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By
Eivind Bjøntegård

Department of Economics
University of Oslo

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The Composite Mean Regression as a Tool in Production Studies

by Eivind Bjøntegård [1]
Department of Economics
University of Oslo

Abstract

Johansen (1972) explains how a short run macro production function can be derived on the basis of a distribution of micro production units with respect to fixed input coefficients. The present note points out that the composite mean regression, introduced by Frisch (1929), can be useful in analysing some of the production models in Johansen (1972). The focus is on complementary, alternative and marginally independent production factors at the macro level in a production model which assumes efficient allocation of given quantities of inputs.

Keywords: Composite mean regression, complementary, alternative and marginally independent production factors, macro production functions.

1.   Introduction

The purpose of this note is to demonstrate that the composite mean regression in Frisch (1929) can be useful in analysing a production model in Johansen (1972). Johansen explains how a short run macro production function can be derived on the basis of a distribution of micro production units with respect to fixed input coefficients. Frisch (1965) introduces the concept of marginal dependency between production factors, i.e. complementary, alternative and marginally independent factors. The composite mean regression is applicable in discussing this dependency at the macro level within a production model stated in section 4.1 in Johansen. This model assumes efficient allocation of given input quantities among the micro production units. It may be denominated the standard model. In section 3.6 Johansen formulates a generalized model where the degree of capacity utilization in the micro units is a nondecreasing function of the quasi rent

the micro units are able to earn. It can be shown that the marginal productivities of the macro production function in this general model are equal to the regression coefficients in an appropriate composite mean regression. However, in order to save on symbols and formulas the presentation of this case is omitted from the present note.

The regression model is presented in section 2 and the production model in section 3 below. In this last section definitions of complementary, alternative and marginally independent factors are given together with a further discussion. The production model is restricted to three production factors and the regression model accordingly to two variables. Extension to an arbitrary number of factors and variables should be straighforward.

2.   The composite mean regression

Frisch (1929) is working with discrete variables. Here his idea is adopted to a continuous distribution. The variables z_1 and z_2 are assumed to be distributed on the positive orthant according to the density function

(2.1) $h(z_1, z_2)$.

The regression line whose coefficients g_1 and g_2 are to be determined, is written as

$$g_0 + g_1 z_1 + g_2 z_2 = 0 ,$$

where $g_0 \neq 0$ is an arbitrarily fixed constant. Frisch requires the regression to satisfy some invariance conditions for transformation of the variables. This requirement can be disregarded here and g_0 put equal to -1 without loss of generality. The regression line to be determined can thus be written as

(2.2) $-1 + g_1 z_1 + g_2 z_2 = 0 .$

The composite mean regression is obtained by choosing g_1 and g_2 to minimize the expression

(2.3) $\iint (-1 + g_1 z_1 + g_2 z_2)^2 \, h(z_1 z_2) \, dz_1 dz_2 ,$

where it is integrated for all positive values of $h(z_1, z_2)$. The first order conditions for a minimum are

$$\iint 2(-1 + g_1 z_1 + g_2 z_2) \, z_1 h(z_1, z_2) \, dz_1 dz_2 = 0$$

$$\iint 2(-1 + g_1 z_1 + g_2 z_2) \, z_2 h(z_1, z_2) \, dz_1 dz_2 = 0$$

Assuming the second order conditions fulfilled and introducing the first and second order moments

(2.4)
$$M_i = M(z_i) = \iint z_i h(z_1, z_2) \, dz_1 dz_2 \qquad i = 1,2$$

$$M_{ij} = M(z_i z_j) = \iint z_i z_j h(z_1, z_2) \, dz_1 dz_2 \qquad i,j = 1,2$$

the solutions for g_1 and g_2 can be obtained from the first order conditions

(2.5)
$$g_1 = \frac{1}{M_{11} M_{22} - M_{12}^2} (M_1 M_{22} - M_2 M_{21})$$

$$g_2 = \frac{1}{M_{11} M_{22} - M_{12}^2} (M_2 M_{11} - M_1 M_{12}) .$$

3.   Marginal dependency - the standard model

Johansen assumes fixed input coefficients in the micro production units. Let ξ_1, ξ_2, ξ_3 indicate input per unit of output of the three inputs and let

(3.1) $f(\xi_1, \xi_2, \xi_3)$

be a continuous capacity distribution. The domain of f( ) is in the interior of the positive orthant in the ξ_1, ξ_2, ξ_3-space. The meaning of (3.1) is that in a 3-dimensional region $\Delta\xi_1$, $\Delta\xi_2$, $\Delta\xi_3$ around the point ξ_1, ξ_2, ξ_3, is located an output capacity approximately equal to $f(\xi_1, \xi_2, \xi_3) \Delta\xi_1 \Delta\xi_2 \Delta\xi_3$. This capacity requires an amount of input no. i equal to $\xi_i f(\xi_1, \xi_2, \xi_3) \Delta\xi_1 \Delta\xi_2 \Delta\xi_3$ when operated.

Letting q_1, q_2, q_3 be input prices in terms of output, the quasi rent earned by micro units with input coefficient ξ_1, ξ_2, ξ_3 can be defined as

(3.2) $r = 1 - q_1\xi_1 - q_2\xi_2 - q_3\xi_3$

and a zero quasi rent plane as

(3.3) $q_1\xi_1 + q_2\xi_2 + q_3\xi_3 = 1 \quad or \quad \xi_1 = \dfrac{1}{q_1}(1 - q_2\xi_2 - q_3\xi_3)$.

Efficient allocation of variable inputs is obtained when micro units earning a positive quasi rent are operated at full capacity while the others are left idle. Total output, X, and total factor use, V_i, for the production sector as a whole can then be written as

$$X = \int\int \int_0^{\frac{1}{q_1}(1 - q_2\xi_2 - q_3\xi_3)} f(\xi_1, \xi_2, \xi_3)\, d\xi_1 d\xi_2 d\xi_3$$

(3.4)

$$V_i = \int\int \int_0^{\frac{1}{q_1}(1 - q_2\xi_2 - q_3\xi_3)} \xi_i f(\xi_1, \xi_2, \xi_3)\, d\xi_1 d\xi_2 d\xi_3 \quad i = 1,2,3 .$$

In (3.4) the limits of integration for ξ_2 and ξ_3 may be set equal to 0 and ∞. After having integrated over ξ_1 from 0 to $1/q_1$ $(1 - q_2\xi_2 - q_3\xi_3)$ the integrand of the integral over ξ_2 and ξ_3 is zero for values of these variables close to the origin or above the plane (3.3).

Assuming that a matrix, M, to be defined later, is nonsingular (positive definite), q_1, q_2, q_3 may be conceived of as functions of V_1, V_2, V_3 in the last equations of (3.4). Inserting into the first equation gives

(3.5) $X = F(V_1, V_2, V_3)$.

This is the short run macro production function in Johansen.

Frisch (1965, p. 58) defines the product accelerations as the second order derivatives of F. Two factors are complementary if an increase in the quantity of one factor increases the marginal productivity of the other, i.e. if $\dfrac{\delta^2 F(V_1, V_2, V_3)}{\delta V_i \delta V_j} > 0, \; i \neq j.$ The factors are alternative when this second order derivative is negative and marginally independent when it is zero.

According to some remarks on page 192 and 193, Frisch (1965) considers the definitions above to be technical concepts as opposed to complementarity and

alternativity in demand. The definition will be applied here although the Johansen macro production function (3.5) is a highly economic concept.

Regarding X and V_1, V_2, V_3 in (3.4) as functions of q_1, q_2 and q_3 and letting $\delta X / \delta q_1$ and $\delta V_i / \delta q_1$ denote partial derivatives, one gets

(3.6)
$$\frac{\delta X}{\delta q_1} = -\frac{1}{q_1} \int\int \xi_1 f(\xi_1, \xi_2, \xi_3) \, d\xi_2 d\xi_3$$

$$\frac{\delta V_i}{\delta q_1} = -\frac{1}{q_1} \int\int \xi_1 \xi_i f(\xi_1, \xi_2, \xi_3) \, d\xi_2 d\xi_3, \quad i = 1,2,3$$

Let us further introduce the integral

(3.7)
$$J = \frac{1}{q_1} \int\int f(\xi_1, \xi_2, \xi_3) \, d\xi_2 d\xi_3 \ .$$

In the integrals (3.6) and (3.7) ξ_1 is to be set equal to $1/q_1 \, (1 - q_2\xi_2 - q_3\xi_3)$ according to (3.3). Replacing (3.3) by (3.2) in (3.4) X can also be considered a function of r. Partial derivation gives

$$\frac{\delta X}{\delta r} = -J, \ when \ r = 0.$$

J can be interpreted as total capacity density along the plane (3.3).

Let us also introduce the density function.

(3.8)
$$\frac{1}{q_1 J} f(\frac{1}{q_1} (1 - q_2\xi_2 - q_3\xi_3), \xi_2, \xi_3)$$

and first and second order moments

(3.9)
$$M_i = M(\xi_i) = \frac{1}{q_1 J} \int\int \xi_i f(\xi_1, \xi_2, \xi_3) \, d\xi_2 d\xi_3 \qquad i = 1,2,3$$

$$M_{ij} = M(\xi_i \xi_j) = \frac{1}{q_1 J} \int\int \xi_i \xi_j f(\xi_1, \xi_2, \xi_3) \, d\xi_2 d\xi_3 \quad i,j = 1,2,3$$

where ξ_1 again is to be replaced by $1/q_1 \, (1 - q_2\xi_2 - q_3\xi_3)$.

Using the definitions above and differentiating in (3.4) gives

$$dX = -JM_1 dq_1 - JM_2 dq_2 - JM_3 dq_3$$

(3.10)

$$dV_i = -JM_{i1} dq_1 - JM_{i2} dq_2 - JM_{i3} dq_3, \quad i = 1,2,3.$$

It is convenient to rewrite (3.10) with vector and matrix notations

(3.11) $dX = -Jm'dq$

(3.12) $dV = -JM\,dq$

where m = $[M_i]_{3x1}$, dq = $[dq_i]_{3x1}$, dV = $[dV_i]_{3x1}$, M = $[M_{ij}]_{3x3}$,
and for later use we write q = $[q_i]_{3x1}$.
It can be shown that the matrix M is positive definite provided there is sufficient spread among the marginal input coefficients in (3.3).

Since ξ_1, ξ_2, ξ_3 satisfy (3.3), one can derive

(3.13)
$$1 = q'm$$
$$m = Mq.$$

From (3.12), (3.13) and the fact that M = M'

(3.14) $q'dV = -Jq'M\,dq = -Jm'dq$

which together with (3.11) gives

(3.15) $dX = q'dV$

which proves that the marginal productivities are equal to the factor prices,
$$\frac{\delta F(V_1,V_2,V_3)}{\delta V_i} = q_i.$$

Since the factor prices are functions of the factor quantities, the second order derivatives of F or the product accelerations in the terminology by Frisch, can be obtained from (3.10) or (3.12) as $\frac{\delta q_i(V_1,V_2,V_3)}{\delta V_j}$.

Let $W = \left[\frac{\delta^2 F(\ )}{\delta V_j \delta V_i}\right] = \left[\frac{\delta q_i(\ )}{\delta V_j}\right]$ be the Hessian of F. From (3.12)

(3.16) $W = -\dfrac{1}{J} M^{-1}.$

Since M is positive definite, W is negative definite.

From (3.16):

(3.17) $\dfrac{\delta q_2}{\delta V_1} = \dfrac{1}{J|M|} \begin{vmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{vmatrix}.$

It is easily seen that with only two factors, no. 1 and no. 2, (3.17) becomes

(3.18) $\dfrac{\delta q_2}{\delta V_1} = \dfrac{M_{12}}{J(M_{11}M_{22} - M_{12}^2)} > 0,$ where J and M_{ij} are defined for the two factor case as in Johansen p 57.

It is clear that in the two factor case, the factors are always complementary.

Returning to (3.17), using (3.13) and manipulating the determinant in the numerator[2] gives:

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$$K = \begin{vmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{vmatrix} = \dfrac{1}{q_1} \begin{vmatrix} q_1 M_{12} & q_1 M_{13} \\ M_{32} & M_{33} \end{vmatrix}.$$

Adding q_3 times the second row to the first

$$K = \dfrac{1}{q_1} \begin{vmatrix} q_1 M_{12} + q_3 M_{32} & q_1 M_{13} + q_3 M_{33} \\ M_{32} & M_{33} \end{vmatrix}.$$

Using $q_1 M_{12} + q_3 M_{32} = M_2 - q_2 M_{22}$ and $q_1 M_{13} + q_3 M_{33} = M_3 - q_2 M_{23}$ from (3.13)

$$K = \dfrac{1}{q_1}\left(\begin{vmatrix} M_2 & M_3 \\ M_{32} & M_{33} \end{vmatrix} - q_2 \begin{vmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{vmatrix} \right).$$

Putting $|M_1| = M_{22}M_{33} - M_{32}^2$ outside the parenthesis

$$K = \dfrac{|M_1|}{q_1}\left(\dfrac{\begin{vmatrix} M_2 & M_3 \\ M_{32} & M_{33} \end{vmatrix}}{|M_1|} - q_2 \right).$$

Inserting in (3.17) gives (3.19).

$$(3.19) \quad \frac{\delta q_2}{\delta V_1} = \frac{|M_1|}{q_1 J |M|} \left(\frac{\begin{vmatrix} M_2 & M_3 \\ M_{32} & M_{33} \end{vmatrix}}{|M_1|} - q_2 \right), \qquad$$

where $|M_1|$ is the principal minor $M_{22} M_{33}$ - $M_{23}^2 > 0$.

Comparing the first expression in the parenthesis in (3.19) with (2.5) it is seen that this is equal to the regression coefficient g_2 in the composite mean regression

$$(3.20) \quad -1 + g_2 \xi_2 + g_3 \xi_3 = 0$$

when the density function (3.8) and definitions (3.9) are applied.

(3.19) can thus be written as

$$(3.21) \quad \frac{\delta q_2}{\delta V_1} = H(g_2 - q_2) \quad \text{where H is a positive number.}$$

The following conclusions can be drawn from (3.21): Factors no. 1 and no. 2 are complementary when $g_2 > q_2$. If $g_2 < q_2$ they are alternative and they are marginally independent if $g_2 = q_2$.[3]

By computing $\frac{\delta q_3}{\delta V_1}$ we will get similar conclusions for factors no. 1 and no. 3. However, factor no. 1 cannot be alternative to both no. 2 and no. 3. If this were the case, the regression line (3.20) will lie northeast of the line

$$1 = q_2 \xi_2 + q_3 \xi_3$$

in the ξ_2, ξ_3-plane and wholly outside the region where the density function (3.8) takes on positive values. In such a situation the measured deviations from the regression line cannot be minimized. It can thus be concluded that factor no. 1 is complementary to either no. 2 or no. 3. If factor no. 1 and no. 2 are alternative, then factor no. 3 is

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[3]

Letting q_1 be exogenous and V_1 endogenous in (3.4) one gets $X = F^* (q_1, V_2, V_3)$ and $V_1 = v (q_1, V_2, V_3)$. It can be shown that g_2 in (3.21) satisfies

$$g_2 = \frac{\delta F^*(q_1, V_2, V_3)}{\delta V_2},$$

but proceeding along these lines doesn't give more insight into marginal dependency than can be derived from a general production function independently of its origin.

complementary to both no. 1 and no. 2.

The conclusion that factor no. 1 is complementary to at least one of the other factors can reasonably be assumed to hold for an arbitrary number of n factors. When every factor is complementary to at least one of the others, there must be at least n-1 positive cross accelerations. This is a minimum requirement that leaves wide scope for alternativity when the number factors becomes large. The number of negative cross accelerations may then be quite large. On page 59 in Frisch (1965) the number of different cross accelerations is stated to be n(n-1)/2. Deducting the minimum number of positive accelerations according to the reasoning above, we get the highest possible number of negative cross accelerations

$$k = (n-1)\,(\frac{n}{2} - 1).$$

When n = 2, k = 0 and when n = 3, k = 1. This is in accordance with the previous results. For n = 4, 5, 10 and 20 k becomes 3, 6, 36 and 171 respectively.

We conclude the note by repeating the results for the two and three factor cases remembering that the numbering of factors is arbitrary: When there are only two production factors, these are always complementary. In the three factor case factor no. 1 is complementary to either no. 2 or no. 3. If factor no. 1 and no. 2 are alternative, factor no. 3 is complementary to both no. 1 and no. 2.

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