Fyll inn oppgavekode og emneinformasjon på alle skissearkene — Fill out question code and test information on every sheet

Oppgavekode / Question code	Dato / Date	Emnekode / Subject code	Kandidatnummer / Candidate number	Oppgavenummer / Question number	Sidetall / Page number
6442919	12.05.23	ECON4140	17471	1	1

Tegneområde — Drawing area

1)

a) $\bar{A} = \begin{pmatrix} 1 & 2 \\ -5/2 & R \end{pmatrix}$

$\det(\bar{A}) = 1 \cdot R - 2 \cdot (-5/2)$
$= R + 5 \neq 0$ if
$R \neq -5$. Then $r(\bar{A}) = 2$

If $R = -5$, $r(\bar{A}) = 1$ $(R_0 = -5)$

b) $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\bar{A}\vec{u} = \begin{bmatrix} 1 & 2 \\ -5/2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -5+7 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

Then $\mu = \underline{\underline{2}}$

c) • When $R = 7$ $(\neq -5)$, $\det(\bar{A}) = 7 + 5 = 12$

$tr(\bar{A}) = 8$, $\left(\frac{tr\bar{A}}{2}\right)^2 = 4^2 = 16$

Then $0 < \det\bar{A} < \left(\frac{tr\bar{A}}{2}\right)^2$ we have two distinct eigenvalues with same sign as $\left(\frac{tr\bar{A}}{2}\right)^2$ (positive).

We have one already, $\mu = 2$.
The other is: $tr\bar{A} - \mu = 8 - 2 = 6$.
They are <u>both</u> real.

When $R = -5$, $\det\bar{A} = 0$, $tr\bar{A} = 1 - 5 = -4$

So $\det\bar{A} = 0 \neq tr\bar{A}$, and we have two distinct eigenvalues, $\lambda_1 = 0$, and $\lambda_2 = tr\bar{A} = -4$.
They are <u>both</u> real (if $0 \in \mathbb{R}$)

→ rest of c) next page

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Tegneområde / Drawing area

Rest of c)

$R=-5,\ \lambda_1=0\quad \begin{bmatrix}1 & 2\\ -5/2 & -5\end{bmatrix}=\vec{0}$

$\sim\begin{bmatrix}1 & 2\\ -5/2 & -5\end{bmatrix}\ (5/2) \sim \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} = \quad 1v_1+2v_2=0$

$2v_2=-v_1$

$v_2=-\frac{1}{2}v_1$

eigenvector $\begin{bmatrix}1\\ -\frac{1}{2}\end{bmatrix}$ $\quad v_1=1\quad v_2=-\frac{1}{2}$

d) Definiteness $\quad Q(\vec{x})=\vec{x}'\bar{A}\vec{x}$

$\bar{A}=\begin{pmatrix}1 & 2\\ -5/2 & R\end{pmatrix}$ must be symmetric: $\frac{1}{2}\left(\begin{pmatrix}1 & 2\\ -5/2 & R\end{pmatrix}+\begin{pmatrix}1 & -5/2\\ 2 & R\end{pmatrix}\right)$

$=\frac{1}{2}\begin{bmatrix}2 & -\frac{1}{2}\\ -1/2 & 2R\end{bmatrix}=\begin{bmatrix}1 & -\frac{1}{4}\\ -\frac{1}{4} & R\end{bmatrix}$

$\det(\bar{A}^{sym})=R-(-\frac{1}{4})(-\frac{1}{4})=R-\frac{1}{16}\rightarrow$ Positive definite if $R>\frac{1}{16}$ (since all leading princip. minors >0)

If $R-\frac{1}{16}<0$, it is indefinite (since even order minor <0).

If $R-\frac{1}{16}=0$, then positive semidefinite, since $a_{11}>0$ and the last minor $=0$.

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Tegneområde — Drawing area

2)

a) (S) I $\dot{x} = x + 2y + R$ II $\dot{y} = -\frac{5}{2}x + Ry$

I: $2y = \dot{x} - x - R$, $y = \frac{1}{2}\dot{x} - \frac{1}{2}x - \frac{1}{2}R$

$\dot{y} = \frac{1}{2}\ddot{x} - \frac{1}{2}\dot{x}$

insert into II: $\frac{1}{2}\ddot{x} - \frac{1}{2}\dot{x} = -\frac{5}{2}x + R\left(\frac{1}{2}\dot{x} - \frac{1}{2}x - \frac{1}{2}R\right)$

$\frac{1}{2}\ddot{x} - \frac{1}{2}\dot{x} - \frac{R}{2}\dot{x} + \frac{5}{2}x + \frac{R}{2}x = -\frac{1}{2}R^2$

$\frac{1}{2}\ddot{x} - \frac{1-R}{2}\dot{x} + (R+5)x = -\frac{1}{2}R^2 \quad |\cdot 2$

$\ddot{x} - (1-R)\dot{x} + 4x = -R^2$, $\ddot{x} - (R+1)\dot{x} + (R+5)x = -R^2$

b) $R = -1$, $a = -(R+1) = 0$ $b = R+5$, $b = 4$

$\frac{1}{4}a^2 - b = 0 - 4 < 0$ Complex root (no real roots).

$\alpha = \frac{1}{2}a = 0$, $\beta = \sqrt{4} = 2$ $e^{0t} = 1$

$x = A\cos 2t + B\sin 2t$

$RHS = -R^2 = -(-1)^2 = -1$, $u^* = \frac{-1}{4}$

$x = A\cos 2t + B\sin 2t - \frac{1}{4}$, $Q = -\frac{1}{4}$.

If RHS were $\cos\frac{\pi}{t}$, we could try and fit

$u^* = C\sin\frac{\pi}{t} + D\cos\frac{\pi}{t}$.

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Tegneområde — Drawing area

2) c) Difference eq:

$x_{t+2} - (R+1)x_{t+1} + (R+5)x_t = 0$ homogeneous.

Particular solution: $D \cdot 2^t$

$x_0 = 0$, $2C \cdot (5/2)^0 + D \cdot 2^0 = 0$

$2C + D = 0$ $\quad 2C = -D$

$C = -\frac{1}{2}D$

$x_1 = 2023$

$2C \cdot (\frac{5}{2})^1 + D \cdot 2^1 = 2023$

$5C + 2D = 2023$, $\quad C = -\frac{1}{2}D$

$5(-\frac{1}{2}D) + 2D = 2023$

$2D - \frac{5}{2}D = 2023 \to \frac{4-5}{2}D = -\frac{1}{2}D = 2023$

$C = -\frac{1}{2}(-4046) = \underline{\underline{2023}}$ $\qquad D = \underline{\underline{-4046}}$

Particular solution:

$2 \cdot (2023) \cdot (\frac{5}{2})^t + (-4046)2^t$

$= 4046(\frac{5}{2})^t - 4046 \cdot 2^t$

$= \underline{4046((\frac{5}{2})^t - 2^t)}$

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Tegneområde Drawing area

2d) $R = -3$

$x_{t+2} + 2x_{t+1} + 2x_t = -9$

(D) Difference equation: $a = 2 \quad b = 2$

Globally asymptotically stable iff

$|a| < 1 + b$ and $b < 1$, $a = -(R+1) = -(-3+1) = 2$

$b = R + 5 = 2$

$|-(R+1)| < 1 + R + 5$

$|2| < 1 + 2 = 3$

$R + 5 < 1$

$2 < 1$

No, <u>not</u> globally asympt. stable.

(E) $\ddot{x} - (-3+1)\dot{x} + (-3+5)x = -(-3)^2$

$\ddot{x} + 2\dot{x} + 2x = -9$

Is globally asymptotically stable iff $a > 0$ and $b > 0$

$2 > 0$ and $b > 0$

E <u>is</u> globally asymptotically stable

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Tegneområde Drawing area

3)

$$\max_{u \in (-\infty,\infty)} \int_{2023}^{2024} \left(\frac{5}{4}x^2 - u^2\right) dt \quad s.t \quad \dot{x} = x + 2u - 1$$

$x(2023) = \bar{x} > 0$

$x(2024)$ free

a) • Maximum principle:

Hamiltonian: $H = \frac{5}{4}x^2 - u^2 + p(x + 2u - 1)$

• Our optimal u^* is the one that max. H

• For the continuous function p, we have

$$\dot{p} = -\frac{\partial H}{\partial x} = -\left(\left(2 \cdot \frac{5}{4}\right)x + p\right) = -\frac{5}{2}x - p$$

• Transversality condition: Since $x(2024)$ free (terminal condition), we have that $p(2024) = 0$.

• The system of differential equations:

$\dot{x} = x + 2u - 1$

$\dot{p} = -\frac{5}{2}x - p$

• Arrow conditions: Concavity of H in x

$$H = \underbrace{\frac{5}{4}x^2 + px}\ - u^2 + 2pu - p$$

$H'_x = \frac{5}{2}x + p$

$H''_{xx} = \frac{5}{2} > 0$

Not concave.

They don't hold.

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Tegneområde — Drawing area

3) b)

$u(2024)$: The u that max when no tomorrow

$\frac{5}{4}x^2 - u^2$. $u(2024)=0$

c) Conditions in a: $\dot{x} = x+2u-1$ $\frac{\partial H}{\partial u} = -2u+2p=0$

$R=-1$ $\dot{p} = -\frac{5}{2}x - p$ $u=p$

If $p=-1$, then $u=1$ (by FOC of Hamiltonian)

then we have

$$\left.\begin{aligned}\dot{x} &= x+2-1\\ \dot{p} &= -\tfrac{5}{2}x+1\end{aligned}\right\}$$ Same as (S) in problem 2,

and in 2a) I already showed how to obtain the second order differential equation for x

$$\ddot{x} - (R+1)\dot{x} + (R+5)x = -R^2$$

$$\ddot{x} + 4x = -1$$

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3969681	12.5.23	ECON4140	17471	4	8

Tegneområde — Drawing area

4)

a) $J_T(x) = -x_T$ (max J_T for no tomorrow)

$J_{T-1}(x) = \max\{f(u) + J_T(xu)\}$

Case 1: $f(u) = -1$ if $u = 1/2$

Case 2: $f(u) = 0$ if $u = 1$

$J_{T-1}(x) = \max\{-1 + (-\frac{1}{2}x)\} = \max\{-(1 + \frac{x}{2})\}$

$J_{T-1}(x) = \max\{0 + (-x)\} = \max\{-x\}$

which is max?

$-1 - \frac{x}{2} > -x$

$-\frac{x}{2} + x > 1$

$\frac{1}{2}x > 1 \rightarrow x > 2$

When $x > 2$, then $-(1+\frac{x}{2})$ is max.

Just as $W(x)$.

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Tegneområde / Drawing area

b) Proof by induction:

Base case: $N=1$

$$J_T(x) = W(x) = \begin{cases} -x & x \in (0,2] \\ -(1+\frac{x}{2}) & x \in (2,4) \end{cases}$$

holds because $J_T = \max J_T$

General case: Assume it holds for $N=k$. Then for $N=k+1$

$J_{T-k-1} = \max\{f(u) + J_{T-k}(ux)\}$, [WTS: $J_{T-k-1} = W(x)$]

Case 1: $u = 1/2$, $f(u) = -1$

Case 2: $u = 1$, $f(u) = 0$

Case 1: $\max\{-1 + (\underbrace{-x}_{w(x)=-x}\frac{1}{2})\} = \max\underbrace{\{-(1+\frac{x}{2})\}}$

Case 1: $\max\{-1 + (\underbrace{-(1+\frac{\frac{1}{2}x}{2})}_{w(x) = -(1+\frac{x}{2})})\}$

$= \max\{-1 - 1 - \frac{1}{4}x\} = \max\{-2 - \frac{1}{4}x\}$

$= \max\{-\frac{1}{2}(1+\frac{x}{2})\}$

Case 2: $\max\{0 + (-x)\} = \underbrace{\max\{-x\}}$

Case 2: $\max\{0 + (-(1+\frac{x}{2}))\} = \underbrace{\max\{-(1+\frac{x}{2})\}}$

$J_{T-k-1} = J_{T-k} = W(x) = \{ \dots$

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9987936	12.05.23	ECON4140	17471	5	10

Tegneområde | Drawing area

5)

$S = \{v_1, \dots, v_{2023}\}$ distinct point v_i

S is convex if $\forall v_i$ in S,

$\lambda \vec{v}_i + (1-\lambda)\vec{v}_j \in S \quad \forall \vec{v}_i, \vec{v}_j$ in S and all $\lambda \in [0,1]$.

S needs to contain all line segment connecting the two distinct points.

Just a singleton $B = \{v_1\}$ for example, is convex.

S is the union of all the singleton sets of all the points, and that is not neccesarily convex.

As I understand the problem, S is a "collection" of distinct points. It is a finite set, and it does not neccessarily contain mid-points. If $S \subseteq \mathbb{R}^1$ and S is an interval (the v's are interval), then it is convex.

But, if S is $S \subseteq \mathbb{R}^n$ it is not convex.