

# Constructing Counterfactuals from Cointegration Relationships

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# Introduction

- ▶ Policy changes or macroeconomic shocks affect large economic entities such as national economies, cities, sectors.
- ▶ Candidate controls come from similar socio-economic environments.
- ▶ Their outcomes may admit stable long-run relationships  $\Rightarrow$  cointegration with the control unit(s).
- ▶ The counterfactual is cointegrating relationship that predicts the treated unit from the controls
- ▶ Dynamic effects of the event are deviations from the long-run relationship.

# Which control units?

A valid control

- (i) **Relevance:** it is driven by the same long-run forces as the treated unit.
- (ii) **Exclusion:** it is unaffected by the event, directly or indirectly.

**“Cointegrated” alone guarantees neither:**

- ▶ many sets cointegrate with the treated unit so which is the counterfactual?
- ▶ a cointegrated unit can still be affected by the treated unit (indirect/spillover effects).

*So which controls?*

# This paper

**Choose controls by two criteria :**

- ▶ **Irreducible cointegration**  $\Rightarrow$  delivers relevance (and identification)
- ▶ **Weak exogeneity**  $\Rightarrow$  delivers exclusion (no long-run feedback)

... plus a **search algorithm** that finds such sets in a set of candidate controls.

Revisited on Abadie, Diamond & Hainmueller (2015): German reunification, 1990.

## Related work

Two most related work:

- ▶ **Bai, Li & Ouyang (2014):** cointegrated counterfactuals; controls chosen by fit / narrative / robustness checks.
- ▶ **Harvey & Thiele (2021):** multivariate structural time-series model with efficiency gains; pairwise cointegration with the treated unit.

**This paper:** a criterion for choosing control units with possibly several common trends.

# The model

Potential outcomes:  $y_{it}^0$  (untreated),  $y_{it}^1$  (treated). Only unit 1 is treated, at time  $\tau$ ; the effect is

$$\Delta_t = y_{1t}^1 - y_{1t}^0, \quad t = \tau, \dots, T.$$

We observe:

- ▶ controls:  $y_{it} = y_{it}^0, \quad i = 2, \dots, N$ ;
- ▶ the treated:  $y_{1t} = y_{1t}^0 \ (t < \tau), \quad y_{1t} = y_{1t}^1 \ (t \geq \tau)$ .

The one missing piece:  $y_{1t}^0$  for  $t \geq \tau$  is the counterfactual, to be constructed from the controls.

## The model: common trends

For  $t = 1, \dots, T$ :

$$y_t^0 = \mu + B \mu_t + \varepsilon_t, \quad \mu_t = \mu_{t-1} + \eta_t,$$

with  $K < N$  common stochastic trends  $\mu_t$  and loadings  $B$ .

A1.  $\text{rank}(B) = K \Rightarrow K$  distinct trends.

A2.  $\Sigma_\eta$  positive definite  $\Rightarrow$  the trends are not cointegrated.

A3.  $\varepsilon_t \sim I(0) \Rightarrow$  no idiosyncratic trends.

*Together: outcomes are  $I(1)$ , driven by  $K$  common trends.*

# The model: Cointegration

**Result 1.** There is an  $N \times (N - K)$  matrix  $B_{\perp}$  with  $B'B_{\perp} = 0$  and

$$B'_{\perp} y_t^0 \sim I(0) \implies N - K \text{ cointegrating relationships.}$$

- ▶ Assumption: the cointegration is **preserved absent the intervention** (invariance property).
- ▶ Any cointegrating vector,  $\beta_1 \in \text{col}(B_{\perp})$ , with non-zero weight on unit 1 gives a counterfactual:

$$\beta'_1 y_t \sim I(0), \quad t = \tau, \dots, T.$$

## The model: Identification

$$\beta = \begin{bmatrix} 1 & \beta_{2a} \\ \mathbf{0}_{N-K-1,1} & \beta_{2b} \\ \beta_{1c} & \beta_{2c} \end{bmatrix},$$

$\beta_1$  is identified via exclusion restrictions (control unit selection) by the rank condition ( $\text{rank}(\beta_{2b}) = N - K - 1$ )

- ▶ **Generic** identification: it holds for a typical  $\beta_{2b}$ .
- ▶ It is a condition on excluded units, so on **the full system**.

**But the rank condition can't tell you which subset to pick. It is not operational for selection.**

## Motivating example

$N = 5$ , two trends; **unit 5 is stationary**. Unit 1 treated.

$$y_{1t} = \mu_{1t} + \varepsilon_{1t}, \quad y_{2t} = \mu_{1t} + \varepsilon_{2t}, \quad y_{3t} = \mu_{2t} + \varepsilon_{3t}, \quad y_{4t} = \mu_{2t} + \varepsilon_{4t}, \quad y_{5t} = \varepsilon_{5t}.$$

- ▶ Valid counterfactual:  $y_{2t}$
- ▶  $y_{2t} + y_{5t}$ ,  $y_{2t} + (y_{3t} - y_{4t})$ , ... are *all* cointegrated with  $y_{1t}$
- ▶ But irrelevant units **contaminate** the counterfactual.
- ▶ In applications, all have near-perfect fit  $\Rightarrow$  fit can't say much.

*Only  $\{1, 2\}$  is irreducible as every other set has a droppable unit.*

# Irreducible cointegration

**Definition (Davidson, 1998).** A cointegrated set is *irreducibly cointegrated* (IC) if dropping any one variable leaves a set that is *not* cointegrated.

## Why IC is the right criterion

- ▶ Every control unit is essential  $\Rightarrow$  **only relevant controls**.
- ▶ An IC set has cointegration **rank 1**  $\Rightarrow$  a single cointegrating vector, so  $\beta_1$  is identified.

*Irreducibility brings sparsity, relevance, and identification.*

## Motivating example: no long-run feedback

Unit 1 treated, unit 2 control; one trend:

$$y_{1t} = \mu_t + \varepsilon_{1t} + \delta_t \mathbb{I}(t \geq \tau), \quad y_{2t} = \mu_t + \varepsilon_{2t}, \quad \varepsilon_{2t} = \boxed{a_{21}} \varepsilon_{1,t-1} + a_{22} \varepsilon_{2,t-1} + u_{2t}.$$

Counterfactual =  $y_{2t}$ , so  $\hat{\Delta}_t = y_{1t} - y_{2t}$ :

- ▶  $a_{21} = 0$ :  $E[\hat{\Delta}_t] = \delta_t$  **at every horizon**  $t \geq \tau$ .
- ▶  $a_{21} \neq 0$ :  $E[\hat{\Delta}_{\tau+1}] = \delta_{\tau+1} - a_{21}\delta_{\tau}$ . **Biased.**

## No long-run feedback

**Formally**, if the model admits a steady-state Kalman filter, then it has VECM representation:

$$\Delta y_{c,t} = \underbrace{\alpha_c \beta_1' y_{t-1}}_{\text{long-run}} + \underbrace{\sum_i \phi_i' \Delta y_{t-i}}_{\text{short-run}} + e_t, \quad \boxed{\alpha_c = 0}.$$

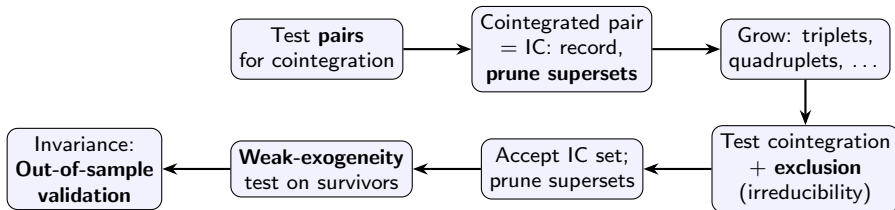
- ▶ **No long-run feedback:** the control doesn't react to long-run deviations, so it doesn't react to the treatment. (This is weak exogeneity w.r.t. the cointegrating relationship, Engle, Hendry & Richard (1983))
- ▶ Other types of **short-run** temporal dependence (the  $\Delta y_{t-i}$  terms) may arise; assumed to be negligible.

Then a valid counterfactual needs **super exogeneity** = weak exogeneity + **invariance** (the relationship holds absent the event).

# Search algorithm

Exponentially many candidate subsets ( $2^N - N - 1$ ;  $> 10^6$  at  $N=20$ ).

Key fact (Davidson 1998): **a superset of a cointegrated set can't be IC**. So search bottom-up and prune.



Tests: **Johansen trace** + large-VAR pre-test (Bykhovskaya & Gorin, 2024); LR tests for exclusion and weak exogeneity; out of sample prediction; inference via **bootstrap** or Bartlett-corrected asymptotics for speed.

# A Small Simulation Exercise

A simulation to check the search before trusting it on data.

- ▶ **DGP:**  $N = 16$  series,  $K = 12$  common trends ( $\Rightarrow 4$  IC);  $T = 50 + 10$ ; 1000 simulations
- ▶ **Inference:** asymptotic with **Bartlett small-sample corrections** (bootstrap  $\sim 30$  min vs.  $\sim 2$  min with corrections).
- ▶ **Metric:** how often the true IC sets are recovered (recall), and how many false positives are admitted (precision).

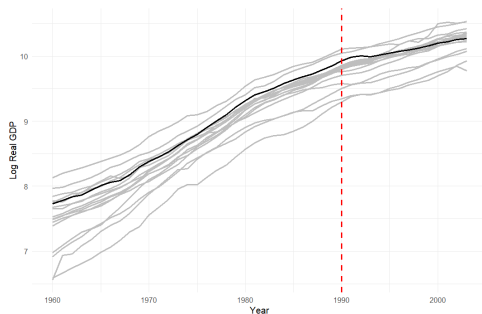
|                                   | Recall | Precision |
|-----------------------------------|--------|-----------|
| Asymptotic (Bartlett-corrected)   | 0.77   | 0.24      |
| Asymptotic + validation in search | 0.59   | 0.43      |

**Note:** Bootstrap inference is expected to perform better than asymptotic inference aided by small-sample corrections

# Application: the reunification of Germany in 1990

Revisiting Abadie, Diamond & Hainmueller (2015).

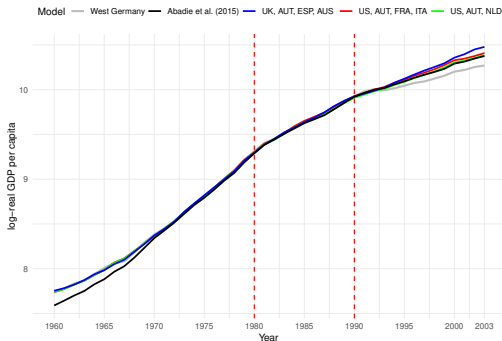
- ▶ **Data:** log GDP per capita, 1960–2003, 16 OECD countries as controls.
- ▶ **Search sample:** 1960–1980; find IC sets.
- ▶ **Validation sample:** 1981–1990; out-of-sample check before treatment.
- ▶ **Algorithm:** bootstrap inference, 5% level for all tests.



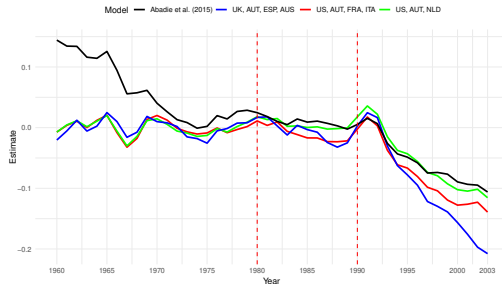
# The counterfactual and the effect

The search returns IC sets (no weak exogeneity!) containing Germany; their cointegrating relationship predicts German GDP from the controls.

West Germany vs IC Counterfactuals



Estimates from Different IC Subsets of Controls



# Conclusion

- ▶ Choosing controls is an **identification problem**, not a fitting problem.
- ▶ **Irreducible cointegration** solves it operationally: unique, minimal, only-relevant control sets identified by the rank condition.
- ▶ **Weak exogeneity** rules out long-run feedback
- ▶ A feasible **bottom-up search** makes this work in practice.
- ▶ **Application:** West German reunification; estimated effects are in line with the literature.