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P1

a) Differentiate: (K, L, t, ω)

① $aL^a\sqrt{K} + bL^b\sqrt{K} = \omega L e^{rt}$

$\to a^2L^{a-1}\sqrt{K}dL + \frac{1}{2}aL^aK^{-1/2}dK + b^2L^{b-1}\sqrt{K}dL + \frac{1}{2}bL^bK^{-1/2}dK =$

$Le^{rt}d\omega + \omega e^{rt}dL + r\omega Le^{rt}dt + (\cancel{t\omega Le^{rt}dr})$ (NO)

② $L^a + L^b = \sqrt{K}e^{rt}$

$\to aL^{a-1}dL + bL^{b-1}dL = \frac{1}{2}K^{-1/2}e^{rt}dK + r\sqrt{K}e^{rt}dt$

b) Approx. $L(1, 2(a+b)+h)$ $dt = 1$ $d\omega = h$

First reeplace the point $(\overset{K}{4}, \overset{L}{1}, \overset{t}{0}, \overset{\omega}{2(a+b)})$

① $a^2 \cdot 2dL + \frac{a}{2} \cdot \frac{1}{2}dK + b^2 \cdot 2dL + \frac{b}{2} \cdot \frac{1}{2}dK =$

$\underbrace{e^{r \cdot 0}}_{1} h + 2(a+b)\underbrace{e^{r0}}_{1}dL + r \cdot 2(a+b) \cdot \underbrace{L}_{1}\underbrace{e^{r \cdot 0}}_{1} \cdot 1$

② $(\cancel{e^0 \cdot d\omega})^{NO}$

$adL + bdL = \frac{1}{4}dK + 2re^0 \cdot 1$

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P1

Continuation of (b)

(1) $2a^2 dL + \frac{a}{4} dK + 2b^2 dL + \frac{b}{4} dK = h + 2(a+b)dL + r\,2(a+b)$

(2) $(a+b)dL = \frac{dK}{4} + 2r$

from (2) $dK = 4(a+b)dL - 8r$

reeplacing in (1)

$\Rightarrow (2a^2 + 2b^2)dL + a((a+b)dL - 2r) + b((a+b)dL - 2r) = h + 2(a+b)dL + r\,2(a+b)$

$((2a^2 + 2b^2) + a(a+b) + b(a+b) - 2a - 2b)dL = 2r(a+b) + h + 2r(a+b)$

$(2a^2 + 2b^2 + a^2 + ab + b^2 + ba - 2a - 2b)dL = 4r(a+b) + h$

$$dL = \frac{4r(a+b) + h}{(3a^2 + 3b^2 + 2ab - 2(a+b))}$$

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P2

a) $K \neq 0$ (positive or negative)

i) $\lim\limits_{x \to 0^+} \frac{1}{x^K \ln x} = \frac{1}{0^K \ln 0} \begin{cases} K>0 & \frac{1}{0 \cdot -\infty} \\ K<0 & \frac{1}{\infty \cdot -\infty} = 0 \end{cases}$

$\Rightarrow$ for $K>0$;

$\lim\limits_{x \to 0^+} \frac{\frac{1}{x^K}}{\ln x} = \frac{\infty}{-\infty} \xrightarrow{L.H.} \frac{-K x^{-K-1}}{\frac{1}{x}} = -K x^{-K} = \cdot$

$\lim\limits_{x \to 0^+} \Rightarrow -K \frac{1}{x^K} = \frac{-K}{0} = -\infty$

ii) $\lim\limits_{x \to +\infty} \frac{1}{x^K \ln x} \begin{cases} K>0 & \frac{1}{\infty \cdot \infty} = 0 \\ K<0 & \frac{1}{0 \cdot \infty} \end{cases}$

for $K<0$:

$\lim\limits_{x \to +\infty} \frac{x^{-K}}{\ln x} \xrightarrow{L.H} \underbrace{-K \cdot \frac{1}{x^K}}_{\text{from}(i)} = \overbrace{-K}^{(+)} \cdot \overbrace{x^{-K}}^{(+)} = +\infty$

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iii) $\lim_{x \to 0^+} \frac{1}{x^K (\ln x)^{2018}}$ $\begin{cases} K>0 & \frac{1}{0 \cdot +\infty} \\ K<0 & \frac{1}{\infty \cdot +\infty} = 0 \end{cases}$

for $K>0$

$$\lim_{x \to 0^+} \frac{x^{-K}}{(\ln x)^{2018}} = \frac{\infty}{\infty} \xrightarrow{L.H.} \frac{-K x^{-K-1}}{\frac{2018 (\ln x)^{2017}}{x}}$$

$$\lim_{x \to 0^+} \Rightarrow \frac{-K x^{-K}}{2018 (\ln x)^{2017}} \xrightarrow{L.H.} \frac{K^2 x^{-K-1}}{\frac{(2018)(2017)(\ln x)^{2016}}{x}}$$

$$\lim_{x \to 0^+} \Rightarrow \frac{K^2 x^{-K}}{(2018)(2017)(\ln x)^{2016}} \quad \ldots$$

Apply L.H rule 2018 times:

$$\lim_{x \to 0^+} \frac{K^{2018} x^{-K}}{2018! (\ln x)^0} = \underbrace{\frac{K^{2018}}{2018!}}_{(+)} \frac{1}{x^K} = +\infty$$

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P2

(b)

(i) $\int_x^1 \frac{1}{u(1-\ln u)}\,du = \ln(1-\ln x) \qquad 0<x<e$

$\rightarrow$ Integration by substitution:

- $\ln u = v$ replacing

$\frac{1}{u}du = dv$

$$\int_x^1 \frac{1}{1-v}\,dv = \Big|_x^1 -\ln(1-v)$$

back to the origral value:

$$\int_x^1 \frac{1}{u(1-\ln u)}\,du = \Big|_x^1 -\ln(1-\ln u)$$

$$= \underbrace{-\ln(1-\underbrace{\ln 1}_{0})}_{0} + \ln(1-\ln x)$$

$$\int_x^1 \frac{1}{u(1-\ln u)}\,du = \ln(1-\ln x)$$

* Error:
Not from x to 1;
v runs from $\ln x$ to $\ln 1$.
* Doing indefinite integrals first would have avoided this fault.
* Antiderivative:
$-\ln|1-v|$, which equals $-\ln(1-v)$ since the limits for v are $\ln x$ ($< \ln e = 1$) and 0.
* The answer ends up correct.

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P2

(b)

ii) $\frac{1}{2}\int e^{v}\ln((1+e^{v})^{2})dv = (1+e^{v})(\ln(1+e^{v})-1)+C$

By substitution:

$1+e^{v}=u$
$e^{v}dv=du$ $\Big\}$ $\frac{1}{2}\int \ln(u)^{2}du$

$= \frac{\cancel{2}}{\cancel{2}}\int \ln(u)du = u\ln u - u + C$

$= u(\ln u - 1) + C$

Back the original value:

$\frac{1}{2}\int e^{v}\ln((1+e^{v})^{2})dv = (1+e^{v})(\ln(1+e^{v})-1)+C$

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P2

(c) $x \in (0, e)$

$$\dot{x} = x(1-\ln x)\, e^t \ln\left((1+e^t)^2\right)$$

Preferably:
Check for zeroes
before dividing.

$$\int \frac{dx}{x(1-\ln x)} = \int e^t \ln\left((1+e^t)^2\right) dt$$

* Constant solution when $x \equiv 0$ and $x \equiv e$

otherwise we integrate:

From (b):

$$-\ln(1-\ln x) = 2(1+e^t)(\ln(1+e^t)-1) + \underbrace{2C}_{-A}$$

$$\ln(1-\ln x) = A - 2(1+e^t)(\ln(1+e^t)-1)$$

Incomplete solution:
In this course, you should solve for x.

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P3

$$A_t = \begin{pmatrix} 1 & t & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & -t & -1 \end{pmatrix} \quad \text{and} \quad 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(a)

i)

$$A_t^2 = A_t \cdot A_t = \begin{pmatrix} 1 & t & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & -t & -1 \end{pmatrix}\begin{pmatrix} 1 & t & 0 & 2 \\ 0 & -1 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & -t & -1 \end{pmatrix}$$

TAKE NOTE:
To the right, the matrix multiplication is *explained* in terms of the dot product that make up elemen (1,1). ***Good!***

$$A_t^2 = \begin{array}{c} a_{11} \\ \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \end{array} \quad //$$

Dot product:

$$a_{11} = (1)(1) + (t)(0) + (0)(0) + (2)(-2) = -3 \; //$$

ii)

$$|A' A_t| = |A'_t|\,|A_t| = |A_t|\,|A_t| = |A_t^2|$$

$$|A_t^2| = -3 \begin{vmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -3(-3)(9-0) = 81 \; //$$

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P3

(a)

(iii) $I1(I1)'$

$$I1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{4\times1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{4\times1}$$

$$I1(I1)' = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{4\times1} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}_{1\times4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}_{4\times4}$$

(iv) $$A_t 1 = \begin{pmatrix} 3+t \\ 1 \\ -1 \\ -(3+t) \end{pmatrix}_{4\times1} \quad (A_t1)' = \begin{pmatrix} 3+t & 1 & -1 & -(3+t) \end{pmatrix}_{1\times4}$$

$$\underset{1\times4}{(A_t1)'}\underset{4\times1}{(A_t 1)} = \left((3+t)^2 + 1 + 1 + (3+t)^2 \right) = \left(2(3+t)^2 + 2 \right)$$

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P3

(b)

from (a) part (i):

$$A_t^2 = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = -3 \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{I_{4\times4}}$$

Then:

$(A_t \cdot A_t = -3I)A^{-1}$ Multiply A^{-1} (when it exists) by the right hand.

$A_t = -3A_t^{-1}$

$\Rightarrow \quad A_t^{-1} = \underbrace{-\frac{1}{3}}_{\to s} A_t = sA_t$ ✓

(c) $A_t x = 1$ for one solution to exist A_t must be invertible:

$A_t^{-1}(A_t x = 1) \Rightarrow x = A_t^{-1} 1$

from (b): $x = -\frac{1}{3}\underbrace{A_t \cdot 1}_{\text{from a) iv)}}$

$$x = -\frac{1}{3}\begin{pmatrix} 3+t \\ 1 \\ -1 \\ -(3+t) \end{pmatrix}$$ ✓

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P3

d) If $|M| = d \quad (\neq 0) \qquad M_{n\times n}$

$$M^{-1} = \frac{1}{|M|} C'$$

$$\Rightarrow \left(M^{-1} = \frac{1}{d} C' \right) M$$

Multiply by M from the right.

$$I_{n\times n} = \frac{1}{d} C' M$$

$|d\, I_{n\times n}| = |C' M|$ determinants

$$d^n = |C'|\,|M|$$

$$d^n = |C|\, d$$

$$|C| = \frac{d^n}{d} = d^{n-1}$$

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P4

• Let $c > 0$ u is continuosly differentiable.

$$\max u(x,y) \quad s.t. \quad c - u(1-x, 1-y) = 0 \qquad (L)$$

And:

$$\max u(x,y) \quad s.t. \quad c - u(1-x, 1-y) \leq 0 \quad 0 \leq x \leq 1 \quad 0 \leq y \leq 1 \quad (K)$$

a) * $u(1-x, 1-y) = g(x,y)$

i) • Lagrange Conditions of (L):

MISSING: The constraint is part of the "Lagrange conditions".

$$\mathcal{L}: u(x,y) - \lambda_1 (c - g(x,y))$$

F.O.C:

The "−1" is inaccurate, the way g is defined. The rightmost equality holds true. (Same for derivatives wrt. both x and y.)

$$\frac{\partial \mathcal{L}}{\partial x} = u'_x + \lambda_1 g'_x \cdot -1 = 0 \Rightarrow u'_x = \lambda_1 u'_x(1-x, 1-y)$$

$$\frac{\partial \mathcal{L}}{\partial y} = u'_y + \lambda_1 g'_x \cdot -1 = 0 \Rightarrow u'_y = \lambda_1 u'_y(1-x, 1-y)$$

• Kuhn-tucker for (K)

$$\mathcal{L}: u(x,y) - \lambda_1 (c - u(1-x, 1-y)) - \lambda_2(-x) - \lambda_3(x-1) - \lambda_4(-y) - \lambda_5(y-1)$$

Preferably, write (x,y) and (1−x, 1−y) as it matters in this problem. The top of the next page improves!

$$\frac{\partial \mathcal{L}}{\partial x} = u'_x + \lambda_1 u'_x \cdot -1 + \lambda_2 - \lambda_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = u'_y + \lambda_1 u'_y \cdot -1 + \lambda_4 - \lambda_5 = 0$$

Slackness:

$\lambda_1 \geq 0$, $\lambda_1 = 0$ if $c - u(1-x, 1-y) < 0$
$\lambda_2 \geq 0$, $\lambda_2 = 0$ if $0 < x$
$\lambda_3 \geq 0$, $\lambda_3 = 0$ if $x < 1$
$\lambda_4 \geq 0$, $\lambda_4 = 0$ if $0 < y$
$\lambda_5 \geq 0$, $\lambda_5 = 0$ if $y < 1$

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ii)

F.O.C of L: replace $(x, y) = (\frac{1}{2}, \frac{1}{2})$

- $U'_x(x, y) = h_1 U'_x(1-x, 1-y)$

$U'_x(\frac{1}{2}, \frac{1}{2}) = h_1 U'_x(1-\frac{1}{2}, 1-\frac{1}{2})$

$\underbrace{\underbrace{U'_x(\frac{1}{2}, \frac{1}{2})}_{} = h_1 \underbrace{U'_x(\frac{1}{2}, \frac{1}{2})}_{}}_{=}$ $\quad h_1 = 1$

the same for

$U'_y(x, y) = h_1 U'_y(1-x, 1-y)$ $\quad h_1 = 1$

$U'_y(\frac{1}{2}, \frac{1}{2}) = h_1 U'_y(\frac{1}{2}, \frac{1}{2})$

because the F.O.C hold and the constraint should also hold.

It is false that the point $(\frac{1}{2}, \frac{1}{2})$ will satisfy the Lagrange conditions as long as the constraint holds, it will satisfy the point if it also satisfies the F.O.C. We can have points that satisfy the constrain, but not the F.O.C.

In this case $(\frac{1}{2}, \frac{1}{2})$ satisfies both.

So this answer is a bit "unexpected" and must be read with caution:
* Likely the first part is intended to address the general theory, while the last sentence ("In this case [...]") addresses this particular case.
* Without that last sentence, it would be quite a flaw.

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$\to \overset{0}{c} - (2(e+1-x) - e^{2(x)} - (1+e)e^{1+2(1-y)})$

P4

b) $U(x,y) = 2(e+x) - e^{2(1-x)} - (1+e)e^{1-2y}$ $\quad c = 0$

i) point $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$ *

$\frac{\partial L}{\partial x} = 2 + 2e^{2(1-x)} + \lambda_1\left(-2 - 2e^{2x}\right) + \lambda_2 - \lambda_3 = 0$

$\frac{\partial L}{\partial y} = 2e^{2-2y} + \lambda_1\left(-2e^{2-2(1-y)}\right) + \lambda_4 - \lambda_5 = 0$

reeplace $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $c = 0$

$L'_x = 2 + 2e^{2-1} + \lambda_1 \cdot -2(1 + e^{2 \cdot 1/2}) + \lambda_2 - \lambda_3 = 0$

$2 + 2e - 2\lambda_1(1 + e) + \lambda_2 - \lambda_3 = 0$ ... (1)

$L'_y = 2e - 2\lambda_1 e + \lambda_4 - \lambda_5 = 0$

$2e - 2\lambda_1 e = \lambda_5 - \lambda_4$ ... (2)

<u>slackness</u> :

$\lambda_1 \geq 0$, $\underbrace{c}_{0} - \underbrace{U(1-x, 1-y)}_{0} = 0$ ✓

$\lambda_2 \geq 0$, $\lambda_2 = 0$ $\quad 0 < \frac{1}{2}$

$\lambda_3 \geq 0$, $\lambda_3 = 0$ $\quad x < 1$

$\lambda_4 \geq 0$, $\lambda_4 = 0$ $\quad 0 < \frac{1}{2}$

$\lambda_5 \geq 0$, $\lambda_5 = 0$ $\quad \frac{1}{2} < 1$

* $\left(\frac{1}{2}, \frac{1}{2}\right)$ indeed satisfies all the conditions ✓

reeplacing in (2) : $2e - 2\lambda_1 e = 0 - 0 \Rightarrow \lambda_1 = 1$ ✓ ≥ 0

in (1) : $\cancel{2+2e} - \cancel{2-2e} + 0 - 0 = 0$

$0 = 0$ ✓

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P4

b)

iii) $\Delta c \rightarrow -0,03 = \varepsilon$

by the envelope theorem:

$$\frac{\partial f^*(c)}{\partial c} \quad \frac{\partial \mathcal{L}^*}{\partial c} = \lambda$$

$$\Rightarrow \frac{\partial u^*}{\partial c} = \frac{\partial \mathcal{L}^*}{\partial c} = \lambda_1 = 1$$

Δ then the optimal value changes approx by

$$\simeq \lambda_1 . \varepsilon = 1 . -0.03 = 0.03$$