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A NOTE ON IMPOSING STRONG COMPLEMENTARY SLACKNESS CONDITIONS IN DEA

by

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Abstract: A new DEA model has been introduced recently combining the primal and the dual models in order to impose strong complementary slackness conditions. It was claimed that a reference set that contains the maximum number of efficient units can then be determined. The model is very interesting as a theoretical idea. However, not only does the computational burden increase significantly, but it seems also that the basic matrices may be inherently ill-conditioned, leading to wrong results. Numerical experiments have been carried out on two real datasets of medium size with 163 and 920 units. These experiments show pervasive existence of ill-conditioned matrices leading to obviously wrong estimates of efficiency scores, and units declared as efficient reference units while actually being inefficient.

Keywords: Data envelopment analysis, BCC model, DEA/SCSC model, strong complementary slackness conditions

JEL classifications: C61, D20

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1. Introduction

In a series of recent papers (Sueyoshi and Sekitani 2007a, b; Sueyoshi and Sekitani 2009; Sueyoshi and Goto 2010) a new DEA model was introduced combining the primal and dual DEA models and imposing strong complementary slackness conditions. A main purpose of the new model, termed DEA/SCSC in Sueyoshi and Goto (2010, p. 3), was to identify all possible optimal solutions, i.e. to find all units in the reference sets for each unit under study.

The new DEA model is very interesting as a theoretical idea. However, Sueyoshi and Sekitani (2007a, p. 1941) and (2009b, p.783) underlined a drawback with the method that it increases the computational burden, thus the proposed formulation needs a considerable computation time in solving a large data set. Indeed, there seems to be some serious numerical problems with their approach. By including constraints securing that strong complementary slackness conditions are obtained, the size of the problem increases significantly in comparison with the standard DEA model introduced in Banker et al. (1984) (BCC). Thus, the size of the proposed model and the inherent problem of comparability of measurement units may result in ill-conditioned basic matrices.

Although the wish to find complete solutions are mentioned by many DEA researchers, Cooper et al. (2006, p. 125) warn against trying to find all solutions by stating “Chasing down all optimal solutions can be onerous.” The purpose of this note is to follow up this remark and address problems that may be encountered by applying a model that promises to find all optimal solutions, by conducting computational experiments, addressing middle-sized problems using two real-life datasets. The method of Sueyoshi and Sekitani seemingly works correctly for small datasets, like the constructed set in Sueyoshi and Sekitani (2009, p. 782), consisting of six units with two inputs and a single output, but in our experience not for medium-sized problems.

We will only use the BCC models as the reference models and only consider radial efficiency measures. It is underlined in Sueyoshi and Sekitani (2007b, pp. 558,559); (2009, p. 782); Sueyoshi and Goto (2010, p. 4) that the new model shall give identical efficiency scores as found by solving the BCC model. We will use this as a criterion when evaluating the results of the new model.

The plan of the paper is as follows: In Section 2 we present the BCC models and the DEA/SCSC model. The computational experiments including a comparison between BCC and DEA/SCSC results for efficiency scores, reference sets and dual variables are presented in Section 3. Concluding remarks and ideas for future research are presented in Section 4.

2. Strong complementary slackness

We will limit our investigation to a BCC model extended to the DEA/SCSC model, following Sueyoshi and Sekitani (2007b).[1] The primal and dual versions of the BCC model, specified as input-oriented, are:

$$\begin{aligned}
&\min \theta \\
&\text{subject to} \\
&\sum_{j=1}^{n} X_j \lambda_j \leq \theta X_0 \\
&\sum_{j=1}^{n} Y_j \lambda_j \geq Y_0 \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geq 0 , \quad j = 1, \ldots, n
\end{aligned} \tag{1a}$$

[1] Sueyoshi and Sekitani (2007a) and Sueyoshi and Sekitani (2009) investigated non-radial models like the additive model, the latter paper also investigating the BCC model, while Sueyoshi and Sekitani (2007b) investigated only the BCC model.

$$\begin{aligned}
&\max\,(u^T X_0 + u_0)\\
&\text{subject to}\\
&u^T Y_j - v^T X_j + u_0 \le 0,\ j = 1,...,n \qquad \text{(1b)}\\
&v^T X_0 = 1\\
&v_k \ge 0,\ k = 1,\ldots,m,\ \ u_i \ge 0,\ i = 1,\ldots,r
\end{aligned}$$

Sueyoshi and Sekitani (2007a), (2007b) proposed to use strong complementary slackness conditions (SCSC) of the linear programming as a computational procedure in order to find all vertices of every face in DEA model. Their main model, DEA/SCSC, in Sueyoshi and Sekitani (2007b) takes the following form:

$$\max\left\{ \eta \;\middle|\; \begin{array}{l}
\theta X_o - \sum_{j=1}^{n}\lambda_j X_j \ge 0,\ \sum_{j=1}^{n}\lambda_j Y_j \ge Y_o,\ \sum_{j=1}^{n}\lambda_j = 1,\\
\lambda_j \ge 0,\ j = 1,\ldots,n,\\
v^T X_o = 1, -v^T X_j + u^T Y_j + u_0 \le 0,\ j = 1,\ldots,n,\\
v \ge 0, u \ge 0,\\
\theta = u^T Y_o + u_0,\\
\lambda_j + v^T X_j - u^T Y_j - u_0 \ge \eta,\ j = 1,\ldots,n,\\
v - \sum_{j=1}^{n}\lambda_j X_j + \theta X_o \ge \eta,\\
u + \sum_{j=1}^{n}\lambda_j Y_j - Y_o \ge \eta, \quad \eta \ge 0
\end{array}\right\} \qquad (2)$$

The first four conditions are from the primal model (1a), and the next three conditions are from the dual model (1b). The condition $\theta = u^T Y_o + u_0$ locks the solution of the efficiency score of the primal model to the optimal value of the objective function of the dual model. The last three conditions express the SCSC constraints. In order to secure that strong complementarity is obtained the variable η is entered as the objective function in (2) and also in the three last constraints (Sueyoshi and Sekitani 2007b, p. 559; 2009, p. 782).

Their method is very interesting approach as a theoretical idea. However, it may not be efficient from computational point of view, especially for the large-scale problems. The size of the model (2) increases significantly in comparison with the BCC model; to be more exact, the size of model (2) is measured by the total number of rows multiplied with the total number of columns $(2m+2r+2n+3)\times(m+r+n+3)$, where the number of inputs is *m*, the number of outputs is *r*, and the number of production units is *n*. Remember that the size of the BCC model is $(m+r+1)\times(n+1)$, and $(m+r)$ is usually much less than n.

Moreover, economic interpretation of some constraints of model (2) does not make sense because in model (2) one has to add variables measured in quite different units during the solution process; this is without meaning. The two aspects pointed out above may result in ill-conditioned basic matrices.

3. Numerical experiments

We first investigated the behaviour of the DEA/SCSC model by using a constructed dataset of only five units and two inputs and a single output taken from Krivonozhko and

*Table 1. Constructed data**

Variables	A	B	C	D	E	F
Input 1	5/4	1	3	5	2	4
Input 2	5/4	3	1	5	2/3	4
Output	9/8	3/2	3/2	3	1/2	3/2

*Source: Krivonozhko and Førsund (2009)

Førsund (2009).[2] This dataset is of about the same dimension as the constructed datasets in Sueyoshi and Sekitani (2009, p.782).[3] The solutions of the two models for the efficiency score and the dual variables are identical with efficiency scores both equal to

[2] To the dataset shown in Figure 1 there we add an inefficient unit *F* (4, 4, 3/2).
[3] We also run this dataset having six units and two inputs and a single output, using our software and got the same results (Sueyoshi and Sekitani (2009, p.783), confirming the correct performance of the DEA/SCSC model for small datasets.

0.5, all dual variables for input and output constraints are positive and equal, and the dual variables for the convexity constraint are equal. However, the units of the reference sets differ, having one more unit in the reference set of the DEA/SCSC model in addition to the same two reference units in the BCC model. This is to be expected. Naturally, the weights then differ. In principle, if the reference set is unique we should get the same reference set for the two models. However, when there are multiple reference sets the DEA/SCSC model will give us all the reference units, so we would expect the latter model to give us more reference units. The reference units appearing in the solution of the BCC model should then be included in the reference set given by the DEA/SCSC model.

In order to investigate our suspicions about what will happen when using larger real datasets, we conducted computational experiments using two middle-sized models. For the first model, call it Model 1, we took the data for electricity utilities in Sweden 1987; see Førsund et al. (2007). Max, min and mean statistics are shown in Table 1. The number of production units in this model is 163.

Table 2. Data for electricity utilities, Sweden 1987

Variables	**Mean**	**St. deviation**	**Min**	**Max**	**Unit 1**	**Unit 104**
Outputs						
MWh low voltage	286057	3454887	9190	4895138	160604	219398
MWh high voltage	665979	46644285	0	65966223	47863	46140
No of customers low voltage	22841	225909	695	422793	11720	17302
No of customers high voltage	36	641	0	908	18	18
Inputs						
Labour, man years	133	6493	2	9189	35	58
Km of low voltage lines	1168	21159	21	30033	716	883
Km of high voltage lines	989	40783	8	57733	177	560
Transformer capacity in kVA	155434	1801496	4000	2554000	79000	136700

In our computational experiments we used optimisation software CPLEX (Moré and Wright 1993), one of the best optimisation programs, and software FrontierVision, a specially elaborated program by our group at the Institute for Systems Analysis, Moscow, for DEA models that enables one to visualize the multidimensional frontier with the help of the construction of two- and three-dimensional sections of the frontier.

Sueyoshi and Sekitani (2007b) recommended solving model (2) for every production unit in the model. We have followed this recommendation. Consider the results for two electricity distribution units, 1 and 104, chosen randomly. Inputs and outputs for these units are given in Table 2. Solving the BCC input-oriented model (1a) for unit 104, CPLEX software produces the following optimal solution:

$$\theta^* = 0.5869,\ \lambda^*_{139} = 0.5257,\ \lambda^*_{157} = 0.1913,\ \lambda^*_{255} = 0.211,\ \lambda^*_{271} = 0.072, \qquad (3)$$

where λ^*_j is the j th optimal variable, all other λ-variables are equal to zero. Observe that the FrontierVision program produces the same solution (3).

Solving the DEA/SCSC model (2) for unit 104, CPLEX software produces the following optimal solution:

$$\theta^* = 0.5924,\ \lambda^*_{22} = 0.02,\ \lambda^*_{139} = 0.46,\ \lambda^*_{144} = 0.0057,\ \lambda^*_{157} = 0.13,$$

$$\lambda^*_{203} = 0.019,\ \lambda^*_{246} = 0.086,\ \lambda^*_{255} = 0.185,\ \lambda^*_{271} = 0.088, \qquad (4)$$

here again λ^*_j is the j th optimal variable, all other λ-variables in the optimal solution are equal to zero.

The efficiency scores are close, but not identical as should follow from the restrictions in (2). At first sight solution (4) gives us a reference set for inefficient unit 104. We note that all four reference units of the BCC model are also reference units in the DEA/SCSC model, but also four more are included in the latter model. However, it turned out that unit 144 and unit 203 in the reference set of the DEA/SCSC model are inefficient; their efficiency scores are 82 % and 95 %, respectively calculated in the BCC model. Moreover, there is internal inconsistency within the DEA/SCSC model because in these model units 144 and 203 have efficiency scores of 83 % and 97 %, respectively, although all the reference units had to have efficiency scores of 1.

Inspecting the solutions of the dual variables for the two models set out in Table 3 we see

Table 3. Solutions for dual variables for unit 104 solving the BCC model and the DEA/SCSC model

Model	Dual variables for inputs				Dual variables for outputs			
	v_1	v_2	v_3	v_4	u_1	u_2	u_3	u_4
BCC	$3.52 \cdot 10^{-3}$	$2.33 \cdot 10^{-4}$	0	$4.32 \cdot 10^{-6}$	$3.18 \cdot 10^{-6}$	0	0	0
DEA/SCSC	$4.11 \cdot 10^{-3}$	$2.03 \cdot 10^{-4}$	0	$4.25 \cdot 10^{-6}$	$2.6 \cdot 10^{-6}$	0	$7.41 \cdot 10^{-6}$	0

that the solutions for some of the dual variables are equal to zero, but one less in the DEA/SCSC model.

Figure 1 represents an intersection of the eight-dimensional production possibility set

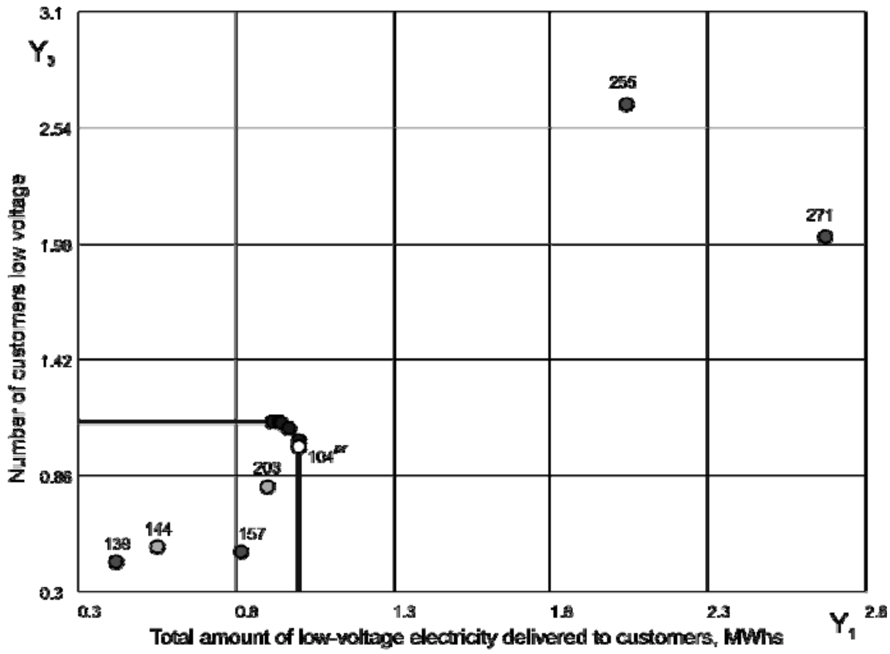

Figure 1. Input isoquant for unit 104^{pr}, the projection of unit 104 onto the frontier

with a two-dimensional plane for unit 104^{pr}, the projection of unit 104 on to the frontier in the BCC input-oriented model, where the directions of the plane are determined by the following outputs: total amount of low-voltage electricity delivered to customers and number of customers low voltage. The light and dark points in the figure denote projections of units from the reference sets onto the two-dimensional plane. Unit 139, unit 157, unit 255 and unit 271 belong to the reference set obtained with the help of the BCC input-oriented model (1a). The light unit 144 and unit 203 are inefficient; they are included in the reference set according to model (2).

Increasing the scale of Figure 1 in order to observe the details we obtain Figure 2.

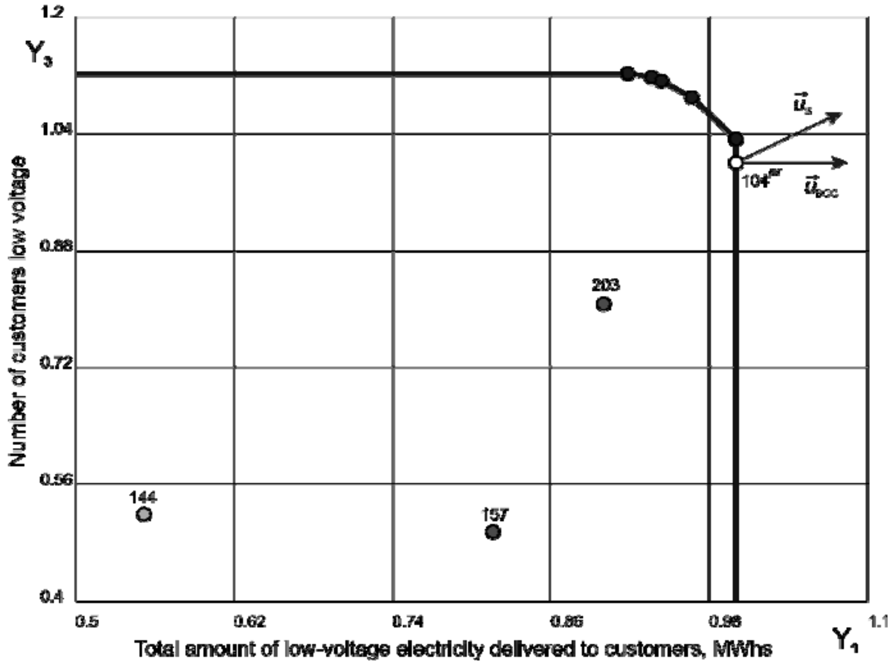

Figure 2. Vectors of supporting hyperplanes for unit 104^{pr}

Vector $\vec{u}_{BCC}$ is the vector of dual variables that determines the supporting hyper-plane at point 104^{pr}. Vector $\vec{u}_{BCC}$ is perpendicular to the slack facet at point 104^{pr}, since component u_{3BCC} of vector $\vec{u}_{BCC}$ is zero (Table 3), and the corresponding slack variable s_3^+ is nonzero, this agrees completely with the strong complementary slackness conditions, see Cooper et al. (2006). However vector $\vec{u}_S$ is not perpendicular to the slack facet since both u_1 and u_3 are positive (u_2 and u_4 zero), according to model (2) (Table 3). So, the strong complementary slackness conditions are not satisfied, in the sense that both variables from the dual pair u_3 and s_3^+ are positive, this is because the model (2) generates ill-conditioned basic matrices during the solution process.

The point is that when “astute” mathematicians write strong complementary slackness conditions, they consider them just as conditions, i.e. they keep in mind that only one variable of the dual pair of variables is nonzero. The situation is quite another if somebody uses SCSC model as a solution procedure, in this case one has to add quite different variables measured in different units during the solution process.

Solving the input-oriented problem BCC model (1a) for unit 1, the CPLEX program produced the following optimal variables:

$$\theta^* = 0.78\,,\ \lambda^*_{82} = 0.063\,,\ \lambda^*_{119} = 0.012\,,\ \lambda^*_{139} = 0.269\,,$$
$$\lambda^*_{246} = 0.220\,,\ \lambda^*_{255} = 0.041\,,\ \lambda^*_{286} = 0.403, \qquad (5)$$

all other optimal λ-variables are equal to zero. FrontierVision gives us the same result.

However, solving DEA/SCSC model (2) by CPLEX program for unit 1, we obtained the following result:

$$\theta^* = 0.805\,,\ \lambda^*_{33} = 0.072\,,\ \lambda^*_{42} = 0.065\,,\ \lambda^*_{103} = 0.100\,,\ \lambda^*_{139} = 0.223\,,$$
$$\lambda^*_{157} = 0.027\,,\ \lambda^*_{185} = 0.012\,,\ \lambda^*_{190} = 0.047\,,\ \lambda^*_{246} = 0.098\,, \qquad (6)$$
$$\lambda^*_{255} = 0.155\,,\ \lambda^*_{275} = 0.013\,,\ \lambda^*_{286} = 0.183\,,$$

all other λ-variables in the optimal solution are equal to zero.

The efficiency scores are close, but not identical. Again, it seems that we obtain the reference set for unit 1. In the BCC model unit 1 has 6 reference units and 11 in the DEA/SCSC model, and three of them appear in both in sets. However, again one unit in the DEA/SCSC reference set, Unit 185, turned out to be inefficient; its efficiency score is 83 % in the BCC model.

Inspecting the solutions of the dual variables for the two models set out in Table 4 we see that the solutions for four of the dual variables are equal to zero. The DEA/SCSC model produces three dual variables for outputs equal to zero, two more than for the BCC model. This is a rather strange result and one may suspect that such a result is created by ill-conditioned matrices.

Table 4. Solutions for dual variables for Unit 1 solving the BCC model and the DEA/SCSC model

Model	Dual variables for inputs				Dual variables for outputs			
	v_1	v_2	v_3	v_4	u_1	u_2	u_3	u_4
BCC	$2.89 \cdot 10^{-3}$	0	$1.35 \cdot 10^{-3}$	$8.33 \cdot 10^{-6}$	$4.84 \cdot 10^{-6}$	$1.16 \cdot 10^{-6}$	0	$1.76 \cdot 10^{-3}$
DEA/SCSC	$2.4 \cdot 10^{-3}$	0	$1.26 \cdot 10^{-3}$	$8.7 \cdot 10^{-3}$	$5.6 \cdot 10^{-6}$	0	0	0

For Model 2 we took the data from 920 Russia bank's financial accounts for January 2009, where we use the following inputs and outputs for the BCC model:

Inputs: working assets; time liabilities; demand liabilities.

Outputs: equity capital; liquid assets; fixed assets.

Max, min and mean statistics for banks are shown in Table 5.

Table 5. Data for banks Russia 2008

Variables	**Mean**	**St. deviation**	**Min**	**Max**	**Unit 1**	**Unit 353**
Outputs						
Liquid assets, ths rubles	4279490	30304201	73	717402532	82362674	35755186
Equity capital, ths rubles	2205806	23572632	423	632286730	28283056	24829951
Fixed assets, ths rubles	608481	7414069	42	221058541	4551402	6225750
Inputs						
Demand liabilities, ths rubles	11318997	140641585	0	4184548095	102656087	74148463
Time liabilities, ths rubles	18289244	162725433	1	4213176749	424810499	191632992
Working assets, ths rubles	24587080	230385425	0	6233536293	484631606	249211165

We have solved the BCC input-oriented model (1a) and the DEA/SCSC model (2) for a large group of banks.

Consider a typical solution for unit 353 chosen in a random manner. Solving the BCC input-oriented model (1a) for unit 353 by CPLEX software we obtain the following optimal variables:

$$\theta^* = 0.72\,, \lambda^*_{47} = 0.0886\,, \lambda^*_{180} = 0.7686\,,\ \lambda^*_{199} = 0.0542\,,\ \lambda^*_{476} = 0.0794\,,\ \lambda^*_{951} = 0.009 \qquad (7)$$

all other optimal λ-variables are equal to zero. FrontierVision produced the same result.

Solving model (2) for unit 353 by CPLEX software, we obtain the following optimal variables:

$\theta^* = 1.16$, $\lambda^*_{67} = 0.8994$, $\lambda^*_{126} = 0.00086$, $\lambda^*_{140} = 0.00095$, $\lambda^*_{199} = 0.0303$, $\lambda^*_{528} = 0.0674$, (8)

all other λ-variables are equal to zero.

The solutions for the efficiency score are quite different; in fact the efficiency score in the DEA/SCSC model is outside the range giving meaning being greater than 1. Again, production units associated with variables (8) do not form a reference set since $\theta^* > 1$ and unit 67, unit 126 and unit 140 are inefficient; their efficiency scores are 93%, 2%, 17%, respectively.

Figure 3 depicts an intersection of the six-dimensional production possibility set with a

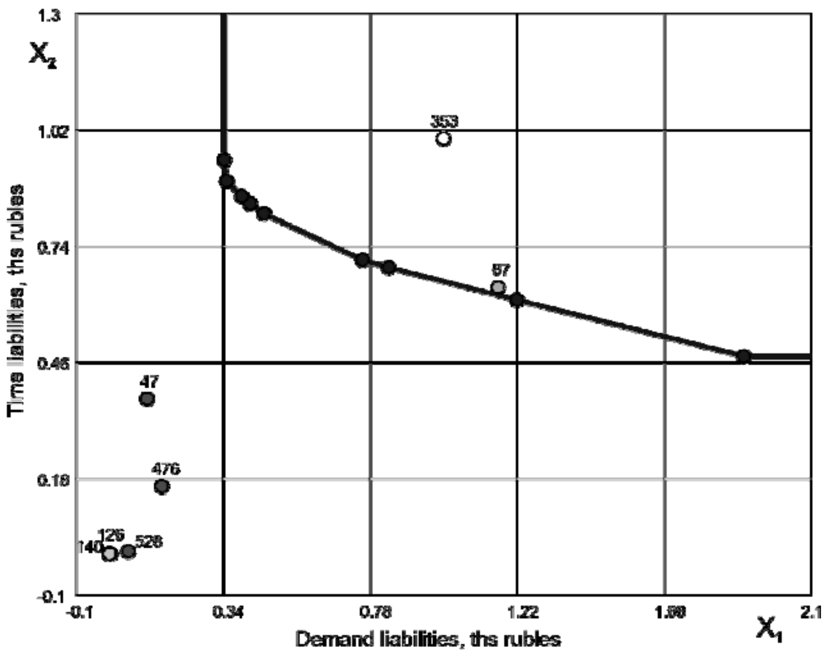

Figure 3. Output isoquant for bank 353

two-dimensional plane for bank 353, where the direction of the plane is determined by the following inputs: demand liabilities and time liabilities. The light unit 67, unit 126 and unit 140 are inefficient. These units are included in the reference set according to model (2).

All sections of the multidimensional production possibility set with two-dimensional planes were constructed using software FrontierVision.

As a second unit from the bank data we picked unit no. 1. Solving the BCC input-oriented model (1a) for Unit 1 by CPLEX software we obtain the following optimal variables:

$$\theta = 0.6436,\ \lambda_{199} = 0.1417,\ \lambda_{418} = 0.0345,\ \lambda_{528} = 0.0705,\ \lambda_{951} = 0.1183, \qquad (9)$$

all other optimal λ- variables are equal to zero.

Solving model (2) for Unit 1 by CPLEX software, we obtain the following optimal variables:

$$\theta = 0.6802,\ \lambda_{48} = 0.1629,\ \lambda_{199} = 0.1199,\ \lambda_{476} = 0.0402,\ \lambda_{528} = 0.5201,$$

$$\lambda_{898} = 0.0496,\ \lambda_{951} = 0.1069, \qquad (10)$$

all other optimal λ-variables are equal to zero. The efficiency scores differ, and of the four units appearing in the reference set in the BCC model three appears also in the reference set of the DEA/SCSC model together with three new ones. However, again this optimal solution do not form a proper reference set, since unit 48 and unit 898 are inefficient, their efficiency scores are equal to 91 % and 78 %, respectively.

Inspecting the solutions of the dual variables for the two models set out in Table 4 we see

Table 6. Solutions for dual variables for unit 1 solving the BCC model and the DEA/SCSC model

Model	Dual variables for inputs			Dual variables for outputs			Convexity constr.
	v_1	v_2	v_3	u_1	u_2	u_3	u_o
BCC	$7.51 \cdot 10^{-9}$	0	$4.71 \cdot 10^{-10}$	$2.61 \cdot 10^{-9}$	$8.00 \cdot 10^{-9}$	0	-0.027
DEA/SCSC	$1.34 \cdot 10^{-8}$	0	$2.06 \cdot 10^{-9}$	0	$2.42 \cdot 10^{-8}$	0	-1

that the solutions for some of the dual variables are equal to zero, especially for outputs, having one more zero than for the BCC model, again indicating problems with obtaining a proper optimal solution.

Let us try to reveal the reasons why CPLEX program does not produce reliable solutions for model (2). Consider some constraints of model (2)

$$u + \sum_{j=1}^{n} \lambda_j Y_j - Y_o \geq \eta. \tag{11}$$

In Model 1 components of output vector Y_j are measured in the following units: MWh low voltage, MWh high voltage, number of customers, low voltage, and number of customers, high voltage, respectively. Components of the weight vector u is measured in the following units: (efficiency score)/MWh, (efficiency score)/(number of high-voltage customers), (efficiency score)/(number of low-voltage customers)[4]. The variables λ_j and η are dimensionless. Hence in relation (11) one has to add (or subtract) quite different variables during the solution process, for example: (efficiency score) / MWh, MWh, and a dimensionless variable. So, economic interpretations of some constraints of model (2) do not make sense. If different units of measurement are chosen this changes the numerical sensitivity of the basic matrices.

Moreover, the size of basic matrices of model (2) is significantly increased. For Model 1 the size of basic matrices (considered as square matrices) is 345×345, at the same time the size of the basic matrices of the BCC model (1a) is 9×9. For the Model 2 the size of the basic matrices of the DEA/SCSC problem (2) is 1855×1855, and the size of the basic matrices for the BCC model (1a) is 7×7.

As a consequence, the *condition number* of the basic matrices for model (2) also increases significantly. Remember that the condition number of a matrix (Wilkinson, 1965) characterizes the sensitivity of the solution of a system of linear equations with respect to this matrix and the right-hand side. The more value of the condition number corresponds to the more ill-conditioned matrix. The condition number of a square nonsingular matrix A is determined as $\| A \|_2 \| A^{-1} \|_2$.

[4] The efficiency score is dimensionless or measured in percent.

In our computational experiments with the BCC model and the DEA/SCSC model we calculated condition numbers for basic matrices of the BCC model (1a) and model (2) with the help of the software Mathematica 6.0. On the average, the increase of condition number values of model (2) in comparison with the BCC model (1a) is by factor 10^2 to 10^3.

Thus, basic matrices of model (2) are ill-conditioned even for the middle-sized problems. This explains why CPLEX program may not produce correct solutions using the DEA/SCSC model (2).

4. Conclusions

The motivation for introducing the DEA/SCSC model by Sueyoshi and Sekitani (2007a, b) and (2009) was to obtain complete solutions of efficient units being in the reference set of an inefficient unit. Although Sueyoshi and Sekitani (2009) and (2007b) pointed to increases the computational burden, no mentioning of potentially ill-conditioned basic matrices was done. However, we have demonstrated, first by pointing out the magnitude of the increase in the dimension of the basic matrices, and then by carrying out numerical experiments on medium-sized real data, that ill-conditioned matrices may easily occur and make valueless solutions offered by the DEA/SCSC model. Efficiency scores of the models differed in spite of the restriction that should obtain equality, and the DEA/SCSC model declared units, that where inefficient in the BCC model, actually belonging to the reference set when solving the DEA/SCSC model, i.e., the units were efficient according to the DEA/SCSC model, again violating the theoretical restriction.

We should also mention the inherent problem of the dimension of the variables in the strong complementary slackness constraints of the DEA/SCSC model. When a model violates such a fundamental feature that only variables with the same unit of measurement can be added up, then it is to be expected that ill-conditioned basic matrices may occur.

Clearly, for the DEA/SCSC model to be applied successfully for real datasets larger than artificial small-scale data sets a special solution algorithm is required. The computational procedure proposed in Sueyoshi and Sekitani (2007b) is not really helpful because Step 1 there assumes that an optimal solution to (2) is found. Further research is warranted.

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