



CURVED TRENDS

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Two complementary approaches:

- Testing for structural breaks

- 0 In intercept (no trend in model)
- 1 In intercept (with trend in model)
- 2 In slope (with trend in model)
- 3 In intercept and slope (with trend in model)

Bai and Perron (1998), Perron and Yabu (2009), Kejriwal and Perron (2010) and others, following Andrews (1993).

- Saturation estimation: modelling structural breaks and outliers

- 1 IIS: Impulse indicator saturation
- 2 SIS: Step indicator saturation
- 3 TIS: Trend indicator saturation

Hendry, Johansen, and Santos (2008), Johansen and Nielsen (2009), Castle, Doornik, Hendry, and Pretis (2015), Nielsen and Qian (2025), Castle, Doornik, Hendry, and Pretis (2020)

The natural progression is from step to trend: step is differenced broken trend.

A model with a linear trend makes a strong assumption about the future

- Local linear trend model (STAMP), exponential smoothing with trend (Holt);
- Dampen trends/growth rates: first 'principle' of Castle, Doornik, and Hendry (2021);
- Theta method of Assimakopoulos and Nikolopoulos (2003): trend halved – did well in M3 competition.

Doornik, Castle, and Hendry (2026b)(2026a)

use broken logarithm of the trend (or broken trend) as a forecast correction for UK inflation *after only a few systematic errors*.

The log function is attractive but inconvenient.

- Motivation
- **Inverse hyperbolic sine**
- **Testing for breaks**
- **Indicator saturation**
- **Forecasting**
- Conclusions

Consider the **inverse hyperbolic sine** function ($\operatorname{arcsinh}$ or asinh):

$$\operatorname{asinh}(x) = \int_0^x \frac{dt}{(1+t^2)^{1/2}}, \quad \mathbb{R} \rightarrow \mathbb{R}$$

$$\operatorname{asinh}(-x) = -\operatorname{asinh}(x),$$

$$\operatorname{asinh}(0) = 0,$$

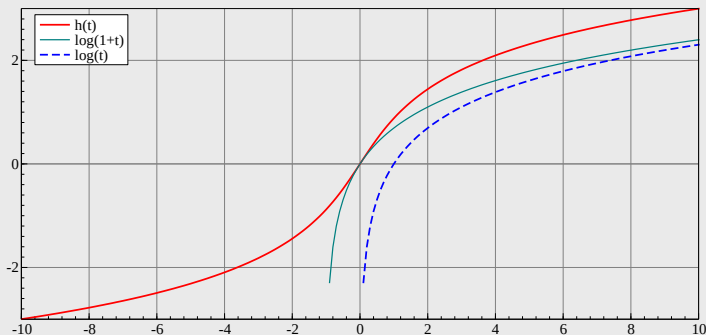
$$\sinh(\operatorname{asinh}(x)) = x.$$

Power series:

$$\operatorname{asinh}(x) = x - \frac{1}{2} \frac{x^3}{3} + \frac{1}{24} \frac{3x^5}{5} - \dots \quad |x| < 1,$$

$$\operatorname{asinh}(x) = \log(2x) + \frac{1}{2} \frac{1}{2x^2} - \frac{1}{24} \frac{3}{4x^4} + \dots \quad |x| > 1.$$

$$\operatorname{asinh}(x) \equiv h(x) = \log \left[x + (x^2 + 1)^{1/2} \right]. \quad (1)$$



computation

Use (1) except when:

- $x < 0$ and large: use $h(-x)$ to avoid cancellation,
- x small: use power series to avoid cancellation.

$h(x) = \log \left[x + (x^2 + 1)^{1/2} \right]$ is a convenient transformation:

- 1 defined over the real line,
- 2 log-like,
- 3 can be applied directly to TIS trends with break after period S :

$$\text{TIS: } \tau_S^{*'} = (-S+1, -S+2, \dots, -2, -1, 0, 0, \dots, 0),$$

$$\text{HIS: } h_S^{*'} = h(\tau_S^{*'}).$$

Note on construction of TIS trends

TIS trend for date S is zero at S , negative and upward trending before S . And zero from $S + 1$, where an unbroken trend would continue 1, 2, 3.... This is convenient for forecasting: no TIS trend extrapolates into the future.

Only the full trend (if included) extrapolates.

What happens in sample stays in sample.

Outline of procedure

- 1 Decide on max no breaks, break model, ε (spacing, e.g. 10%), α .
- 2 Search for 1 break with smallest RSS (max test at min RSS);
- 3 Test
- 4 If significant search for two breaks with smallest RSS (hard) (no conditioning on date of first break) ...
- 5 Otherwise stop.

Two potential issues:

- 1 Specific to general
Step 2 consistent for level breaks if number underspecified
- 2 Need search algorithm
Bai and Perron (2003) give algorithm using $T \times T$ matrix.
Does not strictly apply when only trend breaks.
Doornik (2022) provides alternative.

A small simulation illustrates the issues.

- 1 Model 1: level breaks
- 2 Model 2: trend breaks
- 3 Model 3: level and trend break
- 4 Model 4: log-trend breaks
- 5 Model 5: level and log-trend break

DGP matches model, $T = 100$, $M = 10\,000$, $\epsilon \sim \text{IIN}[0, 1]$.
Two breaks at $1/3$ and $2/3$ of sample.

More formally for $t = 1, \dots, T$

$$y_t = \eta_1(1)1_{\{t > T/3\}} + \eta_1(2)1_{\{t > 2T/3\}} + \eta_2(1)D_t(T/3) + \eta_2(2)D_t(2T/3) + \epsilon_t, \epsilon_t \sim N[0, 1].$$

With broken trend $[t-B]1_{\{t > B\}}$ or $h([t-B]1_{\{t > B\}})$ and the $j = 1, 2$ breaks:

DGP Model 1: $\eta_1(j) = 1, \eta_2(j) = 0,$

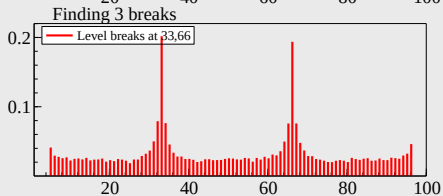
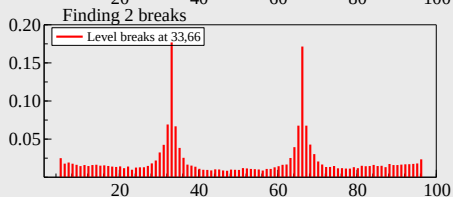
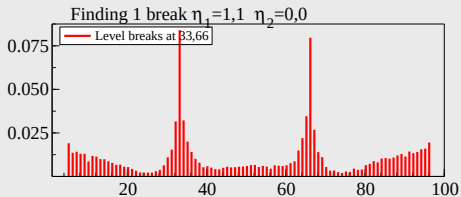
DGP Model 2: $\eta_1(j) = 0, \eta_2(j) = 1, D_t(B) = [t-B]1_{\{t > B\}}$

DGP Model 3: $\eta_1(j) = 1, \eta_2(j) = 1, D_t(B) = [t-B]1_{\{t > B\}}$

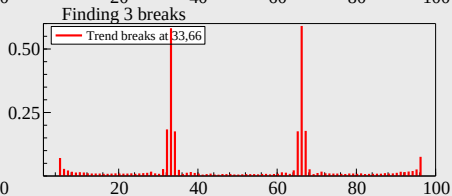
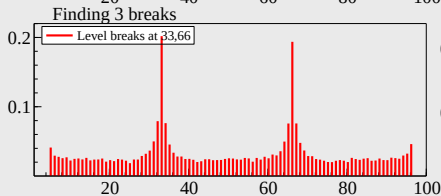
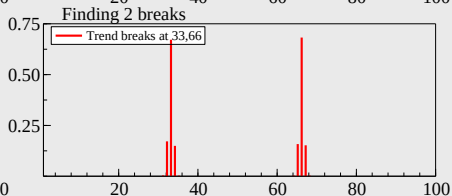
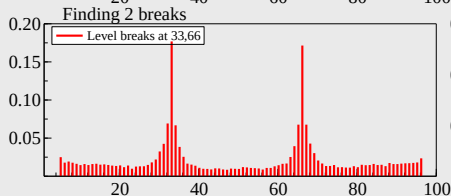
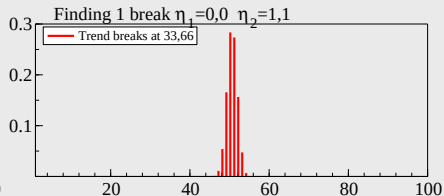
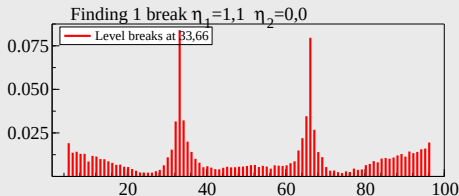
DGP Model 4: $\eta_1(j) = 0, \eta_2(j) = 1, D_t(B) = h([t-B]1_{\{t > B\}})$

DGP Model 5: $\eta_1(j) = 1, \eta_2(j) = 1, D_t(B) = h([t-B]1_{\{t > B\}})$

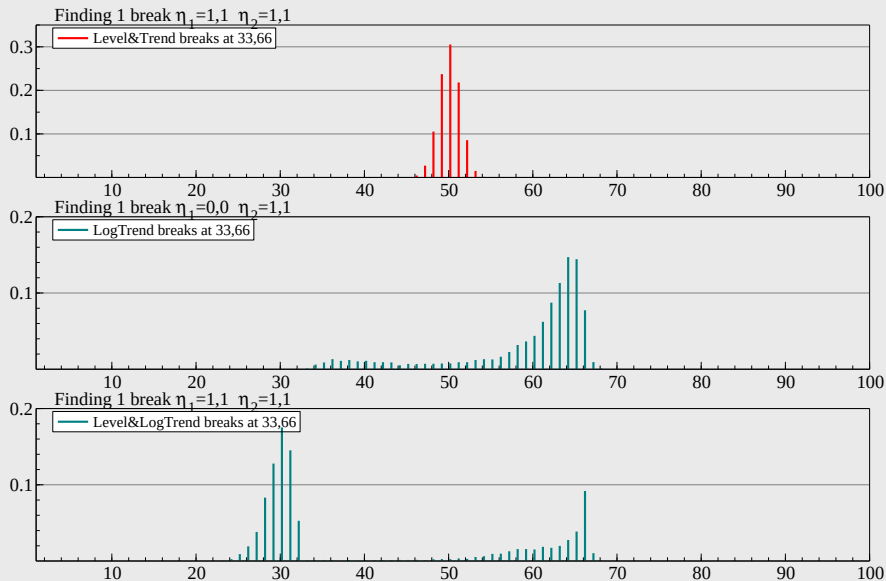
Model 1: searching for 1,2,3 breaks



Model 1 (left). Model 2 (right) inconsistent selection



Model 3 (top), 4 (middle), 5 (bottom): searching for 1 break



$M = 10\,000$, $T = 100$, $\varepsilon = 0.1$:

Testing for 1 break:

	Model 1	Model 2	Model 4
Rejection frequency at 5% and no break			
size	0.052	0.064	0.063
Rejection frequency at 5% and 2 breaks			
$\eta_1 = 3, 3$; $\eta_2 = 0, 0$:	0.26		
$\eta_1 = 3, -3$; $\eta_2 = 0, 0$:	1.00		
$\eta_1 = 0, 0$; $\eta_2 = 1, 1$:		1.00	0.99
$\eta_1 = 0, 0$; $\eta_2 = 1, -1$:		0.02	0.74

Castle, Doornik, Hendry, and Pretis (2020) derive the analytical non-centrality of the test for a trend that starts at $B + 1$. This is based on a linear trend:

$$\begin{aligned}\tau_{B,t} &= [t - B]1_{t > B}, \quad t = 1, \dots, T \\ (\tau_{B,1}, \dots, \tau_{B,T}) &= (0, \dots, 0, 1, 2, 3, \dots, T - B), \\ h_{B,t} &= h(\tau_{B,t}).\end{aligned}$$

Starting from

$$y_t = \mu + \lambda \tau_{B,t} + \epsilon_t, \quad \epsilon_t \sim \text{IIN}[0, \sigma_\epsilon^2],$$

the second moment matrix can be orthogonalized in two ways:

a estimate in deviations from the mean

$$y_t = \mu_a + \lambda (\tau_{B,t} - \bar{\tau}_B) + \epsilon_t,$$

b maintain the zeros in the demeaned trend:

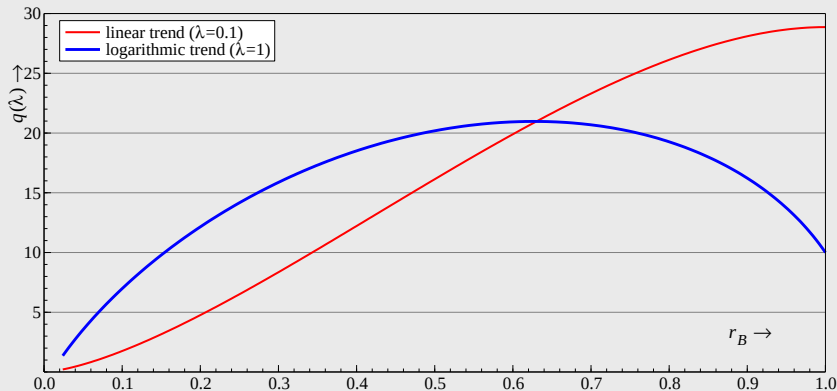
$$y_t = \mu_b + \lambda \left(\tau_{B,t} - \frac{T}{B} \bar{\tau}_B 1_{\{t \leq B\}} \right) + \epsilon_t,$$

Squared non-centrality for testing $t_\lambda = 0$, where $r_B = (T - B)/T$:

	linear trend	logarithmic trend
a: deviations	$(4 - 3r_B) r_B^3 \frac{T^3}{12} \frac{\lambda^2}{\sigma_\epsilon^2}$	$[(\log(T r_B))^2 (1 - r_B) + r_B] r_B T \frac{\lambda^2}{\sigma_\epsilon^2}$
b: maintain zeros	$r_B^3 \frac{T^3}{12} \frac{\lambda^2}{\sigma_\epsilon^2}$	$r_B T \frac{\lambda^2}{\sigma_\epsilon^2}$
a=b: $B = 0, r_B = 1$	$\frac{T^3}{12} \frac{\lambda^2}{\sigma_\epsilon^2}$	$T \frac{\lambda^2}{\sigma_\epsilon^2}$

The logarithmic trend terms were found by trial and error and numerical checking up to $T = 10^9$. More accurate result replaces $\log^2(T r_B)$ with $\log^2(T r_B) [1 - 0.75 \log \log(T r_B) / \log(T r_B)]^{0.25}$.

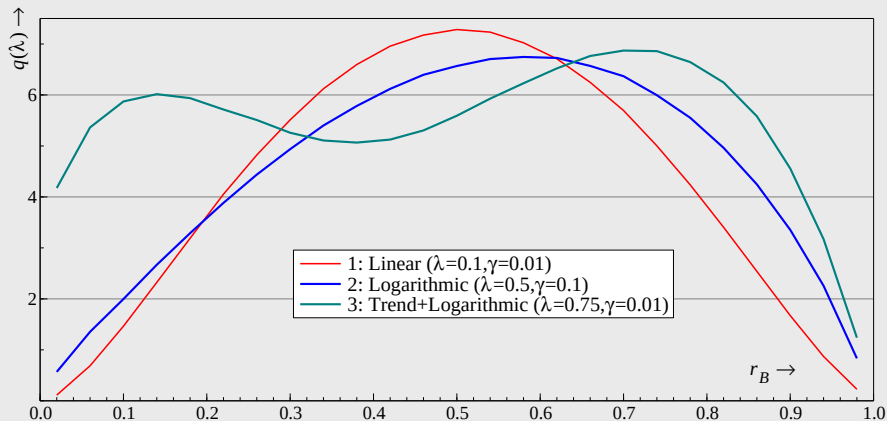
Non-centrality as the break moves through the sample,
 $\sigma_\epsilon^2 = 1$, $T = 100$, $\lambda = 0.1$ (linear trend), $\lambda = 1$ (logarithmic trend).



1: $y_t = \mu + \gamma t + \lambda \tau_{B,t} + \epsilon_t,$

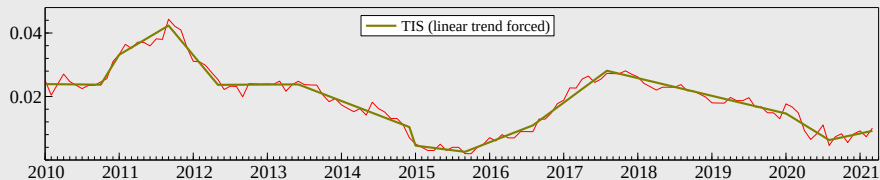
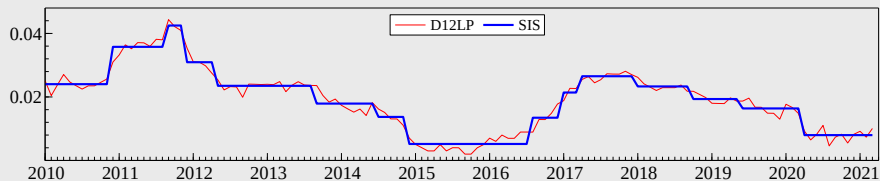
2: $y_t = \mu + \gamma h(t) + \lambda h(\tau_{B,t}) + \epsilon_t,$

3: $y_t = \mu + \gamma t + \lambda h(\tau_{B,t}) + \epsilon_t.$

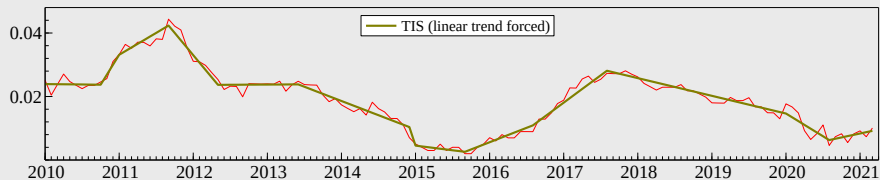
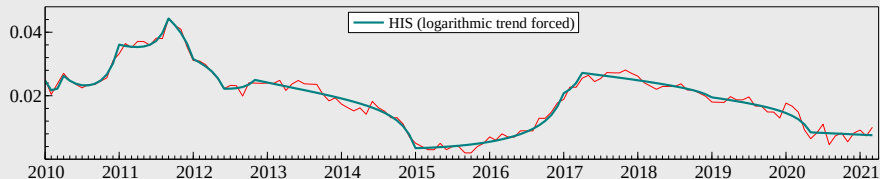
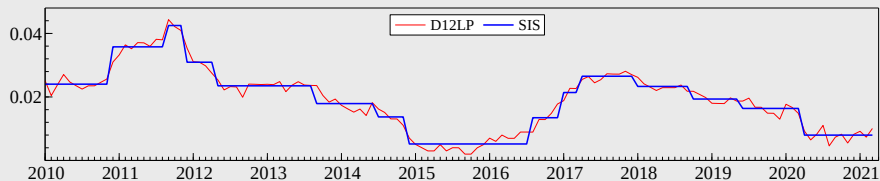


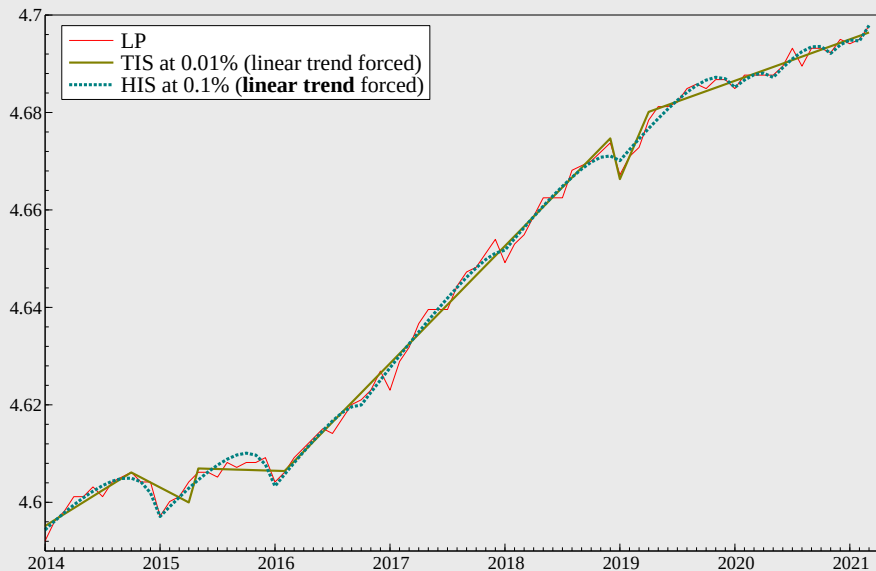
LP: monthly data for 2010(1)–2024(3) on the log of the UK CPIH.
Applying SIS, HIS, TIS to $D12LP = \Delta_1 2 \log(P_t)$ over 2010(1) to 2024(3), $T = 135$:

- SIS at 0.1% with intercept forced,
Step saturation
- TIS at 0.1% with intercept and linear trend t forced,
Linear trend saturation
- HIS at 0.1% with intercept and logarithmic trend $h(t)$ forced.
Logarithmic trend saturation (inverse hyperbolic sine)



SIS vs HIS vs TIS D12LP





Motivated by Doornik, Castle, and Hendry (2026b), we wish to contrast fixing a forecasting model with either partial trends or logarithmic trend.

- Using $h([t-B]1_{\{t>B\}})$ rather than $\log(1 + [t-B]1_{\{t>B\}})$,
- Using a fixed procedure without mixing logarithmic and linear.

Aim is to compare the two.

First estimate model up to T_{old} and fix intervention type, then:

- 1 **Extend sample** by $T = T_{old} + S$, re-estimate, forecast:
this is the new 1-step forecast;
- 2 **Add intervention** to this model, and test significance at α :
Either $\tau_{T-D,t} = [t - (T-D)]1_{\{t>T-D\}}$ or $h(\tau_{T-D,t})$
- 3 **If significant** keep intervention in model and forecast:
this is new multi-step sequence, also replaces new 1-step;
- 4 **Otherwise** remove intervention.

Three control parameters: S, D, α .

E.g. $D = 3$: active in last 3 obs. in sample (and out of sample).

Using $D = 3$, model estimated up to $T+3$ and h_t as trend from $T+1$:

H_0^F Forecast Chow test (Model M_1 or M_2);

H_0^R Replacement test (Model M_2);

H_0^I Intervention test (Model M_3).

T	T + 1	T + 2	T + 3	
M_1	$\iota_1 1_{\{t=T+1\}}$	$\iota_2 1_{\{t=T+2\}}$	$\iota_3 1_{\{t=T+3\}}$	$H_0^F : \iota_1 = \iota_2 = \iota_3 = 0$
M_2	0 ←	$\iota_2 1_{\{t=T+2\}}$ ηh_t	$\iota_3 1_{\{t=T+3\}}$ →	$H_0^F : \eta = \iota_2 = \iota_3 = 0$ $H_0^R : \iota_2 = \iota_3 = 0$
M_3	←	ηh_t	→	$H_0^I : \eta = 0$

Add intervention if:

- CDH: H_0^F rejects and H_0^R does not reject; encompassing decides shape of h_t function.
- Here: H_0^I rejects; form of h_t fixed in advance.

Difference (e.g.): one forecast failure out of three.

Extending Doornik, Castle, and Hendry (2026b) with the missing case of the logarithmic trends.

$$\begin{aligned} \text{DGP-t } y_t &= 5.5 + \lambda \tau_{80,t}^* + 0.05t + 0.025\epsilon_t, & \epsilon_t &\sim \text{IIN}[0, 1], \\ \text{DGP-h } y_t &= 5.5 + \lambda h(\tau_{80,t}^*) + 0.05h(t) + 0.025\epsilon_t, & t &= 1, \dots, T+H. \end{aligned}$$

$$\begin{aligned} \text{Model-t } y_t &= \beta_0 + \beta_1 t + \eta \tau_{80,t} + \{\iota_j 1_{\{t=T+j\}}, j=2..H\} + u_t, \\ \text{Model-ht } y_t &= \beta_0 + \beta_1 t + \eta h(\tau_{80,t}) + \{\iota_j 1_{\{t=T+j\}}, j=2..H\} + u_t, \\ \text{Model-h } y_t &= \beta_0 + \beta_1 h(t) + \eta h(\tau_{80,t}) + \{\iota_j 1_{\{t=T+j\}}, j=2..H\} + u_t. \end{aligned}$$

- Model-ht is mis-specified for DGP-t.
- Model-h with DGP-h not considered in paper.

Rejection frequency at $\alpha = 0.01$, $T = 80$, $H = 3$

λ	H_0^F $\eta=\iota_2=\iota_3=0$	H_0^R $\iota_2=\iota_3=0$	H_0^I $\eta=0$	$H_1^F \& H_0^R$
DGP-t and Model-t				
-0.1	1.0000	0.0095	1.0000	0.9905
-0.05	0.9999	0.0118	1.0000	0.9999
-0.025	0.6207	0.0111	0.7993	0.6207
-0.001	0.0099	0.0095	0.0107	0.0099
DGP-t and Model-ht				
-0.1	1.0000	0.1517	1.0000	0.8483
-0.05	0.9999	0.0359	1.0000	0.9999
-0.025	0.6207	0.0173	0.7921	0.6207
-0.001	0.0099	0.0102	0.0102	0.0099
DGP-h and Model-h				
-0.1	1.0000	0.0003	1.0000	0.9997
-0.05	1.0000	0.0138	1.0000	0.9862
-0.025	0.9114	0.0220	0.9746	0.9114
-0.001	0.0110	0.0106	0.0120	0.0110

Data is monthly UK CPIH inflation from 2010(1) onwards. Initial model estimated up to 2021(3), using:

1 TIS/LT

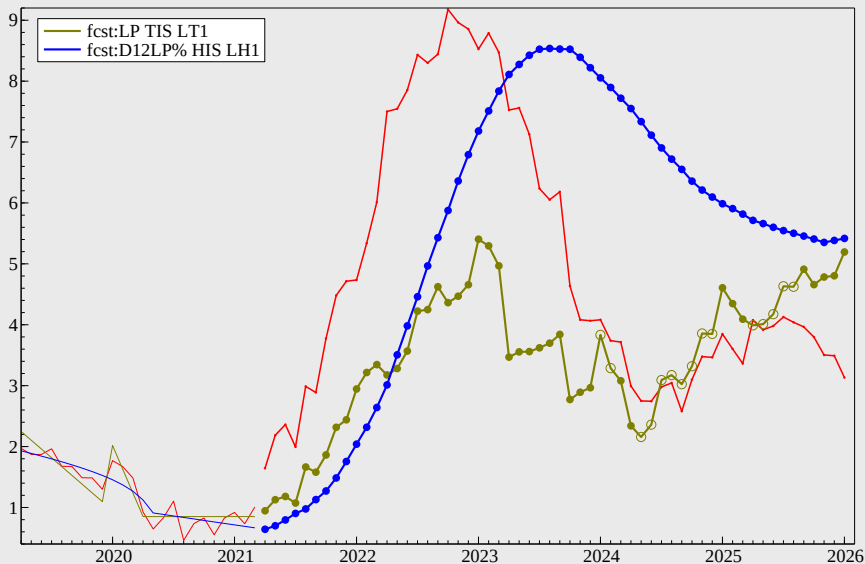
- TIS with *Autometrics* at $\alpha_a = 0.0001$ for $LP = \log(P_t)$ and forced trend and intercept.
- Interventions $\tau_{T-D,t}$.
- Forecasts derived for $D12LP\% = 100\Delta_{12} \log(P_t)$.

2 HIS/LH

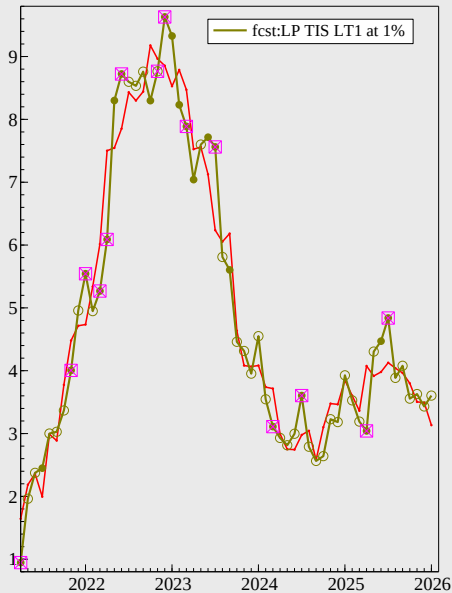
- HIS with *Autometrics* at $\alpha_a = 0.001$ for $D12LP\%$ and forced logarithmic trend and intercept.
- Interventions $h(\tau_{T-D,t})$.

$S = 1, D = 3$.

1-step forecasts, re-estimating every period (no interventions)

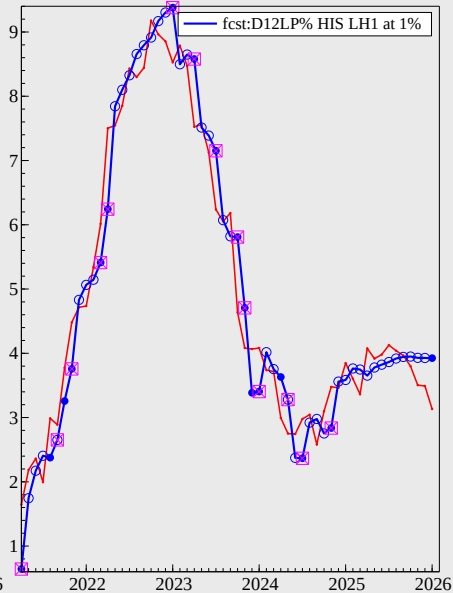


1-step forecasts, $\alpha = 0.01$, re-estimating with interventions



Doornik (Oxford)

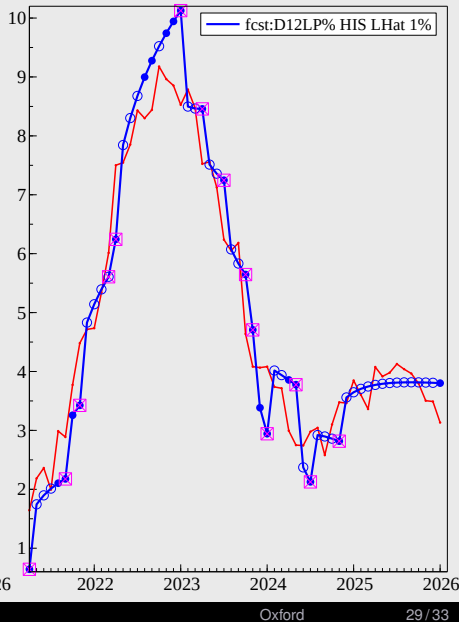
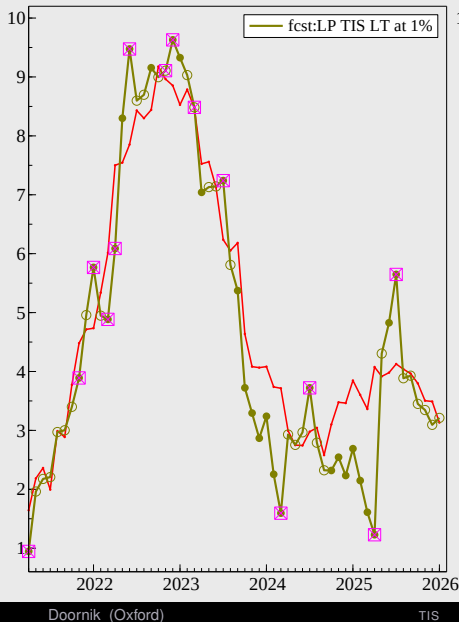
TIS



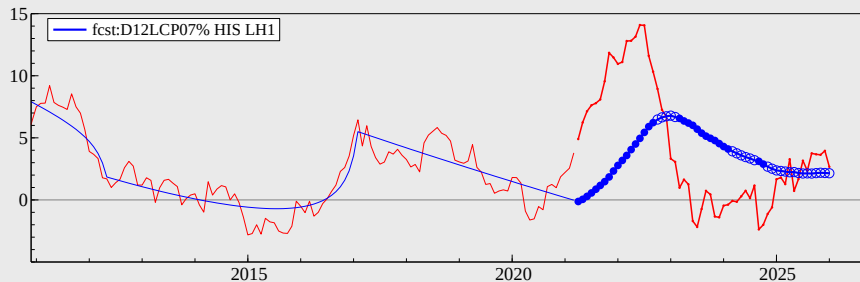
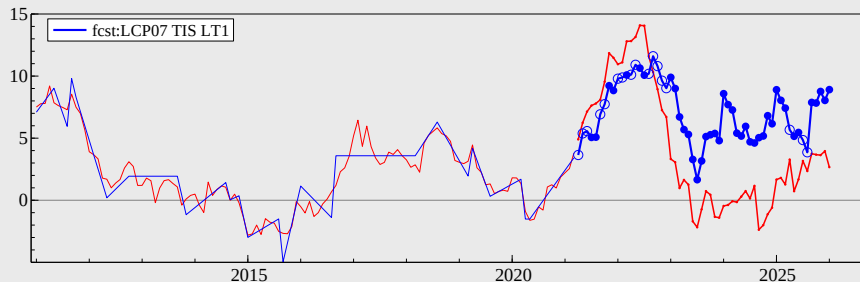
Oxford

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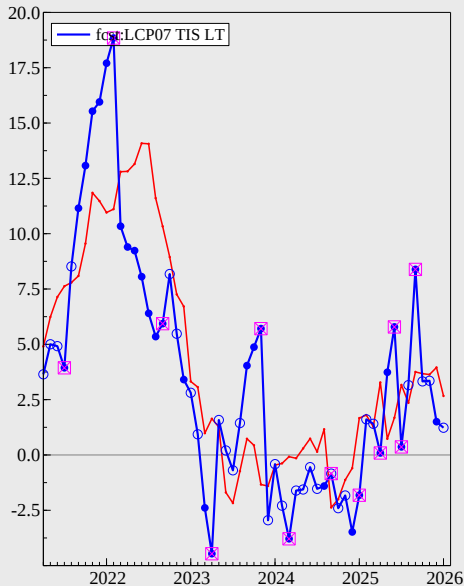
Multi-step forecasts between interventions, $\alpha = 0.01$



Transport component: 1-step forecasts, re-estimating every period (no interventions), $\alpha = 0.0$

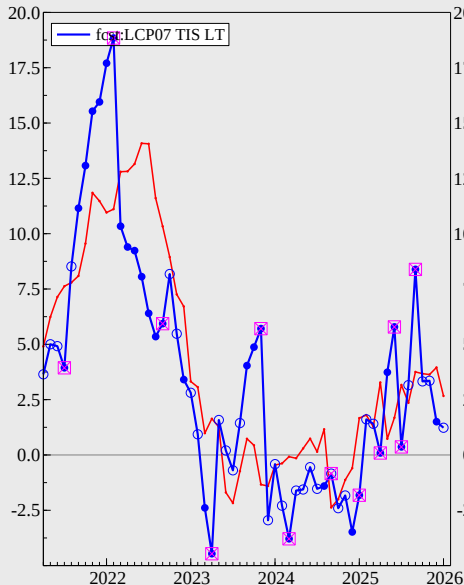


$\alpha = 0.05$



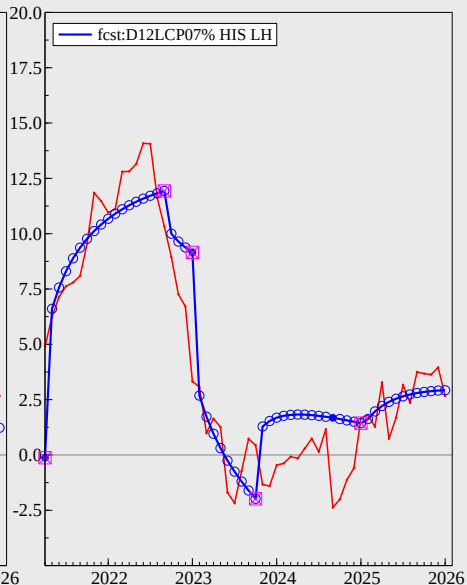
Transport component: Multi-step forecasts between interventions

$\alpha = 0.05$



Doornik (Oxford)

TIS



Oxford

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Investigated logarithmic trend

Inverse hyperbolic sine convenient function

Four settings

- Test for break at unknown date
- Saturation
- Forecasting
- (Trend cycle decomposition)



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Thank You

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