for different values of the $\rho$ coefficient. When $\rho = 1$, $\sigma_v$ is estimated at 0.0012, while for $\rho = 0.97$, the estimate from the baseline model, the estimate for $\sigma_v$ increases to values that are close to the ones assumed in the baseline set-up. However, the uncertainty around $\sigma_v$ is very large, and the marginal likelihood is hardly affected by these alternative approaches.

Finally, we replace the system of SURE approach for the forecasting model with an equation-by-equation estimation approach for the seven forward variables. This is achieved by retaining only the main diagonal in the matrix $\Sigma$ derived in Equation (9). Although theoretically this estimation approach should deliver less efficient estimates, in our application it works quite well
and the marginal likelihood is slightly improved. The results show that the perceived inflation beliefs behave very similarly to the baseline model, while the beliefs about the real variables tend to display more volatility, and as a consequence, this version is marginally more successful in simulating the post-1984 real moderation.

For this simpler equation-by-equation estimation approach of the PLM, we can also consider constant-gain learning instead of Kalman filter learning. The estimated constant gain turns out to be quite low (0.006) and the updating of the PLM beliefs leads to minor revisions. Despite this low time variability, constant gain learning with AR(2) beliefs delivers a marginal likelihood that is better than under RE and under the no-update model (which was described in the last column of Table 1). The outcome is worse than under Kalman filter learning, probably reflecting slower response of constant gain learning to changes in the underlying data process. Different learning speeds of even asymptotically-equivalent constant gain and Kalman filter algorithms had previously been observed in Sargent and Williams (2005).

VI. Concluding Remarks

The hypothesis of model-consistent expectations, especially in the context of a medium-scale DSGE model, implies that economic agents are extremely well informed both about the structure of the model and the type of shocks that are hitting the economy at each point in time. Therefore, it is not surprising that models with simpler, and probably more realistic, assumptions about the expectations mechanism can improve the empirical fit of these models. In addition, our results suggest that there might be an important role for learning in these expectations: agents update their belief models in line with actual past data and, by doing so, their reactions to exogenous shocks change considerably over time. This process is particularly relevant to understanding the changing dynamics of the inflation process. Even under a constant monetary policy rule, the beliefs of the private agents about the mean and the persistence of the inflation process can vary substantially over time. The additional dynamics from the learning process substitute for the persistence in the exogenous price and wage shocks and the backward-looking indexation in wage-setting, which are both very important in the rational expectations version of the model. The potential important role of learning in inflation dynamics also suggests that private sector inflation expectations should be closely monitored by central bankers.

The specification of the small belief models may of course be criticized as being ad-hoc. We have tried to take into account that problem by allowing agents to consider different small models and to weight them depending on their past forecasting performance. Still, the belief models that we consider might be too restrictive. Introducing evidence from surveys about expectations might
help to pin down the relevant information set used by agents and to overcome this problem. Surveys contain not only information about the one-period-ahead expectations, but also about longer-horizon predictions. Consistent processing of this type of information would require a switch to infinite-horizon learning. This type of information and learning approach might also be necessary to overcome one of the main weaknesses of the Euler equation learning, namely the weak quality of the long-term forecasting.

Two other extensions of the paper are on our research agenda. The learning dynamics can potentially also contribute to an explanation of the great moderation on the real side of the economy. At this stage, our belief models for consumption and investment do not exhibit a clearly declining persistence in the growth rates of these variables. Such beliefs might be necessary to explain the observed moderation in the real volatility. Secondly, we would like to test the time variation that is generated by the learning dynamics against a more general and less restrictive time-varying VAR model.

VII. References


Orphanides, A. and J. C. Williams (2005b), “The decline of activist stabilization policy: Natural rate misperceptions, learning, and expectations”, *Journal of Economic Dynamics and Con-


The model is estimated using seven key macroeconomic time series: real GDP, consumption, investment, hours worked, real wages, prices and a short-term interest rate. GDP, consumption and investment are taken from the US Department of Commerce - Bureau of Economic Analysis database. Real Gross Domestic Product is expressed in Billions of Chained 1996 Dollars. Nominal Personal Consumption Expenditure and Fixed Private Domestic Investment are deflated by the GDP deflator. Inflation is the first difference of the log of the Implicit Price Deflator of GDP. Hours and wages come from the BLS (hours and hourly compensation for the NFB sector for all persons). Hourly compensation is divided by the GDP price deflator in order to get the real wage variable. Hours are adjusted to take into account the limited coverage of the NFB sector compared to GDP (the index of average hours for the NFB sector is multiplied by the Civilian Employment figure (16 years and over). The aggregate real variables are expressed per capita by dividing by the population over 16. All series are seasonally adjusted. The interest rate is the Federal Funds Rate. Consumption, investment, GDP, wages and hours are expressed in 100 times log. The interest rate and inflation rate are expressed on a quarterly basis corresponding with their appearance in the model.

Model Appendix

In this appendix, we summarize the log-linear equations of the model. For a more detailed presentation, we refer to the discussion in SW 2007.

- Consumption Euler equation for the non-separable utility function:

$$\hat{c}_t = c_1 E_t [\hat{c}_{t+1}] + (1 - c_1)\hat{c}_{t-1} + c_2 (\hat{L}_t - E_t [\hat{L}_{t+1}]) - c_3 (\hat{R}_t - E_t [\hat{R}_{t+1}] + \hat{e}_i^R)$$

with $c_1 = 1/(1+\eta)$, $c_2 = c_1(\sigma_c - 1)/(\omega L/C)/\sigma_c$, $c_3 = c_1(1-\eta)/\sigma_c$ where $\eta$ is the external habit parameter adjusted for trend growth $\eta = (\eta/\gamma)$, $\sigma_c$ is the inverse of the intertemporal elasticity of substitution. $\hat{e}_i^R$ is the exogenous AR(1) risk premium process.

- Investment Euler equation:

$$\hat{i}_t = i_1 \hat{i}_{t-1} + (1 - i_1)\hat{i}_{t+1} + i_2 \hat{Q}_t^k + \hat{e}_i^H$$
with $i_1 = 1/(1 + \beta \gamma)$, $i_2 = i_1/(\gamma^2 \phi)$ where $\beta$ is the discount factor adjusted for trend growth $(\beta \gamma^{1-\sigma})$, and $\phi$ is the elasticity of the capital adjustment cost function. $\tau^D_i$ is the exogenous AR(1) process for the investment specific technology.

- Value of the capital stock:

$$\hat{Q}_i^k = -(\hat{R}_i - E[G_{t+1}] + \hat{r}_i^k) + q_1 E[r_1^k] + (1 - q_1) E[Q_{t+1}^k]$$

with $q_1 = r_s^k/(r_s^k + (1 - \delta))$ where $r_s^k$ is the steady state rental rate to capital, and $\delta$ the depreciation rate.

- Aggregate demand equals aggregate supply:

$$\hat{y}_t = \frac{c_* \hat{c}_t}{\hat{y}_t} + \frac{i_* \hat{i}_t}{\hat{y}_t} + \hat{r}_i^k + \frac{r_s^k}{\hat{y}_t} \hat{u}_t$$

$$= \Phi_p \left( \hat{a} \hat{k}_t + (1 - \alpha) \hat{L}_t + \hat{\tau}_i^p \right)$$

with $\Phi_p$ reflecting the fixed costs in production which corresponds to the price markup in steady state. $\hat{\tau}_i^D$, $\hat{\tau}_i^u$ are the AR(1) processes representing exogenous demand components and the TFP process.

- Price-setting under the Calvo model with indexation:

$$\hat{P}_t - i_p \hat{P}_{t-1} = \pi_1 (E_t [\hat{P}_{t+1}] - i_p \hat{P}_t) - \pi_2 \hat{\mu}_t^p + \hat{\tau}_t^p$$

with $\pi_1 = \beta \gamma$, $\pi_2 = (1 - \xi_p \beta \gamma)/(1 - \xi_p)/(\xi_p (1 + (\phi_p - 1) e_p))$, with $\xi_p$ and $i_p$ respectively the probability and indexation of the Calvo model, and $e_p$ the curvature of the aggregator function. The price markup $\hat{\mu}_t^p$ is equal to the inverse of the real marginal $\hat{m} \hat{c}_t = (1 - \alpha) \hat{w}_t + \alpha \hat{r}_t^k - \hat{A}_t$.

- Wage setting under the Calvo model with indexation:

$$\hat{w}_t - i_w \hat{w}_{t-1} = \pi_1 (E_t [\hat{w}_{t+1}] - i_w \hat{w}_t) - \pi_3 \hat{\mu}_t^w + \hat{\tau}_t^w$$

with $\pi_3 = (1 - \xi_w \beta \gamma)/(1 - \xi_w)/(\xi_w (1 + (\phi_w - 1) e_w))$ and wage markup $\hat{\mu}_t^w = \hat{w}_t - w_1 \hat{c}_t + (1 - w_1) \hat{c}_{t-1} - \sigma \hat{L}_t$ with $w_1 = 1/(1 - \eta)$. 
• Capital accumulation equation:

\[ \hat{k}_t = k_1 \hat{k}_{t-1} + (1 - k_1) \hat{u}_t + k_2 \hat{e}_t^{s} \]

with \( k_1 = (1 - (i_s / \bar{k}_s), k_2 = (i_s / \bar{k}_s)(1 + \bar{\beta}^2) \sigma^2 S'' \). Capital services used in production is defined as: \( \hat{k}_t = \hat{u}_t + \hat{k}_{t-1} \)

• Optimal capital utilisation condition:

\[ \hat{u}_t = (1 - \nu) / \psi \hat{r}_t \]

with \( \nu \) is the elasticity of the capital utilisation cost function.

• Optimal capital/labor input condition:

\[ \hat{k}_t = \hat{w}_t - \hat{r}_t^k + \hat{L}_t \]

• Monetary policy rule:

\[ \hat{R}_t = \rho \hat{R}_{t-1} + (1 - \rho) (r_x \hat{\pi}_t + r_y (ygap \hat{a}_t) + r_{\Delta y} \Delta (ygap \hat{a}_t) + \hat{e}_t^r \]

with \( ygap_t = (\hat{y}_t - \Phi \hat{e}_t^y) \).

The following parameters are not identified by the estimation procedure and therefore calibrated: \( \delta = 0.025, \epsilon_p = 10, \epsilon_w = 10, \phi_w = 1.5 \).

Table A1 provides more information on the RE and baseline-AL estimation results, and completes Table 1 in the paper.
### Table A.1—Posterior Estimates for RE and AL models - Complete list of parameters

<table>
<thead>
<tr>
<th></th>
<th>Prior Distribution</th>
<th>RE Posterior</th>
<th>AL-Baseline</th>
<th>RE Posterior</th>
<th>AL-Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>type</td>
<td>mean</td>
<td>st.dev</td>
<td>mean</td>
<td>5%</td>
</tr>
<tr>
<td><strong>st.dev. of the innovations</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>IG 0.1 2</td>
<td>0.45</td>
<td>0.41</td>
<td>0.49</td>
<td>0.46</td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>IG 0.1 2</td>
<td>0.26</td>
<td>0.21</td>
<td>0.30</td>
<td>0.15</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>IG 0.1 2</td>
<td>0.52</td>
<td>0.47</td>
<td>0.56</td>
<td>0.50</td>
</tr>
<tr>
<td>$\sigma_d$</td>
<td>IG 0.1 2</td>
<td>0.42</td>
<td>0.35</td>
<td>0.50</td>
<td>0.45</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>IG 0.1 2</td>
<td>0.22</td>
<td>0.20</td>
<td>0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>$\sigma_f$</td>
<td>IG 0.1 2</td>
<td>0.15</td>
<td>0.13</td>
<td>0.18</td>
<td>0.15</td>
</tr>
<tr>
<td>$\sigma_g$</td>
<td>IG 0.1 2</td>
<td>0.23</td>
<td>0.19</td>
<td>0.26</td>
<td>0.23</td>
</tr>
<tr>
<td><strong>persistence of the exogenous processes:</strong> $\rho = \text{AR}(1), \theta = \text{MA}(1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>B 0.5 0.2</td>
<td>0.93</td>
<td>0.90</td>
<td>0.97</td>
<td>0.99</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>B 0.5 0.2</td>
<td>0.25</td>
<td>0.10</td>
<td>0.40</td>
<td>0.55</td>
</tr>
<tr>
<td>$\rho_c$</td>
<td>B 0.5 0.2</td>
<td>0.98</td>
<td>0.97</td>
<td>0.99</td>
<td>0.97</td>
</tr>
<tr>
<td>$\rho_d$</td>
<td>B 0.5 0.2</td>
<td>0.75</td>
<td>0.66</td>
<td>0.84</td>
<td>0.51</td>
</tr>
<tr>
<td>$\rho_e$</td>
<td>B 0.5 0.2</td>
<td>0.08</td>
<td>0.02</td>
<td>0.15</td>
<td>0.10</td>
</tr>
<tr>
<td>$\rho_f$</td>
<td>B 0.5 0.2</td>
<td>0.85</td>
<td>0.75</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>$\rho_{w}p$</td>
<td>B 0.5 0.2</td>
<td>0.96</td>
<td>0.93</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>$\rho_{w}p$</td>
<td>B 0.5 0.2</td>
<td>0.70</td>
<td>0.55</td>
<td>0.86</td>
<td></td>
</tr>
<tr>
<td>$\rho_{w}P$</td>
<td>B 0.5 0.2</td>
<td>0.88</td>
<td>0.81</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>$a_{s2}$</td>
<td>N 0.5 0.25</td>
<td>0.53</td>
<td>0.40</td>
<td>0.68</td>
<td>0.54</td>
</tr>
<tr>
<td><strong>structural parameters</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>N 4.0 1.5</td>
<td>5.45</td>
<td>3.82</td>
<td>7.02</td>
<td>3.23</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>N 1.5 0.37</td>
<td>1.31</td>
<td>1.10</td>
<td>1.53</td>
<td>1.58</td>
</tr>
<tr>
<td>$\eta$</td>
<td>B 0.7 0.1</td>
<td>0.77</td>
<td>0.70</td>
<td>0.84</td>
<td>0.68</td>
</tr>
<tr>
<td>$\sigma_{1}$</td>
<td>N 2.0 0.5</td>
<td>1.48</td>
<td>0.61</td>
<td>2.37</td>
<td>1.77</td>
</tr>
<tr>
<td>$\tilde{\tau}_{1}$</td>
<td>B 0.5 0.1</td>
<td>0.72</td>
<td>0.65</td>
<td>0.80</td>
<td>0.65</td>
</tr>
<tr>
<td>$\tilde{\tau}_{w}$</td>
<td>B 0.5 0.1</td>
<td>0.77</td>
<td>0.68</td>
<td>0.84</td>
<td>0.84</td>
</tr>
<tr>
<td>$\tau_p$</td>
<td>B 0.5 0.15</td>
<td>0.24</td>
<td>0.10</td>
<td>0.38</td>
<td>0.19</td>
</tr>
<tr>
<td>$\tau_{w}$</td>
<td>B 0.5 0.15</td>
<td>0.51</td>
<td>0.30</td>
<td>0.71</td>
<td>0.21</td>
</tr>
<tr>
<td>$\psi$</td>
<td>B 0.5 0.12</td>
<td>0.52</td>
<td>0.33</td>
<td>0.70</td>
<td>0.56</td>
</tr>
<tr>
<td>$\Phi_p$</td>
<td>N 1.25 0.12</td>
<td>1.62</td>
<td>1.49</td>
<td>1.74</td>
<td>1.56</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>B 0.75 0.1</td>
<td>0.85</td>
<td>0.82</td>
<td>0.88</td>
<td>0.89</td>
</tr>
<tr>
<td>$\tau_x$</td>
<td>N 1.5 0.25</td>
<td>1.88</td>
<td>1.63</td>
<td>2.15</td>
<td>1.75</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>N 0.12 0.05</td>
<td>0.10</td>
<td>0.06</td>
<td>0.14</td>
<td>0.15</td>
</tr>
<tr>
<td>$\tau_{\Delta y}$</td>
<td>N 0.12 0.05</td>
<td>0.16</td>
<td>0.13</td>
<td>0.19</td>
<td>0.14</td>
</tr>
<tr>
<td>$\delta$</td>
<td>G 0.62 0.1</td>
<td>0.74</td>
<td>0.56</td>
<td>0.91</td>
<td>0.64</td>
</tr>
<tr>
<td>100(\beta^{-1} - 1)</td>
<td>G 0.25 0.1</td>
<td>0.19</td>
<td>0.09</td>
<td>0.29</td>
<td>0.17</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>N 0.0 2.0</td>
<td>0.84</td>
<td>-0.63</td>
<td>2.34</td>
<td>0.83</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>N 0.4 0.1</td>
<td>0.41</td>
<td>0.39</td>
<td>0.43</td>
<td>0.41</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>N 0.3 0.05</td>
<td>0.19</td>
<td>0.16</td>
<td>0.22</td>
<td>0.17</td>
</tr>
</tbody>
</table>

1 The IG-distribution is defined by the degree of freedom. 2 The effect of TFP innovations on exogenous demand.