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On the Economics of the Optimal Fallow-Cultivation Cycle

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On the Economics of the Optimal Fallow-Cultivation Cycle

Yngve Willassen[1]

March 2001

Abstract

This paper considers the so-called fallow-cultivation problem in traditional agriculture. The paper specifies the problem and determines the optimal length of the fallow-cultivation cycle. We also show the impact of rising output prices on this cycle. This has been widely discussed in the literature, but we conclude that previous stated results do not hold. The fallow-cultivation problem is an important economic problem, since traditional farming provides the food in most low-income countries. However, it is also worth studying since it is an interesting specimen case in a class of dynamic models in which sensible controls split process variables in distinctive phases.

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1. Introduction

In the present paper we specify and solve the so-called fallow-cultivation problem in traditional agriculture. When fields are used for pasture or crops they are gradually depleted for substances which are essential for the plants to grow. In modern agriculture farmers make up for this wear out effect by applying artificial fertilizer, so they can cultivate the fields more or less continuously. The traditional peasant, however, has no access to fertilizer. When the fields are depleted, they have to lie fallow to recover, and only after a certain length of time can the peasant restart cultivation. Thus, we realize that a main problem facing the peasant is to determine the time pattern for switching: (i) from fallow to cultivation, (ii) from cultivation to fallow.

Indeed, this is an important economic problem since traditional farming provides the food in most low-income countries. In addition this problem can properly be considered a specimen case of a set of interesting economic models, where almost any sensible control strategy splits important process variables into distinctive phases. For example firms running an active training program to improve the working abilities of their staff must decide when employees should work and when they should be in a program. In the active phase the employees work but their efficiency will gradually decrease as their know-how and skill become obsolete. While they are in a training program (the passive phase) the workers do not produce but they build up their abilities (human capital) steadily and thus become more efficient when they start to work. Similar ideas underpin the so-called return migration theory which tries to explain a new trend in the international flows of workers between countries.

The fallow-cultivation model has similarities with Whittle's "Restless Bandits" (Whittle (1988)) and Dixit and Pindyck's mothball models (Dixit and Pindyck (1994)). But in the literature on dynamic economic models it has more in common with Faustmann's model for a rotating forest (Clark (1976)). However, compared to Faustmann's model the fallow-cultivation model is more general. For a rotating forest one optimizes over one option only: when to cut the forest, since when the forest is cut one always return to a standard initial state (Willassen (1998)). In the fallow-cultivation model we optimize over two options: (i) when to start cultivation if the fields lie fallow, and (ii) when to lay the fields fallow if we are cultivating.

Below we shall analyse this model thoroughly. We sketch the following plan of the paper: In section 2 we specify the fallow-cultivation model, in section 3 we derive the feasible strategy and show its optimality. Environmentalists have for years been interested in the effect of changes in output prices on the way peasants run their fields. In section 4 we clarify the impact of varying output prices on the optimal strategy and show that previously published results are not sustainable (Barret (1991), p. 183). In section 5 we discuss the relation between the output price and soil fertility. Finally, in section 6 we illustrate our propositions and results by numerical experiments.

2. Formulation

We consider a specific piece of farmland whose soil fertility is summarized by an index variable (x). Since the farmer does not fertilize the fields, crop cultivation will gradually exhaust the soil causing the index variable (x) to diminish. Eventually the index will reach a level where further cultivation is unprofitable and the farmer will lay the fields fallow. During the fallow period the farmer receives no income, but the soil recuperates.

We realize that we face an optimization problem characterized by two phases:

1. A crop cultivation period (the active phase) in which the farmer collects income by selling the harvests but the fields deteriorate.

2. A fallow period (the passive phase) in which the farmer receives no income but the soil recuperates.

Thus, for a given set of exogenous variables we wish to optimize over the options of being active or being passive. In the modelling of this problem we need the variables:

$x(t)$	-	an index variable indicating the amount of crops which potentially can be produced on the fields at time t.
p	-	the price of crops (assumed constant).
ρ	-	interest rate (constant).
c	-	a constant running cost incurred when cultivating the fields.
d_{ij}	-	fixed costs incurred by switching between the two phases $(i, j = 1, 2)$, so $d_{11} = d_{22} = 0$ while $d_{12} > 0, d_{21} > 0$.

A central feature in this model is that the index $x(t)$ diminishes in the active phase and increases in the passive phase. Specifically, we model the state variable by specifying two linear differential equations. We assume:

$$\dot{x} = -\mu x \qquad \text{(during crop cultivation)} \tag{2.1}$$

$$\dot{x} = \lambda (K - x) \qquad \text{(when the fields lie fallow)} \tag{2.2}$$

In eqs. (2.1)-(2.2) μ and λ are positive constants and K in (2.2) denotes an upper bound on the produce of crops. Finally, to characterize a switching strategy the indicator function χ_t is defined by:

$$\chi_t = \begin{cases} 1 & \text{if the fields are cultivated at time } t \\ 0 & \text{if the fields lie fallow at time } t \end{cases} \tag{2.3}$$

For a given starting point (t, x_t, χ_t) a switching or rotation strategy is given by the double sequence:

$$\omega = (\theta_1, \theta_2, \dots, \theta_k, \dots; x_{\theta_1}, x_{\theta_2}, \dots, x_{\theta_k, \dots}) \tag{2.4}$$

where $\theta_k (k = 1, 2, \dots)$ denotes stopping (switching) times satisfying $\theta_k \leq \theta_{k+1}$, and x_{θ_k} denotes the value of the associated index (state variable) at time θ_k. To the strategy (2.4) we associate a discounted total reward given by:

$$J^\omega = \int_t^\infty (px^\omega(s) - c)\, e^{-\rho s} \chi(s)\, ds - \sum_{j=1}^\infty e^{-\rho\theta_j} \left(d_{12}\chi_{\theta_j} + d_{21}\left(1 - \chi_{\theta_j}\right)\right) \tag{2.5}$$

Finally, we can state the farmer's optimization problem in the following way:

Find for all initial states (t, x_t, χ_t) the value function $\phi(\cdot)$ defined by:

$$\phi(t, x, \chi) = \max_\omega J^\omega(t, x, \chi) \tag{2.6}$$

3. Determination of the optimal fallow-cultivation strategy

From the specification above it follows that the soil fertility (state variable) $x(t)$ is restricted to the interval $[0, K]$. Hence, for any feasible strategy (ω) the discounted reward $J^{(\omega)}$ (2.5) is finite. The value function $\phi(t, x, \chi)$ (2.6) will therefore satisfy the relevant dynamic programming (DP) equation. The following candidate for the value function suggest itself:

$$\phi(t, x, \chi) = e^{-\rho t} F(x, \chi) \tag{3.1}$$

where the value function $F(x, \chi)$ splits into:

$$F(x, \chi) = F_1(x)\, \chi + F_2(x)(1 - \chi) \tag{3.2}$$

In (3.2) $F_1(x)$ denotes the value function attained by being in a crop cultivation phase at (t, x), similarly $F_2(x)$ is the value function attained by being in a fallow phase at (t, x), i.e. $F(x, 1) = F_1(x)$ and $F(x, 0) = F_2(x)$. The value of χ has to be incorporated in the state description in order to deduce the optimality equation for $F(x, \chi)$. Below this has been achieved by deriving the optimality equations for $F_1(x)$ and $F_2(x)$ separately.

Hence, we start at (t, x) and consider a short time interval Δt. By using (2.5) the optimality equations for the value functions for the two phases become:

$$F_1(x) = \max_i \{(px - c)\Delta t - d_{1i} + e^{-\rho\Delta t} F_i(x(t + \Delta t))\} + o(\Delta t) \tag{3.3}$$

$$F_2(x) = \max_i \{e^{-\rho\Delta t} F_i(x(t + \Delta t)) - d_{2i}\} + o(\Delta t) \tag{3.4}$$

where $d_{11} = 0, d_{12} > 0$ and $d_{22} = 0, d_{21} > 0$.

The assumption that $d_{12} > 0$ and $d_{21} > 0$ means that one incurs costs by switching between the two phases.

In order to make the two optimality equations more amenable to analysis we apply standard arguments to the expressions in the curly brackets of (3.3) -(3.4), attaining:

$$F_1(x) = \max_i \{(px - c)\Delta t - d_{1i} + F_i - \rho\Delta t F_i + a_i(x)\Delta t F_{ix}\} + o(\Delta t) \tag{3.5}$$

$$F_2(x) = \max_i \{F_i - \rho\Delta t F_i + a_i(x)\Delta t F_{ix} - d_{21}\} + o(\Delta t) \tag{3.6}$$

where we have used the definitions:

$$F_{ix} = \frac{\partial F_i(x)}{\partial x} \qquad (i = 1, 2) \tag{3.7}$$

$$\dot{x}_i = a_i(x) = \begin{cases} -\mu x & (i = 1) \\ \lambda(K - x) & (i = 2) \end{cases} \tag{3.8}$$

and note that F_i is evaluated at x.

The max operation in the optimality equations means that being in either phase one has the two options: (a) continue in the present phase, or (b) switch to the alternative phase. Hence, being in phase 1 (3.5) has the following implications for the two options: If option (a) is optimal then (3.5) implies:

$$(px - c) - \rho F_1 - \mu x F_{1x} = 0 \tag{3.9}$$

$$F_1(x) \geq F_2(x) - d_{12} \tag{3.10}$$

If switching to phase 2 is optimal (option (b)), then (3.5) implies:

$$F_1(x) = F_2(x) - d_{12} \tag{3.11}$$

$$(px - c) - \rho F_1 - \mu x F_{1x} \leq 0 \tag{3.12}$$

Similarly, being phase 2 the optimality equation (3.6) has the implications: If option (a) is optimal (3.6) implies:

$$-\rho F_2 + \lambda (K - x) F_{2x} = 0 \tag{3.13}$$

$$F_2(x) \geq F_1(x) - d_{21} \tag{3.14}$$

If switching to phase 1 is optimal (3.6) implies:

$$F_2(x) = F_1(x) - d_{21} \tag{3.15}$$

$$-\rho F_2 + \lambda (K - x) F_{2x} \leq 0 \tag{3.16}$$

These DP equations (inequalities) give a natural determination of the optimal fallow-cultivation cycle. When it is optimal to cultivate crops, the value function $F_1(x)$ is determinated by (3.9) implying:

$$F_1(x) = \frac{px}{\rho + \mu} - \frac{c}{\rho} + A_1 x^{-\frac{\rho}{\mu}} \tag{3.17}$$

where A_1 is a positive constant.

In this phase $F_1(x) \geq F_2(x) - d_{12}$ and one continues cultivation until:

$$F_1(x_*) = F_2(x_*) - d_{12} \tag{3.18}$$

where x_* denotes a lower switch-point indicating when it is optimal to shift from cultivation to fallow.

Similarly, when it is optimal to be in fallow phase the value function $F_2(x)$ is determined by (3.13), implying:

$$F_2(x) = A_2 (K - x)^{-\frac{\rho}{\lambda}} \tag{3.19}$$

where A_2 is a positive constant.

In this phase $F_2(x) \geq F_1(x) - d_{21}$ and one remains there until equality is attained (3.15). That is:

$$F_2(x^*) = F_1(x^*) - d_{21} \tag{3.20}$$

where x^* denotes an upper switch-point indicating when one should shift from fallow to cultivation.

Thus the presumed optimal fallow-cultivation strategy ω^* has a simple characterization: There exists two thresholds x_* and x^* with $x_* < x^*$ which together with the indicator χ determine two continuation sets:

$$D_1 = \{(t, x, \chi) : \quad \chi = 1, \quad x \geq x_*\} \tag{3.21}$$

$$D_2 = \{(t, x, \chi) : \quad \chi = 0, \quad x \leq x^*\} \tag{3.22}$$

Thus D_1 and D_2 depict two curves on two parallel sheets in the state variable space.

But more can be deduced from the DP principle. If we define the difference function:

$$G(x) = F_1(x) - F_2(x) \tag{3.23}$$

(3.10), (3.11) and (3.14), (3.15) imply that for all x we have:

$$-d_{12} \leq G(x) \leq d_{21} \tag{3.24}$$

Hence, x_* corresponds to a minimum and x^* to a maximum of $G(x)$. Since $F_1(x)$ and $F_2(x)$ are differentiable, we also have:

$$G'(x_*) = 0 \tag{3.25}$$

$$G'(x^*) = 0 \tag{3.26}$$

The equations (3.18), (3.20) and (3.25), (3.26) make up the so-called high contact principle. This principle will give us a definitive proposal for the optimal fallow-cultivation cycle since it provides feasible values of A_1, A_2 appearing in the value functions $F_1(x)$ and $F_2(x)$ and of the thresholds x_* and x^* appearing in the two continuation sets D_i (3.21)-(3.22).

Finally, it is instructive to verify that the formal procedure presented above solve the problem (2.5)-(2.6). We summarize the facts in the following proposition.

Proposition 3.1 *Suppose that the value function F (3.2) satisfies the DP equation (inequalities) (3.9)-(3.16). Assume further that the high contact principle hold for feasible values of A_1, A_2, x_* and x^* (i.e. the A_i's are positive and $0 < x_* < x^* < K$). Then ϕ given by (3.1) solves the problem (2.5)-(2.6), and the strategy ω^* defined by the continuation sets D_i (3.21)-(3.22) are optimal.*

Proof First note that $\phi(t,x,\chi) = e^{-\rho t}F(x(t),\chi_t)$ is continuously differentiable except at the switching times $\{\theta_1, \theta_2, \ldots, \theta_j, \ldots\}$, and tends to zero as $t \to \infty$. Using the fundamental theorem of integral calculus we have:

$$\begin{aligned}
& J^{\omega}(t,x,\chi) - \phi(t,x,\chi) \\
& \quad = \int_t^{\infty} (px^{\omega}(s) - c)\, e^{-\rho s} ds - \sum_{j=1}^{\infty} e^{-\rho\theta_j} \left(d_{12}\chi_{\theta_j} + d_{21}\left(1 - \chi_{\theta_j}\right)\right) \\
& \quad + \int_t^{\infty} \left(-\rho F(x(s), \chi_s) + a_i(x(s)) F_{ix}\right) ds \\
& \quad + \sum_{j=1}^{\infty} e^{-\rho\theta_j} \left(F\left(x_{\theta_j}, \chi_{\theta_j^+}\right) - F\left(x_{\theta_j}, \chi_{\theta_j^-}\right)\right) \\
& \quad = \int_t^{\infty} \left(-\rho F(x(s), \chi_s) + a_i(x(s)) F_{ix} + (px^{\omega}(s) - c\chi_s)\right) e^{-\rho s} ds \\
& \qquad + \sum_{j=1}^{\infty} e^{-\rho\theta_j} \left(F\left(x(\theta_j), \chi_{\theta_j^+}\right) - F\left(x(\theta_j), \chi_{\theta_j^-}\right) - \left(d_{12}\chi_{\theta_j} + d_{12}\left(1 - \chi_{\theta_j}\right)\right)\right)
\end{aligned} \tag{3.27}$$

where F_{ix} and $a_i(x)$ are defined by (3.7)-(3.8).

By construction (3.12) and (3.16) hold, implying that the integrand is non-positive for all strategies ω and zero for $\omega = \omega^*$. Similarly, since (3.10) and (3.14) are assumed to hold the summands will also be non-positive for all ω, and equal to zero for $\omega = \omega^*$. Hence, we conclude that:

$$J^{\omega}(t,x,\chi) \leq \phi(t,x,\chi) = J^{\omega^*}(t,x,\chi) \tag{3.28}$$

which proves that ω^* in an optimal fallow-cultivation strategy. ■

So much for the formalism, but what about its economic substance? When we have determined the integration constants A_i and the two thresholds as explained above, we have at the same time determined the value functions $F_1(x)$ (3.17) and $F_2(x)$ (3.19) as well as the two continuation sets D_1 and D_2. We also note that $F_1(x)$ is defined on D_1 and $F_2(x)$ on D_2. In fact $F_1(x)$ and $F_2(x)$ summarize the values of the two options: (1) cultivating the fields, (2) letting the fields lie fallow. We also observe that $F_1(x)$ splits into two terms. The first one $\left(\frac{px}{(\rho+\mu)} - \frac{c}{p}\right)$ shows the present value of the future profits accuring by cultivating the fields without disruption, while the second $\left(A_1 x^{\frac{-\rho}{\mu}}\right)$ shows the increment attainable by utilizing the fallow option. It is obvious that A_1 must be positive. When the fields lie fallow there will be no immediate profit, so $F_2(x)$ derives its value exclusively from the future cultivation option. Again it follows that A_2 must be positive. We conclude that A_1 and A_2 for a given price (p) reflect the values of the two options.

Remark 3.1 *The following observation on A_1 and A_2 is also useful. Since A_1 and A_2 are determined by the high contact principle, i.e. eqs. (3.18), (3.20), (3.25), (3.26), it is evident that A_1 and A_2 depend on the price (p). Furthermore , since the reward J^{ω} (2.5) is linear in p, it follows from the general theory of optimization that the value functions F_1 (3.17) and F_2 (3.19) are non-decreasing and convex in p. This implies that $A_i \quad (i = 1, 2)$ are non-decreasing and convex.*

Although the relations between the option values and the output prices are important to the individual farmers, the environmentalists are more concerned about peasant's management of their fields and soil fertility when output prices vary. We shall discuss these questions in the next two sections.

4. Do the thresholds x_* and x^* depend on the output price?

Having shown explicitly how the optimal fallow-cultivation cycle can be deduced, we now wish to examine the relation between the thresholds x_* and x^* and the output price p. Indeed, it was this question which originally sparked off this study. How output prices influence the way peasants cultivate and manage their farmland have always interested environmenalists. Barret (op. cit.) gives a summary of the relevant literature. Evidently, there is no general agreement upon the impact of rising output prices on the peasants behavior. Some authors argue that increasing output prices will stimulate the peasant to strive for still larger profit, which inevitable will exhaust the fields. Contrary to this view others argue that as a result of larger profits peasants can afford to invest in better soil conservative methods. On both sides environmentalists have discussed this question without specifying an explicit model, so the discussion is hardly more than a listing of reasonable pro and con arguments. Barret's study is also an attempt to clarify this problem formally. Though his model seems incomplete since it doesn't include output prices, it reflects the tiring and recuperative aspects of the fallow-cultivation phases. He summarizes his results on the impact of rising output prices in his proposition 3 (p. 183) saying: "An unanticipated permanent increase in the output price will have no effect on the optimal fallow-cultivation cycle, and hence no effect on soil fertility and output".

Barret's arguments may seem reasonable, but it is not evident that his conclusion is correct. In order to clarify this question we shall apply some comparative state analysis to the model specified and studied in the preceding section. We start with the equations constituting the high contact principle, i.e. (3.18), (3.20), (3.25), (3.26).

$$\frac{px_*}{\rho+\mu}-\frac{c}{\rho}+A_1\left(x_*\right)^{-\frac{\rho}{\mu}}-A_2\left(K-x_*\right)^{-\frac{\rho}{\lambda}}=-d_{12} \tag{4.1}$$

$$\frac{px^*}{\rho+\mu}-\frac{c}{\rho}+A_1\left(x^*\right)^{-\frac{\rho}{\mu}}-A_2\left(K-x^*\right)^{-\frac{\rho}{\lambda}}=d_{21} \tag{4.2}$$

$$\frac{p}{\rho+\mu}-\frac{\rho}{\mu}A_1\left(x_*\right)^{-\left(\frac{\rho}{\mu}+1\right)}-\frac{\rho}{\lambda}A_2\left(K-x_*\right)^{-\left(\frac{\rho}{\lambda}+1\right)}=0 \tag{4.3}$$

$$\frac{p}{\rho+\mu}-\frac{\rho}{\mu}A_1\left(x^*\right)^{-\left(\frac{\rho}{\mu}+1\right)}-\frac{\rho}{\lambda}A_2\left(K-x^*\right)^{-\left(\frac{\rho}{\lambda}+1\right)}=0 \tag{4.4}$$

For given values of the economic variables $\{c, d_{12}, d_{21}, p, \rho\}$ and the process parameters $\{\lambda, \mu, K\}$, these equations are solved wrt. $\{x_*, x^*, A_1, A_2\}$. Then we vary the output price p and study its effects by applying the implicit function theorem to (4.1)-(4.4). We attain:

$$\left(x_*\right)^{-\frac{\rho}{\mu}}\frac{\partial A_1}{\partial p}-\left(K-x_*\right)^{-\frac{\rho}{\lambda}}\frac{\partial A_2}{\partial p}=-\frac{x_*}{\rho+\mu} \tag{4.5}$$

$$\left(x^*\right)^{-\frac{\rho}{\mu}}\frac{\partial A_1}{\partial p}-\left(K-x^*\right)^{-\frac{\rho}{\lambda}}\frac{\partial A_2}{\partial p}=-\frac{x^*}{\rho+\mu} \tag{4.6}$$

$$G''\left(x_*\right)\frac{\partial x_*}{\partial p}-\frac{\rho}{\mu}\left(x_*\right)^{-\left(\frac{\rho}{\mu}+1\right)}\frac{\partial A_1}{\partial p}-\frac{\rho}{\lambda}\left(K-x_*\right)^{-\left(\frac{\rho}{\lambda}+1\right)}\frac{\partial A_2}{\partial p}=-\frac{1}{\rho+\mu} \tag{4.7}$$

$$G''(x^*)\frac{\partial x^*}{\partial p}-\frac{\rho}{\mu}\left(x^*\right)^{-\left(\frac{\rho}{\mu}+1\right)}\frac{\partial A_1}{\partial p}-\frac{\rho}{\lambda}\left(K-x^*\right)^{-\left(\frac{\rho}{\lambda}+1\right)}\frac{\partial A_2}{\partial p}=-\frac{1}{\rho+\mu} \tag{4.8}$$

where

$$G''\left(x\right)=\left(\frac{\rho}{\mu}+1\right)\left(\frac{\rho}{\mu}\right)A_1\left(x\right)^{-\left(\frac{\rho}{\mu}+2\right)}-\left(\frac{\rho}{\lambda}+1\right)\left(\frac{\rho}{\lambda}\right)A_2\left(K-x\right)^{-\left(\frac{\rho}{\lambda}+2\right)} \tag{4.9}$$

Since we know that x_* corresponds to a minimum and x^* to a maximum of $G(x)$ we have:

$$G''\left(x_*\right)>0 \tag{4.10}$$

$$G''\left(x^*\right)<0 \tag{4.11}$$

The eqs. (4.5)-(4.8) have a convenient structure. Although, we are mainly interested in

$\partial x_*/\partial p$ and $\partial x^*/\partial p$ it is useful to know $\partial A_1/\partial p$ and $\partial A_2/\partial p$:

$$\frac{\partial A_1}{\partial p} = \frac{(x_*)\,(K - x^*)^{-\frac{\rho}{\lambda}} - (x^*)\,(K - x_*)^{-\frac{\rho}{\lambda}}}{C} \tag{4.12}$$

$$\frac{\partial A_2}{\partial p} = \frac{(x_*)\,(x^*)^{-\frac{\rho}{\mu}} - (x^*)\,(x_*)^{-\frac{\rho}{\mu}}}{C} \tag{4.13}$$

where:

$$C = (\rho + \mu)\,(x^*)^{-\frac{\rho}{\mu}} \left[(K - x_*)^{-\frac{\rho}{\lambda}} - \theta^{\frac{\rho}{\mu}}\,(K - x^*)^{-\frac{\rho}{\lambda}}\right] < 0 \tag{4.14}$$

and

$$\theta = \left(\frac{x^*}{x_*}\right) > 1 \quad \text{since} \quad K > x^* > x_* > 0 \tag{4.15}$$

We observe directly that the numerator of (4.13) is negative, implying that $\partial A_2/\partial p > 0$ since $C > 0$. Thus our pretention (3.30) is immediately verified, but we observe that to confirm (3.29) we need some extra analysis.

At last we are ready to challenge Barret's proposition 3 quoted above. In order to show that it fails we shall argue to a contradiction.

Proposition 4.1 *Suppose that the optimal thresholds x_* and x^* determined by the high contact principle (4.1)-(4.4) satisfy the inequalities $0 < x_* < x^* < K$. Then the derivatives $\partial x_*/\partial p$ and $\partial x^*/\partial p$ are non-zero.*

Proof We shall prove this result by showing that Barret's proposition 3 leads to a contradiction. Thus we assume that:

$$\frac{\partial x_*}{\partial p} = 0 \tag{4.16}$$

$$\frac{\partial x^*}{\partial p} = 0 \tag{4.17}$$

From (4.16)-(4.17) and (4.12)-(4.13) it follows directly that:

$$\frac{\partial^2 A_1}{\partial p^2} = 0 \tag{4.18}$$

$$\frac{\partial^2 A_2}{\partial p} = 0 \tag{4.19}$$

implying that:

$$A_1(p) = a_1 p + b_1 \tag{4.20}$$

$$A_2(p) = a_2 p + b_2 \tag{4.21}$$

for suitable constants a_1, a_2, b_1 and b_2.

The thresholds x_* and x^* are by (4.16)-(4.17) independent of the price p, which implies that (4.1)-(4.4) reduce to linear identities in p. Using (4.20)-(4.21) we deduce from (4.3)-(4.4) that:

$$\frac{b_1}{\mu}(x_*)^{-\left(\frac{\rho}{\mu}+1\right)} = -\frac{b_2}{\lambda}(K - x_*)^{-\left(\frac{\rho}{\lambda}+1\right)} \tag{4.22}$$

$$\frac{b_1}{\mu}(x^*)^{-\left(\frac{\rho}{\mu}+1\right)} = -\frac{b_2}{\lambda}(K - x^*)^{-\left(\frac{\rho}{\lambda}+1\right)} \tag{4.23}$$

Since $0 < x_* < x^* < K$ (4.22)-(4.23) imply that $b_1 = b_2 = 0$. Putting $A_1 = a_1 p$ and $A_2 = a_2 p$ into (4.1)-(4.2) we deduce by similar arguments that:

$$-\frac{c}{\rho} + d_{12} = 0 \tag{4.24}$$

$$-\frac{c}{\rho} - d_{21} = 0 \tag{4.25}$$

implying that:

$$d_{12} + d_{21} = 0 \tag{4.26}$$

Inspecting (4.1)-(4.2) we find that (4.26) implies that the optimal thresholds are equal, i.e. $x_* = x^*$. But this contradicts our hypothesis that $x_* < x^*$ and thus proving proposition 4.1. ■

Now that we have proved that Barret's proposition is not true, it is even more important to derive constructive results showing the effect of the output price on the optimal fallow-cultivation cycle. We summarize the facts in the following proposition.

Proposition 4.2 *(i) Suppose that the optimal thresholds x_* and x^* as well as the option values A_1 and A_2 are determined by the high contact principle. (ii) Suppose also that: $A_1(0) = A_2(0) = 0$. Then the lower threshold $x_*(p)$ is non-decreasing and the upper threshold $x^*(p)$ is non-increasing in the output price p.*

Proof From (4.7)-(4.8) we attain:

$$G''(x_*)\frac{\partial x_*}{\partial p} = \frac{\rho}{\mu}(x_*)^{-\left(\frac{\rho}{\mu}+1\right)}\frac{\partial A_1}{\partial p} + \frac{\rho}{\lambda}(K-x_*)^{-\left(\frac{\rho}{\lambda}+1\right)}\frac{\partial A_2}{\partial p} - \frac{1}{\mu+\rho} \tag{4.27}$$

$$G''(x^*)\frac{\partial x^*}{\partial p} = \frac{\rho}{\mu}(x^*)^{-\left(\frac{\rho}{\mu}+1\right)}\frac{\partial A_1}{\partial p} + \frac{\rho}{\lambda}(K-x^*)^{-\left(\frac{\rho}{\lambda}+1\right)}\frac{\partial A_2}{\partial p} - \frac{1}{\mu+\rho} \tag{4.28}$$

Then, using (4.3)-(4.4) to eliminate $(-\frac{1}{(\mu+\rho)})$ we deduce:

$$G''(x_*)\frac{\partial x_*}{\partial p} = \left(\frac{\partial A_1}{\partial p} - \frac{A_1}{p}\right)\frac{\rho}{\mu}(x_*)^{-\left(\frac{\rho}{\mu}+1\right)} + \left(\frac{\partial A_2}{\partial p} - \frac{A_2}{p}\right)\frac{\rho}{\lambda}(K-x_*)^{-\left(\frac{\rho}{\lambda}+1\right)} \tag{4.29}$$

$$G''(x^*)\frac{\partial x^*}{\partial p} = \left(\frac{\partial A_1}{\partial p} - \frac{A_1}{p}\right)\frac{\rho}{\mu}(x^*)^{-\left(\frac{\rho}{\mu}+1\right)} + \left(\frac{\partial A_2}{\partial p} - \frac{A_2}{p}\right)\frac{\rho}{\lambda}(K-x^*)^{-\left(\frac{\rho}{\lambda}+1\right)} \tag{4.30}$$

If we denote the initial output price by p_0 and expand $A_i(p)$ around p_0 we attain:

$$A_i(p) = A_i(p_0) + \frac{\partial A_i}{\partial p}(p-p_0) + o(p-p_0) \quad i = 1,2 \tag{4.31}$$

Applying (4.31) we observe that:

$$\frac{\partial A_i}{\partial p} - \frac{A_i}{p} = \frac{1}{p_0}\left(\frac{\partial A_i}{\partial p} - A_i\right) + o(p-p_0) \quad i = 1,2 \tag{4.32}$$

where A_i and its derivative are evaluated at p_0.

Owing to the convexity of A_i and assumption (ii) we readily find for p in the neighborhood of p_0 that:

$$\left(\frac{\partial A_i}{\partial p}p - A_i\right) \geq 0 \tag{4.33}$$

From (4.29)-(4.30) we then conclude:

$$G''(x_*)\frac{\partial x_*}{\partial p} \geq 0 \tag{4.34}$$

$$G''(x^*)\frac{\partial x^*}{\partial p} \geq 0 \tag{4.35}$$

Since $G''(x_*) > 0$ and $G''(x^*) < 0$ we finally conclude:

$$\frac{\partial x_*}{\partial p} \geq 0 \tag{4.36}$$

$$\frac{\partial x^*}{\partial p} \leq 0 \tag{4.37}$$

which were to be proved. ■

Remark 4.1 *The proof of proposition 4.1 shows that when the sum of the switching costs* d_{12} *and* d_{21} *is equal to zero, the two thresholds collaps to a single point. Hence, it is more the sum rather than the separate switching costs that matters. Similarly it is plainly obvious that when (*$d_{12} + d_{21}$*) increases the thresholds* x_* *and* x^* *will be pulled apart.*

5. Soil fertility with varying ouput prices

We have shown above that the lower threshold x_* moves upwards while the upper threshold x^* slides downwards when the output price increases. But what happens to soil fertility when the output price rises. In the literature environmentalists air opposing views. An intuitive guess is that rising output prices will encourage soil conservation. But not all agree, i.e. Lipton ((1987), p. 209) argues precisely the opposite case: "... Better farm prices now, if they work as intended, will encourage "soil mining" for quick, big crops now...".

It is tempting to use the model developed above to analyse this question formally. Except, perhaps for initial deviations, the soil fertility index x_t varies solely between the thresholds x_* and x^*. Hence, soil fertility cannot be a time independent quantity. At the same time it is evident that environmentalists associate to soil fertility some kind of an average depletion rate during the cultivation phase. This will be our starting point. From our derivations above we know that the peasant switches from fallow to cultivation when the fertility index x_t crosses the upper threshold x^* and then cultivates the fields until x_t meets the lower threshold x_*. Let τ denote the time it takes for x_t to go from x^* to x_*. We observe from (3.8) that τ is determined by:

$$\tau = \frac{1}{\mu} \int_{x_*}^{x^*} \frac{dy}{y} = \frac{1}{\mu} \ln \left(\frac{x^*}{x_*} \right) \tag{5.1}$$

Evidently, τ depends on the output price. We calculate directly:

$$\frac{\partial \tau}{\partial p} = \tau'_p = \frac{1}{\mu}\left(\frac{x^*_p}{x^*} - \frac{x_{*p}}{x_*}\right) \tag{5.2}$$

where we use the notations $x^*_p = \frac{\partial x^*}{\partial p}$ and $x_{*p} = \frac{\partial x_*}{\partial p}$.

From (4.36) and (4.37) we know that $x^*_p \leq 0$ and $x_{*p} \geq 0$ which imply that $\tau'_p \leq 0$.

As the average fertility rate we define:

$$\bar{x}(p) = \frac{(x^* - x_*)}{\tau} \tag{5.3}$$

The question we wish to answer is this: "What happens to this average when the output price increases"? Obviously, the natural thing to do is to take the partial derivate of $\bar{x}(p)$ with respect to the output price. Since the average fertility rate is defined by (5.3) we easily calculate:

$$\frac{\partial \bar{x}(p)}{\partial p} = \frac{1}{\tau}\left[(x^*_p - x_{*p}) - \frac{(x^* - x_*)\tau'_p}{\tau}\right] \tag{5.4}$$

We already know that $(x^*_p - x_{*p}) < 0, (x^* - x_*) > 0, \tau > 0$ and $\tau'_p < 0$. Hence, the two terms of the bracketed expression will have opposite signs. Even a more refined analysis if this expression did not unconditionally determine its sign.

Hence, to find out what happens to the average fertility rate $\bar{x}(p)$ over a cultivation cycle it is probably best to show the shape of $\bar{x}(p)$ in numerical experiments. Figure (6.6) shows a typical finding of these experiments. The average fertility rate reacts only sluggishly, but is a slowly increasing function of the output price.

6. A numerical illustration

In order to make the contents of our propositions more accessible, we shall now illustrate by graphs how essential variables evolve. It is enough to consider one experiment since different numerical specifications proved to generate quite similar paths.

In this experiment we specified the following numerical values:
For the process parameters: $(\lambda, \mu, K) = (0.05, 0.05, 50)$
For the economic variables: $(c, d_{12}, d_{21}, p, \rho) = (1, 10, 100, p, 0.05)$, and the output price (p) runs from 1 to 2.05.

We report the results in figures (6.1)-(6.6). Figure (6.1) shows the graphs of the two continuation sets D_1 and D_2 (3.21)-(3.22). We observe that D_1 and D_2 depict two curves on two parallel sheets. The stippled straight lines connecting these curves indicate the

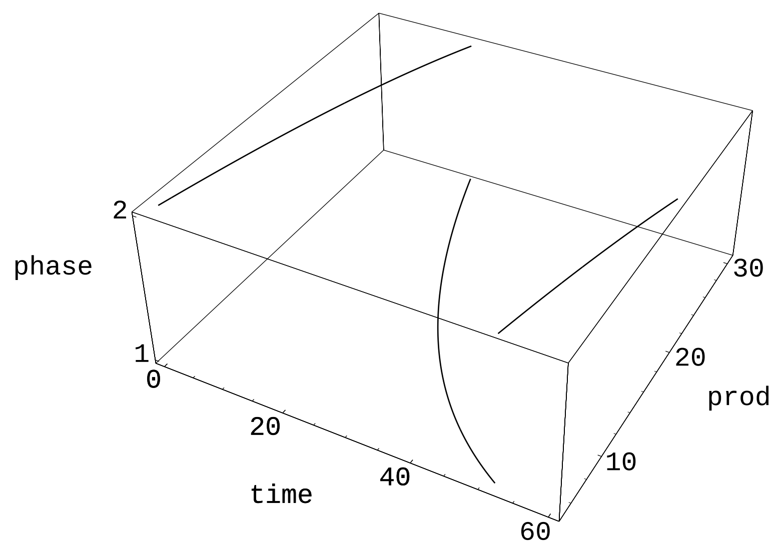

Figure 6.1: The relation between the time and soil fertility (state variable) x in the fallow phase (2) and cultivation phase (1)

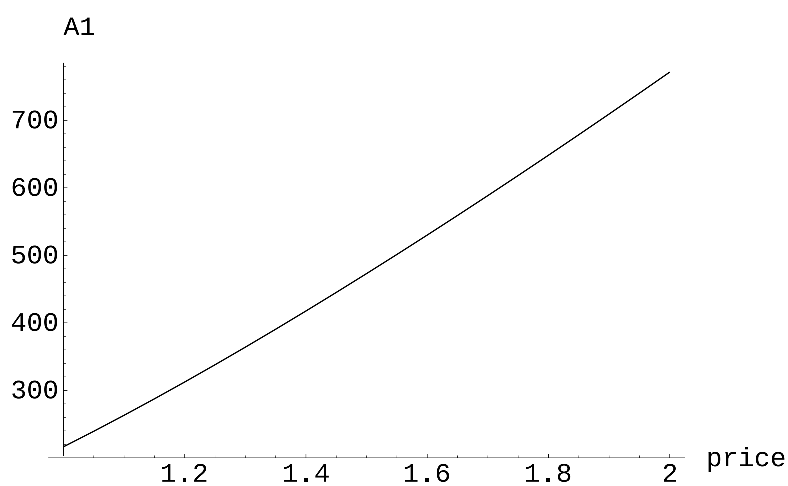

Figure 6.2: The relation between the output price and the option value A_1

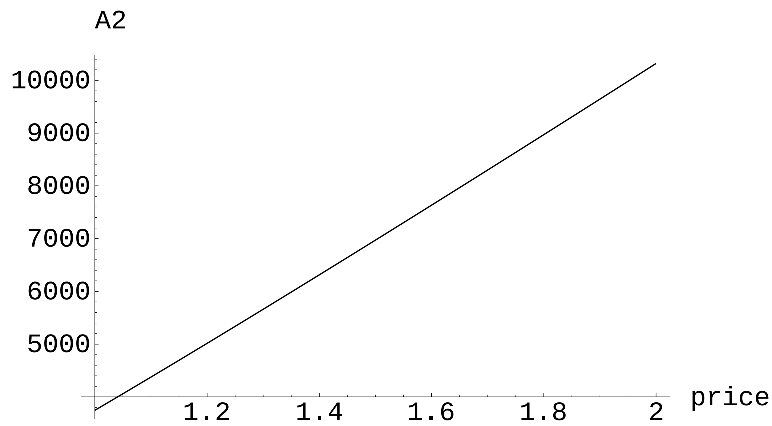

Figure 6.3: The relation between the output price and the option value A_2

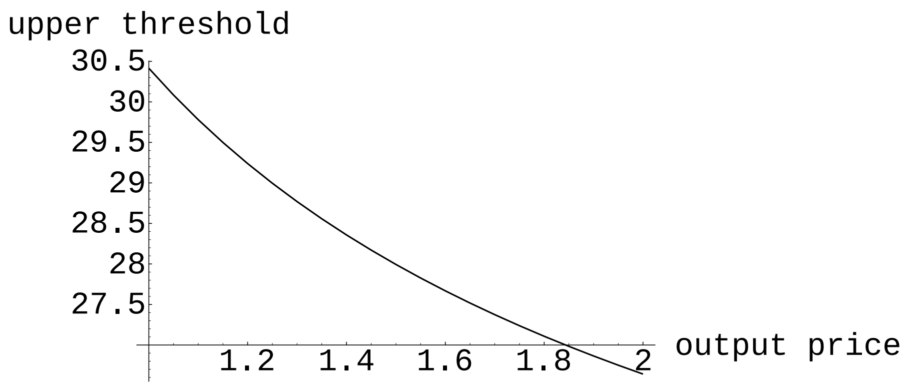

Figure 6.4: The relation between the output price and the upper threshold x^*

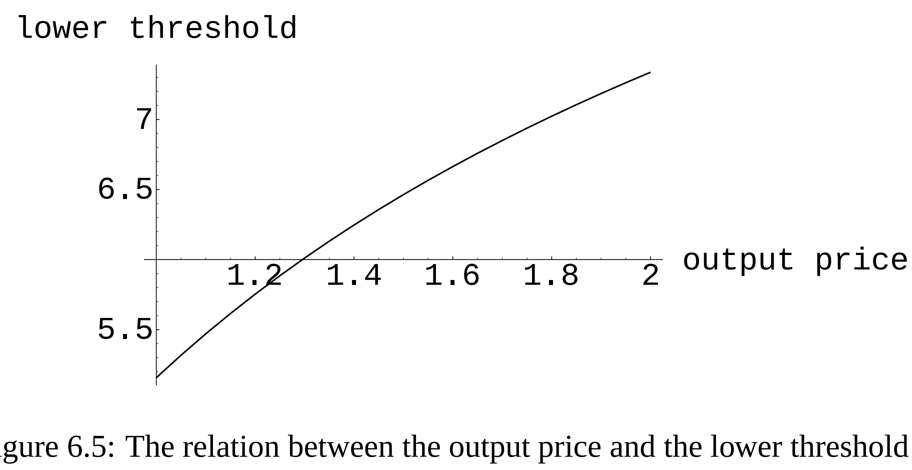

Figure 6.5: The relation between the output price and the lower threshold x_*

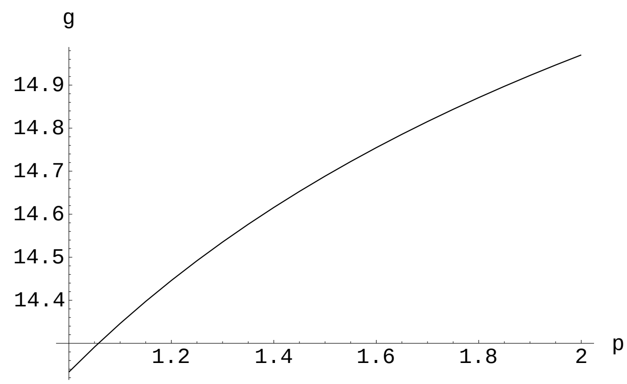

Figure 6.6: The relation between the output price and the average fertility rate (5.3)

switches: (i) from fallow to cultivation, (ii) from cultivation to fallow. Figures (6.2)-(6.3) shows the relations between the output price and the two option values A_1 and A_2. The graphs are obtained by solving (4.1)-(4.4) when p runs through the interval $[1, 2.05]$. The graphs confirm the results of our formal analyses. Figures (6.4)-(6.5) show the relations between the thresholds x^* and x_* and the output price. Also these graphs are attained by solving (4.1)-(4.4) when p traverses the interval $[1, 2.05]$. The graphs confirm our results stated in proposition (4.2). Finally, figure 6.6 shows the dependency of the average fertility rate (5.3) and the output price.

7. Conclusion

In this paper we have provided a formal analysis of the fallow-cultivation problem in traditional agriculture. We have shown how our modelling approach can be used to determine the effect of rising output prices. In order to make the analysis simple we have only studied the pure fallow-cultivation case, i.e. the fields recuperate only by laying fallow. However, it is evident that the present modelling can be extended to include artificial fertilizer by re-specifying the growth equations (2.1)-(2.2). We can also extend the present specification to the case when the index variable (x) follows a one-dimensional diffusion in the two phases.

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References

Barret, S. (1991). Optimal soil conservation and the reform of agricultural pricing policies. Journal of Development Economics 36, 167-187.

Clark, C. W. (1976). Mathematical Bioeconomics, Wiley, New York.

Dixit, A. K. and R. S. Pindyck (1994). Investment under Uncertainty, Princeton University Press. Princeton, New Jersey.

Lipton, M. (1987). Limits of price policy for agriculture: Which way for the World Bank? Policy Development Review 5, 197-215.

Whittle, P. (1988). Restless Bandits: activity allocation in a changing world. In Celebration of Applied Probability (Ed. J. Gani), Journal of Applied Probability, 25A, 287-298.

Willassen, Y. (1998). The stochastic rotation problem: A generalization of Faustmann's formula to stochastic forest growth. Journal of Economic Dynamics and Control 22, 573-596.