Cooperative Advertising in a Dynamic Retail Market Duopoly

Anshuman Chutani ∗, Suresh P. Sethi †

Abstract

Cooperative advertising is a key incentive offered by a manufacturer to influence retailers’ promotional decisions. We study cooperative advertising in a dynamic retail duopoly where a manufacturer sells his product through two competing retailers. We model the problem as a Stackelberg differential game in which the manufacturer announces his shares of advertising costs of the two retailers or his subsidy rates, and the retailers in response play a Nash differential game in choosing their optimal advertising efforts over time. We obtain the feedback equilibrium solution consisting of the optimal advertising policies of the retailers and manufacturer’s subsidy rates. We identify key drivers that influence the optimal subsidy rates and in particular, obtain the conditions under which the manufacturer will support one or both of the retailers. We analyze its impact on profits of channel members and the extent to which it can coordinate the channel. We investigate the case of an anti-discriminatory act which restricts the manufacturer to offer equal subsidy rates to the two retailers. Finally, we discuss two extensions: First, a retail oligopoly with any number of retailers, and second, the retail duopoly that also considers optimal wholesale and retail pricing decisions of the manufacturer and retailer, respectively.

Keywords: Cooperative advertising, Nash differential game, Stackelberg differential game, sales-advertising dynamics, Sethi model, feedback Stackelberg equilibrium, retail level competition, channel coordination, Robinson-Patman act.

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1 Introduction

Cooperative advertising is a common means by which a manufacturer incentivizes retailers to advertise its product to increase its sales. In a typical arrangement, the manufacturer contributes a percentage of a retailer’s advertising expenditures to promote the product. We consider a marketing channel involving a manufacturer and two retailers. We model the problem of the channel as a Stackelberg differential game in which the manufacturer acts as the leader by announcing its subsidy rate to each of the two retailers, who act as followers and play a Nash Differential game in order to obtain their optimal advertising efforts in response to the support offered by the manufacturer.

Cooperative advertising is a fast increasing activity in retailing amounting to billions of dollars a year. Nagler (2006) found that the total expenditure on cooperative advertising in 2000 was estimated at $15 billion, compared with $900 million in 1970 and according to some recent estimates, it was higher that $25 billion in 2007. Cooperative advertising can be a significant part of the manufacturer’s expense according to Dant and Berger (1996), and as many as 25-40% of local advertisements and promotions are cooperatively funded. In addition, Dutta et al. (1995) report that the subsidy rates differ from industry to industry: it is 88.38% for consumer convenience products, 69.85% for other consumer products, and 69.29% for industrial products.

Many researchers in the past have used static models to study cooperative advertising. Berger (1972) modeled cooperative advertising in the form of a wholesale price discount offered by the manufacturer to its retailer as an advertising allowance. He concluded that both the manufacturer and the retailer can do better with cooperative advertising. Dant and Berger (1996) extended the Berger model to incorporate demand uncertainty and considered a scenario where the manufacturer and its retailer have a different opinion on anticipated sales. Kali (1998) examined cooperative advertising subsidy with a threshold minimum advertised price by the retailer, and found that the channel can be coordinated in this case. Huang et al. (2002) allowed for advertising by the manufacturer in addition to coopera-
tive advertising, and justified their static model by making a case for short-term effects of promotion.

Jørgensen et al. (2000) formulated a dynamic model with cooperative advertising, as a Stackelberg differential game between a manufacturer and a retailer with the manufacturer as the leader. They considered short term as well as long term forms of advertising efforts made by the retailer as well as the manufacturer. They showed that the manufacturer’s support of both types of retailer advertising benefits both channel members more than support of just one type, which in turn is more beneficial than no support at all. Jørgensen et al. (2001) modified the above model by introducing decreasing marginal returns to goodwill and studied two scenarios: a Nash game without advertising support and a Stackelberg game with support from the manufacturer as the leader. Jørgensen et al. (2003) explored the possibility of advertising cooperation even when the retailer’s promotional efforts may erode the brand image. Karray and Zaccour (2005) extended the above model to consider both the manufacturer’s national advertising and the retailer’s local promotional effort. All of these papers use the Nerlove-Arrow (1962) model, in which goodwill increases linearly in advertising and decreases linearly in goodwill, and there is no interacting term between sales and advertising effort in the dynamics of sales.

He et al. (2009) solved a manufacturer-retailer Stackelberg game with cooperative advertising using the stochastic sales-advertising model proposed by Sethi (1983), in which the effectiveness of advertising in increasing sales decreases as sales increase. The Sethi model was validated empirically by Chintagunta and Jain (1995) and Naik et al. (2008). Despite the presence of the interactive term involving sales and advertising, He et al. (2009) were able to obtain a feedback Stackelberg solution for the retailer’s optimal advertising effort and the manufacturer’s subsidy rate, and provided a condition for positive subsidy by the manufacturer.

This paper extends the work of He et al. (2009) to allow for retail level competition and provides useful managerial insights on the impact of this competition on the manufac-
turer’s decision. It contributes to the cooperative advertising literature in the following ways. Firstly, most of the cooperative advertising literature uses a one manufacturer, one retailer setting, with the exception of He et al. (2011), who study a retail duopoly. Their formulation, however, is based on the Lanchester setting (see, for e.g., Little (1979)), in which the two competitors split a given total market. We, on the other hand, use Erickson (2009)’s duopolistic extension of the Sethi model, in which competitors could increase their shares of a given total potential market at the same time. We formulate the model as a Stackelberg differential game between the manufacturer as the leader and the retailers as the followers. Furthermore, the two retailers competing for market share play a Nash game between themselves. While our model is considerably more complicated, we are still able to obtain, like in He et al. (2009), a feedback equilibrium solution, sometimes explicitly and sometimes by numerical means. We also explore the conditions under which the manufacturer supports neither of the retailers, just one, or both. We consider the cases when the second retailer also buys from the manufacturer and when he does not. When the second retailer does not buy from the manufacturer, we are able to show the impact of the added retail level competition on the manufacturer’s tendency to support the first retailer. Furthermore, we are able to extend the above threshold conditions and the impact of retail level competition in explicit form, to an extension considering an oligopoly of $N$ identical retailers. Secondly, we investigate in greater detail the issue of supply chain coordination with cooperative advertising. We study the extent to which the supply chain can be coordinated with cooperative advertising, and its effect on the profits of all the parties in the supply chain. Finally, we look into the case of anti-discrimination legislations (such as the Robinson-Patman Act against price discrimination), under which the manufacturer is forced to offer equal subsidy rates to the two retailers. We obtain feedback Stackelberg equilibrium in this case and compare the optimal common subsidy rate with the two optimal subsidy rates without such an act. We also investigate the impact of such legislations on profits of manufacturer, retailers and overall supply chain, and subsequently, its role in coordinating the supply chain.
While most of the analysis is performed with the margins of the retailers and manufacturers as exogenous variables, we also discuss an extension that incorporates wholesale and retail pricing decisions along with subsidy rates and advertising decisions for the manufacturer and the retailers, respectively. Table 1 summarizes a comparison of several research papers in the cooperative advertising area, including ours, and positions our paper on the basis of various issues that are self-explanatory.

The rest of the paper is organized as follows. We describe our model in section 2 and present preliminary results in section 3. We obtain explicit analytical results for a special case of identical retailers in section 4, and that for \( N \) identical retailers in section 5. We perform numerical analysis for the general case in section 6. In section 7, we discuss the issue of channel coordination brought about by cooperative advertising and analyze its effect on the manufacturer’s and retailers’ profits. In section 8, we present an extension in which the manufacturer is required to offer equal subsidy rates, if any, to both retailers. In section 9 we discuss a model in which wholesale prices and retail prices are also decision variables for the manufacturer and the retailers, respectively. We conclude the paper in section 10.

Table 1: A comparison of various research papers based on issues addressed

<table>
<thead>
<tr>
<th>Papers →</th>
<th>Issues addressed ↓</th>
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<tbody>
<tr>
<td>Berger (1972)</td>
<td>✓</td>
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<tr>
<td>Dant &amp; Berger (1996)</td>
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<td>Kali (1998)</td>
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<td>Huang et al. (2002)</td>
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<td>Karray &amp; Zaccour (2005)</td>
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<td>He et al. (2009)</td>
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<tr>
<td>He et al. (2011)</td>
<td>✓</td>
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<tr>
<td>This paper</td>
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Dynamic model | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
Advertising interacting with sales | ✓ | ✓ |
Stochastic demand | ✓ | ✓ |
Pricing decisions | ✓ | ✓ |
Feedback strategy | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
Retail level competition | ✓ |
Channel performance & coordination | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
Anti-discrimination legislation | ✓ |
Proofs of some of the results are relegated to the appendices of the paper. In Appendix D, we discuss uniqueness of the optimal solution obtained in our analysis.

2 The model

We consider a dynamic market channel where a manufacturer sells its product through one or both of two independent and competing retailers, labeled 1 and 2. The manufacturer may choose to subsidize the advertising expenditures of the retailers. The subsidy, expressed as a fraction of a retailer’s total advertising expenditure, is referred to as the manufacturer’s subsidy rate for that retailer. We use the following notation in the paper:

\[ t \quad \text{Time } t \in [0, \infty), \]
\[ i \quad \text{Indicates retailer } i, \ i = 1,2, \text{ when used as a subscript,} \]
\[ x_i(t) \in [0,1] \quad \text{Retailer } i \text{'s proportional market share,} \]
\[ u_i(t) \quad \text{Retailer } i \text{'s advertising effort rate at time } t, \]
\[ \theta_i(t) \geq 0 \quad \text{Manufacturer’s subsidy rate for retailer } i \text{ at time } t, \]
\[ \rho_i > 0 \quad \text{Advertising effectiveness parameter of retailer } i, \]
\[ \delta_i \geq 0 \quad \text{Market share decay parameter of retailer } i, \]
\[ r > 0 \quad \text{Discount rate of the manufacturer and the retailers,} \]
\[ m_i \geq 0 \quad \text{Gross margin of retailer } i, \]
\[ M_i \geq 0 \quad \text{Gross margin of the manufacturer from retailer } i, \]
\[ V_i, V_m \quad \text{Value functions of retailer } i \text{ and of the manufacturer, respectively,} \]
\[ V \quad \text{Value function of the integrated channel;} \]

also \( V_{ix_j} = \frac{\partial V_i}{\partial x_j}, \ i = 1,2, \ j = 1,2, \) and \( V_{mx_i} = \frac{\partial V_m}{\partial x_i} \) and \( V_{xi} = \frac{\partial V}{\partial x_i}, \ i = 1,2. \)

The state of the system is represented by the market share vector \((x_1, x_2)\), so that the
state at time \( t \) is \((x_1(t), x_2(t))\). The sequence of events is as follows. First, the manufacturer announces the subsidy rate \( \theta_i(t) \) for retailer \( i \), \( i = 1, 2, t \geq 0 \). In response, the retailers choose their respective advertising efforts \( u_1(t) \) and \( u_2(t) \) in order to compete for market share. This situation is modeled as a Stackelberg game between the manufacturer as the leader and the retailers as followers and a Nash differential game between the retailers; see Fig. 1. The solution concept we employ is that of a feedback Stackelberg equilibrium. The

![Figure 1: Sequence of Events](image)

The cost of advertising is assumed to be quadratic in the advertising effort, signifying a marginal diminishing effect of advertising. Given the subsidy rates \( \theta_i \), the retailer \( i \)'s advertising expenditure is \((1-\theta_i)u_i^2\). The total advertising expenditure for the manufacturer is \( \theta_1 u_1^2 + \theta_2 u_2^2 \). The quadratic cost function is common in the literature (see, e.g., Deal (1979), Chintagunta and Jain (1992), Jorgensen et al. (2000), Prasad and Sethi (2004), Erickson (2009), and He et al. (2009)).

To model the effect of advertising on sales over time, we use an oligopolistic extension of the Sethi (1983) model, proposed by Erickson (2009). This extension is different from the duopolistic extensions of the Sethi model studied by Sorger (1989), Prasad and Sethi (2004, 2009), and He et al. (2011), where the competitors split a given total market. Here, a gain in the market share of one retailer comes from an equal loss of the market share of the other. In contrast, the Erickson extension permits even a simultaneous increase of the retailers’ shares of a given total market potential. Moreover, Erickson (2009) used his extension to study the competition between Anheuser-Busch, SABMiller, and Molson Coors in the beer
industry.

We adopt the duopolistic version of the Erickson extension as our market share dynamics:

\[
\dot{x}_i(t) = \frac{dx_i(t)}{dt} = \rho_i u_i(t) \sqrt{1 - x_1(t) - x_2(t)} - \delta_i x_i(t), \quad x_i(0) = x_i \in [0, 1], \quad i = 1, 2, \quad (1)
\]

where, for \( i = 1, 2 \), \( x_i(t) \) is the fraction of the total market captured by retailer \( i \) at time \( t \), \( u_i(t) \) is retailer \( i \)'s advertising effort at time \( t \), \( \rho_i \) is the effectiveness of retailer \( i \)'s advertising effort, and \( \delta_i \) is the rate at which market share is lost by retailer \( i \) due to factors such as forgetting and customers switching to other substitutable products.

Because the total market share captured by the manufacturer is \( x_1(t) + x_2(t) \) at any time \( t \), the advertising effort of a retailer acts upon the square-root of the uncaptured market potential. This is the main distinguishing feature of the models, which are extensions of the Sethi model, from the classical Vidale and Wolfe (1957) model, where the advertising effort acts simply upon the uncaptured market potential. Some justification of the square root feature and its empirical validation can be found in Sethi (1983), Sorger (1989), Chintagunta and Jain (1995), Naik et al. (2008), and Erickson (2009a, 2009b). Furthermore, the advertising effort \( u_i \) is subject to marginally diminishing returns modeled by having its cost as \( u_i^2 \), \( i = 1, 2 \). Finally, the subsidy rates do not affect the market share dynamics, as they simply reflect the internal cost sharing arrangements between the manufacturer and the retailers.

Since we are interested in obtaining a feedback Stackelberg solution, the manufacturer announces his subsidy rate policy \( \theta_1(x_1, x_2) \) and \( \theta_2(x_1, x_2) \) as functions of the market share vector \( (x_1, x_2) \). This means that the subsidy rates at time \( t \geq 0 \) are \( \theta_i(x_1(t), x_2(t)) \), respectively, for \( i = 1, 2 \). The retailers then choose their optimal advertising efforts by solving their respective optimization problems in order to maximize the present value of their respective profit streams over the infinite horizon. Thus, retailer \( i \)'s optimal control problem is
\[ V_i(x_1, x_2) = \max_{u_i(t) \geq 0, t \geq 0} \int_0^\infty e^{-rt}(m_i x_i(t) - (1 - \theta_i(x_1(t), x_2(t)))u_i^2(t))dt, \quad i = 1, 2, \quad (2) \]

subject to (1), where we stress that \( x_1 \) and \( x_2 \) are initial conditions, which can be any given values satisfying \( x_1 \geq 0, x_2 \geq 0 \) and \( x_1 + x_2 \leq 1 \). Since retailer \( i \)'s problem is an infinite horizon optimal control problem, we can define \( V_i(x_1, x_2) \) as his so-called value function. In other words, \( V_i(x_1, x_2) \) also denotes the optimal value of the objective function of retailer \( i \) at a time \( t \geq 0 \), so long as \( x_1(t) = x_1 \) and \( x_2(t) = x_2 \) at that time. It should also be mentioned that the problem (1)-(2) is a Nash differential game, whose solution will give retailer \( i \)'s feedback advertising effort, expressed with a slight abuse of notation as \( u_i(x_1, x_2 | \theta_1(x_1, x_2), \theta_2(x_1, x_2)) \), respectively, for \( i = 1, 2 \).

The manufacturer anticipates the retailers’ optimal responses and incorporates these into his optimal control problem, which is also a stationary infinite horizon problem. Thus, the manufacturer’s problem is given by

\[ V_m(x_1, x_2) = \max_{0 \leq \theta_1(t) \leq 1, 0 \leq \theta_2(t) \leq 1, t \geq 0} \int_0^\infty e^{-rt}\left\{ M_1 x_1(t) + M_2 x_2(t) - \theta_1(t) [u_1(x_1(t), x_2(t) | \theta_1(t), \theta_2(t))]^2 - \theta_2(t) [u_2(x_1(t), x_2(t) | \theta_1(t), \theta_2(t))]^2 \right\} dt, \quad (3) \]

subject to for \( i = 1, 2, \)

\[ \dot{x}_i(t) = \rho_i u_i(x_1(t), x_2(t) | \theta_1(t), \theta_2(t)) \sqrt{1 - x_1(t) - x_2(t) - \delta_i x_i(t), x_i(0) = x_i \in [0, 1]}. \quad (4) \]

Here, with an abuse of notation, \( \theta_1(t) \) and \( \theta_2(t) \) denote the subsidy rates at time \( t \geq 0 \) to be obtained. Solution of the control problem (3)-(4) yields the optimal subsidy policy in feedback form expressed as \( \theta_i^*(x_1, x_2) \), \( i = 1, 2 \). Furthermore, we can express retailer \( i \)'s feedback advertising policy, with an abuse of notation, as \( u_i^*(x_1, x_2) = u_i^*(x_1, x_2 | \theta_1^*(x_1, x_2), \theta_2^*(x_1, x_2)) \), \( i = 1, 2 \).
The policies \( \theta_i^*(x_1, x_2) \) and \( u_i^*(x_1, x_2) \), \( i = 1, 2 \), constitute a feedback Stackelberg equilibrium of the problem\((1)-(4)\), which is time consistent, as opposed to an open-loop Stackelberg equilibrium, which, in general, is not. Substituting these policies into the state equations in \((1)\) yields the market share process \((x_1^*(t), x_2^*(t))\), \( t \geq 0 \), and the respective decisions, with notational abuses, as \( \theta_i^*(t) = \theta_i^*(x_1^*(t), x_2^*(t)) \) and \( u_i^*(t) = u_i^*(x_1^*(t), x_2^*(t)) \), \( t \geq 0 \), \( i = 1, 2 \).

3 Preliminary results

We first solve the problem of retailer \( i \) to find the optimal advertising policy \( u_i^*(x_1, x_2 \mid \theta_1(x_1, x_2), \theta_2(x_1, x_2)) \), given the subsidy policies \( \theta_1(x_1, x_2) \) and \( \theta_2(x_1, x_2) \) announced by the manufacturer. The Hamilton-Jacobi-Bellman (HJB) equations for the value functions \( V_i(x_1, x_2) \), and \( i = 1, 2 \), are

\[
 rV_i(x_1, x_2) = \max_{u_i \geq 0} \left[ m_i x_i - (1 - \theta_i(x_1, x_2))u_i^2 + V_{ix_i}(\rho_i u_i \sqrt{1 - x_1 - x_2 - \delta_i x_i}) 
 + V_{ix_3-i}(\rho_{3-i} u_{3-i} \sqrt{1 - x_1 - x_2 - \delta_{3-i} x_{3-i}}) \right], \quad i = 1, 2,
\]

where \( V_{ix_j} \) can be interpreted as the marginal increase in the total discounted profit of retailer \( i, i = 1, 2 \), with respect to an increase in the market share of retailer \( j, j = 1, 2 \).

\textbf{Remark 1:} Before proceeding further, we note that while we restrict \( \theta_1(x_1, x_2) \) and \( \theta_2(x_1, x_2) \) to be nonnegative, it should be obvious that \( 0 \leq \theta_i(x_1, x_2) < 1, i = 1, 2 \). This is because, were the optimal \( \theta_i(x_1, x_2) \geq 1, i = 1, 2 \), the retailer would set \( u_i(x_1, x_2) \) to be infinitely large, resulting in the value function of the manufacturer to be \(-\infty\). This would mean that the manufacturer, who is the leader, would have even less profit than he would by setting \( \theta_1(x_1, x_2) = \theta_2(x_1, x_2) = 0 \). This leads to a contradiction proving that \( \theta_i(x_1, x_2) < 1, i = 1, 2 \). Thus, in what follows, any positive \( \theta_i(x_1, x_2) \) will be an interior solution satisfying \( \theta_i(x_1, x_2) < 1, i = 1, 2 \).

We can now prove the following result characterizing the optimal advertising policy given the subsidy rates of the manufacturer.
Proposition 1: For a given subsidy rate policy \( \theta_i(x_1, x_2), i = 1, 2 \), the optimal feedback advertising decision of retailer \( i \) is

\[
u_i^* = u_i^*(x_1, x_2 | \theta_1, \theta_2) = \frac{V_{ix_i} \rho_i \sqrt{1 - x_1 - x_2}}{2(1 - \theta_i(x_1, x_2))}, \quad i = 1, 2,
\]

and the value function \( V_i(x_1, x_2) \) satisfies

\[
rV_i(x_1, x_2) = \max_{\theta_1 \geq 0, \theta_2 \geq 0} \left[ \sum_{i=1}^{2} \left( M_i x_i - \theta_i u_i^* \right)^2 + V_{mx_i} (\rho_i u_i^* \sqrt{1 - x_1 - x_2 - \delta_i x_i}) \right]
\]

Proof: Since the cost of advertising effort \( u_i \) is \( u_i^2 \), it is clear that there is no sense in having a negative \( u_i, i = 1, 2 \). Thus, we can use the first-order conditions w.r.t \( u_i, i = 1, 2 \), in (5) to obtain (6), and use (6) in (5) to obtain (7). We see that the advertising effort by retailer \( i \) is proportional to the marginal benefit of his own market share, \( i = 1, 2 \). Moreover, the higher is the uncaptured market \( (1 - x_1 - x_2) \), the greater is the advertising effort by retailer \( i \).

After each retailer decides on an optimal response to the manufacturer’s announced subsidy policy, the manufacturer solves his problem, taking into account the retailers’ choices, and decides the optimal subsidy rates. The HJB equation for the manufacturer’s value function \( V_m(x_1, x_2) \) is

\[
rV_m(x_1, x_2) = \max_{\theta_1 \geq 0, \theta_2 \geq 0} \sum_{i=1}^{2} \left[ M_i x_i - \theta_i u_i^* \right]^2 + V_{mx_i} (\rho_i u_i^* \sqrt{1 - x_1 - x_2 - \delta_i x_i})
\]

Using (6), we can rewrite the HJB equation as

\[
rV_m(x_1, x_2) = \max_{\theta_1 \geq 0, \theta_2 \geq 0} \sum_{i=1}^{2} \left[ M_i x_i - V_{mx_i}(1 - \theta_i) \right] + \frac{V_{ix_i}(1 - x_1 - x_2)(2V_{mx_i}(1 - \theta_i) - V_{ix_i} \theta_i) \rho_i^2}{4(-1 + \theta_i)^2}
\]

We can now state the following result characterizing the manufacturer’s optimal subsidy rates policy.
Proposition 2: The manufacturer’s optimal subsidy rates are

\[ \theta_1^*(x_1, x_2) = \max\{\hat{\theta}_1(x_1, x_2), 0\}, \quad \theta_2^*(x_1, x_2) = \max\{\hat{\theta}_2(x_1, x_2), 0\}, \quad (9) \]

where

\[ \hat{\theta}_1(x_1, x_2) = \frac{2V_{mx_1} - V_{1x_1}}{2V_{mx_1} + V_{1x_1}}, \quad \hat{\theta}_2(x_1, x_2) = \frac{2V_{mx_2} - V_{2x_2}}{2V_{mx_2} + V_{2x_2}}, \quad (10) \]

and the manufacturer’s value function \( V_m(x_1, x_2) \) satisfies

\[ rV_m(x_1, x_2) = \sum_{i=1}^{2} \left[ M_i x_i - V_{mx_i} x_i \delta_1 \right. \]
\[ \left. + \frac{(V_{ix_i}(1 - x_1 - x_2)(2V_{mx_i}(1 - \theta_1^*(x_1, x_2)) - V_{ix_i}, \theta_1^*(x_1, x_2))\delta_2^2)}{4(-1 + \theta_1^*(x_1, x_2))^2} \right] \quad (11) \]

Proof: The first-order conditions w.r.t \( \theta_1 \) and \( \theta_2 \) in (8) give \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) as shown in (10). We then, in view of Remark 1, characterize the optimal subsidy rate policy as in (9). Using (9) in (8) gives (11).

A number of important insights follow from our analysis thus far. Equation (10) says that the optimal subsidy rate offered by the manufacturer to a retailer increases as the manufacturer’s marginal profit, with respect to the market share of that retailer increases. The increased subsidy from the manufacturer would also increase the retailer’s advertising effort. Thus, the manufacturer provides more support to the retailer who offers a higher marginal profit from his market share to the manufacturer. On the other hand, as a retailer’s own marginal profit from his own market share increases, then the subsidy rate offered by the manufacturer to that retailer decreases. The intuition behind this result is that the manufacturer would lower his subsidy to the retailer from the knowledge that the retailer has his own incentive to increase his very profitable market share anyway by advertising at a higher rate. Thus, we can expect that the advertising effort by the retailer will increase with the marginal profit of the retailer as well as that of the manufacturer from that retailer. This is confirmed by the following relation, which is derived from (6) and (10).
\[ u_i^*(x_1, x_2) = \rho_i(2V_{mx_i} + V_{ix_i})\sqrt{1-x_1-x_2}. \]

In the dynamic programming equations (7) and (11), we see that their right-hand sides are linear in \( x_1 \) and \( x_2 \), except for \( \theta_1^i(x_1, x_2) \) and \( \theta_2^i(x_1, x_2) \) to be determined. Taking a cue from Sethi (1983), we shall look for linear value functions. That is, we use the forms

\[
V_i(x_1, x_2) = \alpha_i + \beta_i x_i + \gamma_i x_{3-i}, \quad i = 1, 2, \tag{12}
\]

\[
V_m(x_1, x_2) = \alpha + B_1 x_1 + B_2 x_2, \tag{13}
\]

and then try to solve for the coefficients \( \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, B_1 \) and \( B_2 \). With these, we see that

\[
V_{ix_i} = \beta_i, \quad V_{ix_{3-i}} = \gamma_i, \quad \text{and } V_{mx_i} = B_i, \quad i = 1, 2, \tag{14}
\]

and therefore \( \hat{\theta}_i(x_1, x_2) \) and \( \theta_i^*(x_1, x_2), \ i = 1, 2 \), given in (9) and (10) would be constants. Thus, from here on, we shall simply denote them as \( \hat{\theta}_i \) and \( \theta_i^* \), respectively, \( i = 1, 2 \). We substitute (12)-(13) into (7) and (11), set the coefficients of \( x_1 \) and \( x_2 \) and the constant terms to equal zero in the resulting equations, and obtain the following system of equations to be solved for the coefficients in (12)-(13): For \( i = 1, 2 \),

\[
4r\alpha_i = -\frac{\beta_i^2 \rho_i^2}{(-1+\theta_i^*)^2} + \frac{2\beta_{3-i}\gamma_i}{(-1+\theta_i^*)} + \frac{\beta_{3-i}^2}{(-1+\theta_i^*)}, \tag{15}
\]

\[
4r\beta_i = 4m_i - 4\beta_i \delta_i + \frac{2\beta_{3-i} \gamma_i}{(-1+\theta_i^*)} + \frac{\beta_i^2}{(-1+\theta_i^*)}, \tag{16}
\]

\[
4r\gamma_i = -4\gamma_i \delta_{3-i} + \frac{2\beta_{3-i} \gamma_i^2}{(-1+\theta_i^*)} + \frac{\beta_i^2}{(-1+\theta_i^*)}, \tag{17}
\]

\[
r\alpha = -\frac{\beta_i^2 \rho_i^2}{4(-1+\theta_i^*)^2} + \frac{\beta_i B_i \rho_i^2}{2(-1+\theta_i^*)} \frac{\beta_{3-i}^2}{2(-1+\theta_i^*)^2} + \frac{\beta_{3-i} B_{3-i} \rho_{3-i}^2}{2(-1+\theta_i^*)^2}, \tag{18}
\]

\[
rB_i = M_i - B_i \delta_i + \frac{\beta_i^2 \rho_i^2}{4(-1+\theta_i^*)^2} + \frac{\beta_i B_i \rho_i^2}{2(-1+\theta_i^*)^2} + \frac{\beta_{3-i}^2 \rho_{3-i}^2}{4(-1+\theta_i^*)^2} + \frac{\beta_{3-i} B_{3-i} \rho_{3-i}^2}{2(-1+\theta_i^*)^2}, \tag{19}
\]

\[
\theta_i^* = \max\{\frac{2B_i - \beta_i}{2B_i + \beta_i}, 0\}. \tag{20}
\]
Our analysis also reveals a condition under which the manufacturer will support a retailer. To explore this, we let

\[ P_1 = 2V_{mx_1} - V_{1x_1} = 2B_1 - \beta_1 \quad \text{and} \quad P_2 = 2V_{mx_2} - V_{2x_2} = 2B_2 - \beta_2. \]  

(21)

Then, the subsidy rate for retailer \( i \) (given by (9) and (10)) depends on the sign of \( P_i \), \( i = 1, 2 \). Thus, when \( P_i > 0 \), the manufacturer supports retailer \( i \), otherwise he does not.

When \( P_1 \leq 0 \) and \( P_2 \leq 0 \), neither retailer receives any advertising support from the manufacturer. Thus, \( \theta^*_1 = \theta^*_2 = 0 \), and equations (15)-(17) can be solved independently of (18)-(19). By computing the coefficients \( \beta_i \) and \( B_i \), \( i = 1, 2 \), we can then write the conditions for a non-cooperative solution, i.e., \( P_1 \leq 0, P_2 \leq 0 \), in terms of the parameters \( m_i, M_i, \rho_i, \delta_i, i = 1, 2 \). It is also clear that in this case, our model reduces to the duopoly version of the model studied by Erickson (2009).

The explicit solution of the system of equations (15)-(20) is difficult to obtain in general, and therefore, the condition \( P_1 \leq 0, P_2 \leq 0 \) for non-cooperation cannot be given explicitly. However, a simple and intuitive observation can be made as follows. Using \( \theta^*_1 = \theta^*_2 = 0 \), we can rewrite (19) for \( i = 1, 2 \), as

\[ B_i = \frac{2M_i(r + \delta_{3-i}) + (M_i - M_{3-i})\beta_{3-i}\rho^2_{3-i}}{2(r + \delta_i)(r + \delta_{3-i}) + (r + \delta_{3-i})\beta_i\rho^2_i + (r + \delta_i)\beta_{3-i}\rho^2_{3-i}}. \]

Clearly, it makes sense to have the value function \( V_i \) of retailer \( i \) increase with his own market share, \( i = 1, 2 \). Thus, we expect \( \beta_i = \partial V_i / \partial x_i > 0, i = 1, 2 \). In view of this, \( M_2 = 0 \) implies \( B_2 < 0 \), and hence \( P_2 < 0 \) from (21). This verifies and obvious conclusion that when the manufacturer sells through only one retailer, say retailer 1, so that \( M_2 = 0 \), \( P_2 \) is always negative and retailer 2 is never supported.

In the special case of identical retailers, defined when \( m_1 = m_2, \rho_1 = \rho_2, \) and \( \delta_1 = \delta_2 \), some explicit results can be obtained. In addition to this, when \( M_1 = M_2 \), which we refer to as the case of symmetric retailers, we can obtain additional explicit results. Nevertheless,
even in the general case, it is easy to solve the system numerically.

4 Identical retailers

Let \( m_1 = m_2 = m, \rho_1 = \rho_2 = \rho, \) and \( \delta_1 = \delta_2 = \delta. \) We may also assume \( M_1 > M_2 \) without loss of generality. Recall that \( P_i \leq 0, i = 1, 2, \) ensures that no retailers will be supported by the manufacturer. In order to obtain the required condition for no cooperation at all, we set \( \theta_1^* = 0 \) and \( \theta_2^* = 0 \) in the system of equations (15)-(19), and then explicitly solve for all of the coefficients. Specifically, as derived in Appendix A, we obtain for \( i = 1, 2, \)

\[
P_i = \frac{1}{(r + \delta)} \left[ M_i \left( \frac{4(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}} \right) - \frac{1}{(r + \delta)} \left[ M_{3-i} \left( \frac{-2(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}} \right) \right] \right]
\]

and conclude the following result.

**Proposition 3:** When \( P_1, P_2 \leq 0, \) we have a non-cooperative equilibrium in which it is optimal for the manufacturer to not support any retailer. Furthermore, if \( P_1 > 0 \) and \( P_2 \leq 0, \) we have \( \theta_1^* > 0 \) and \( \theta_2^* = 0, \) i.e., the manufacturer supports retailer 1 only.

One can observe that \( P_1 \) and \( P_2 \) are linear in \( M_1 \) and \( M_2. \) In \( P_1, \) the coefficient of \( M_1 \) is positive and that of \( M_2 \) is negative. Thus, \( P_1 \) increases as the margin of the manufacturer from retailer 1 increases, and it decreases as that from retailer 2 increases. As retailer 1 offers a higher margin to the manufacturer, he gets closer to the point of getting advertising support. Moreover, this increase in margin from retailer 1 further hampers the case of retailer 2 getting support from the manufacturer. Indeed, it can be seen that

\[
P_1 - P_2 = \frac{2(M_1 - M_2)}{(r + \delta)},
\]

(23)
which means that $M_1 > M_2$ implies $P_1 > P_2$. Thus, in the case when neither of the retailers are supported, i.e., when $P_1 \leq 0$ and $P_2 \leq 0$, we will never see a situation when a change in the parameters $(M_1, M_2, m, \rho, \delta, r)$ will induce the manufacturer to start supporting retailer 2 only, as long as $M_1 > M_2$. In other words, retailer 1 will be the first to start receiving a positive subsidy rate, whenever changes in the parameters takes place, as $P_1$ would change sign from negative to positive before $P_2$ when $M_1 > M_2$.

The expressions for $P_i, i = 1, 2$, in (22) is still quite complicated. It simplifies a great deal, however, if we assume that the decay coefficient and the discount rate are very small, i.e., $r + \delta \approx 0$. Under this condition, we have

$$\text{sgn}(P_1) = \text{sgn} \left( M_1 - M_2 - \frac{2m}{3} \right) \quad \text{and} \quad \text{sgn}(P_2) = \text{sgn} \left( M_2 - M_1 - \frac{2m}{3} \right),$$

(24)

where $\text{sgn}(y)$ represents the sign of $y$.

Our analysis of the identical retailers case with small $(r + \delta)$ reveals some important insights. We see from (24) that $P_2 < 0$ if $M_1 \geq M_2$, and retailer 2 will not be supported. Thus at any instant at most one retailer will be supported. In addition, retailer 1 will be supported provided that the difference $(M_1 - M_2)$ in the manufacturer’s margins from the retailers is greater than two-thirds of the retailers’ margin.

This last observation above clearly and explicitly brings out the effect of retail level competition when compared to the results obtained in He et al. (2009) which deal with a one manufacturer - one retailer channel. For small $(r + \delta)$, they concluded that if the manufacturer’s margin from the retailer is greater than the retailer’s margin, then the retailer will be supported. In our case, the manufacturer will start supporting retailer 1 as soon as $M_1 - M_2$ exceeds just two-thirds of the retailers’ margin. To make the comparison even clearer, let us assume that $M_2 = 0$, so that retailer 2 does not sell the manufacturer’s product, but he nevertheless competes with retailer 1. Then under the competition, the condition for support is $M_1 \geq 2m/3$, whereas without the presence of retailer 2 it is $M_1 \geq m$. 

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4.1 Symmetric Retailers \((M_1 = M_2 = M)\)

When \(M_1 = M_2 = M\) in addition to the retailers being identical, (22) reduces to

\[
P_1 = P_2 = \frac{6M(r + \delta)}{(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m\rho^2(r + \delta)^2 + m^2\rho^4}} - \frac{-2(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m\rho^2(r + \delta)^2 + m^2\rho^4}}{3(r + \delta)^2}.\]

We can see from the above expression that \(P_1\) and \(P_2\) would quickly approach a negative value as \((r + \delta)\) approaches 0, in fact when \((r + \delta) \approx 0\), it approaches \(-\infty\). This can also be verified by (24) and we find that in this case no retailer will be supported. Figure 2 shows the behavior of \(P_1(= P_2)\) with \((r + \delta)\) for \(M = 1, m = 0.5, \rho = 1\).

![Figure 2: Threshold function \(P_1, P_2\) in the case of symmetric retailers](image)

5 Non-identical retailers: numerical analysis

We perform numerical analysis to study the dependence of the manufacturer’s subsidy rates on the manufacturer’s margins \((M_1, M_2)\) from retailers 1 and 2, respectively, the retailers’ margins \((m_1, m_2)\), the advertising effectiveness coefficients \((\rho_1, \rho_2)\), and the decay coefficients \((\delta_1, \delta_2)\). The base case is \(M_1 = M_2 = 1, m_1 = m_2 = 0.25, \rho_1 = \rho_2 = 1, \delta_1 = \delta_2 = 0.2,\) and \(r = 0.05\). We then vary each parameter one by one to study how it affects \(\theta_1^*\) and \(\theta_2^*\).

The numerical analysis involves only solving a system of equations, which is fairly
straightforward to carry out. In all instances, we find a unique solution to the system. For the case of symmetric retailers, we prove the uniqueness in Appendix D. In what follows, we describe the results obtained from the numerical analysis.

a) Effect of the manufacturer’s margin (Fig. 3): If the manufacturer’s margin from retailer 1 increases, the manufacturer starts offering a higher subsidy rate to retailer 1 and reduces the subsidy rate to retailer 2. Thus, the manufacturer rewards retailer 1 for providing him a higher margin in two ways.

b) Effect of a retailer’s margin (Fig. 4): As the margin of retailer 1 increases, both $\theta_1^*$ and $\theta_2^*$ decrease. The decrease in $\theta_2^*$ is smaller compared to that in $\theta_1^*$. Since the manufacturer’s margins are kept constant, we can say that if the margin of retailer 1 increases relative to the margin he offers to the manufacturer, then the manufacturer starts reducing his subsidy to both retailers, and more drastically with retailer 1 than with retailer 2. Thus, if all other parameters remain the same, the retailer with the higher margin gets a lower subsidy rate.

c) Effect of the advertising effectiveness parameter (Fig. 5): As the advertising effectiveness of retailer 1 increases, the subsidy rates for both retailers decrease. The rate of decrease is higher for retailer 2 than for retailer 1. All other parameters being the same, the retailer with the more effective advertising gets a higher subsidy rate.

d) Effect of the decay coefficient (Fig. 6): As $\delta_1$ increases, the subsidy rates for both retailers increase. However, $\theta_2^*$ is more sensitive to increase in $\delta_1$ than $\theta_1^*$.

6 Cooperative advertising and channel coordination

In this section, we analyze the role of cooperative advertising as a tool to better coordinate the channel and improve the overall channel profit. We evaluate and compare the channel profit in three cases. First, we consider an integrated channel where the advertising decisions are based on the maximization of total profit for the manufacturer and the retailers together. Second, we consider a decentralized channel with the optimal subsidy rates. Third, we
consider a decentralized channel without cooperative advertising.

In the first case, given the retailers’ and manufacturer’s margins \((m_1, m_2, M_1, M_2)\), the optimization problem for the integrated channel can be written as follows:

\[
V(x_1, x_2) = \max_{u_1(t) \geq 0, u_2(t) \geq 0, t \geq 0} \int_0^\infty e^{-rt} ((M_1 + m_1)x_1(t) + (M_2 + m_2)x_2(t) - u_1^2(t) - u_2^2(t))dt \quad (25)
\]

subject to

\[
\dot{x}_i(t) = \frac{dx_i(t)}{dt} = \rho_i u_i(t) \sqrt{1 - x_1(t) - x_2(t)} - \delta_i x_i(t), \quad x_i(0) = x_i \in [0, 1], \ i = 1, 2. \quad (26)
\]
The HJB equation for the value function \( V \) is

\[
 rV(x_1, x_2) = \max_{u_1 \geq 0, u_2 \geq 0} [(M_1 + m_1)x_1 + (M_2 + m_2)x_2 - u_1^2 - u_2^2 + V_{x_1} \dot{x}_1 + V_{x_2} \dot{x}_2],
\]

(27)

where \( \dot{x}_1 \) and \( \dot{x}_2 \) are given by (26). Using (26) in the HJB equation (27) and applying the first-order conditions for maximization w.r.t. \( u_1 \) and \( u_2 \) give the following result.

**Proposition 4:** For the integrated channel, the optimal feedback advertising policies are

\[
 u_1^* = \frac{1}{2} \rho_1 V_{x_1} \sqrt{1 - x_1 - x_2}, \quad u_2^* = \frac{1}{2} \rho_2 V_{x_2} \sqrt{1 - x_1 - x_2},
\]

(28)

and the integrated channel’s value function satisfies

\[
 4rV(x_1, x_2) = 4(M_1 + m_1 - V_{x_1} \delta_1)x_1 + 4(M_2 + m_2 - V_{x_2} \delta_2)x_2 + (1 - x_1 - x_2)(V_{x_1}^2 \rho_1^2 + V_{x_2}^2 \rho_2^2).
\]

(29)

Once again, we conjecture a linear value function of the form \( V(x_1, x_2) = \alpha^I + \beta_1^I x_1 + \beta_2^I x_2 \), where \( \alpha^I, \beta_1^I = V_{x_1} \) and \( \beta_2^I = V_{x_2} \) are constants, and solve the following system of equations:

\[
 4r \alpha^I = \beta_1^I \rho_1^2 + \beta_2^I \rho_2^2,
\]

(30)

\[
 4r \beta_i^I = 4(m_i + M_i) - 4 \beta_i^I \delta_i - \beta_i^I \rho_i^2 - \beta_i^I \rho_3^2 - \beta_i^I \rho_3^2 - \delta_i - \rho_3^2 - \rho_3^2 - \delta_i, \quad i = 1, 2.
\]

(31)

The set of equations (30)-(31) is obtained by comparing the coefficients of \( x_1 \) and \( x_2 \), and the constant terms in (29) with \( \beta_1^I (= V_{x_1}) \), \( \beta_2^I (= V_{x_2}) \), and \( \alpha^I \), respectively.

In the second case, we have a decentralized channel with cooperative advertising, for which we define the channel value function as \( V^c(x_1, x_2) = V^c_m(x_1, x_2) + V^c_r(x_1, x_2) \), where \( V^c_m \) is the manufacturer’s value function (given by (13)) and \( V^c_r \) is the total value function of both retailers (obtained by (12) for \( i = 1, 2 \)).

In the third case, namely, a decentralized channel with no cooperation, the channel value function is defined as \( V^n(x_1, x_2) = V^n_m(x_1, x_2) + V^n_r(x_1, x_2) \), where \( V^n_m \) and \( V^n_r \) are
the manufacturer’s value function and the sum of the two retailers’ value functions in the non-cooperative setting, respectively. These are computed by simply setting $\theta_1^* = \theta_2^* = 0$ in (15)-(19) and then using (12)-(13).

Before we proceed further, let us observe that the manufacturer is the leader and he obtains his optimal subsidy rates by maximizing his objective function. Therefore, it should be obvious that

$$V_c(x_1, x_2) \geq V_n(x_1, x_2). \quad (32)$$

Thus, it remains to study the effect of cooperative advertising on the retailers’ profits and the total channel profit. First, we examine this in the simple case of symmetric retailers. We now present the following result.

**Proposition 5:** *In the case of symmetric retailers, the value functions $V_r^c(x_1, x_2)$ and $V_r^n(x_1, x_2)$ depend only on the sum $(x_1 + x_2)$, and can thus be expressed as $V_r^c(x_1 + x_2)$ and $V_r^n(x_1 + x_2)$, with a slight abuse of notation. Furthermore, we have*

$$V_r^c(x_1 + x_2) \geq V_r^n(x_1 + x_2). \quad (33)$$

The proof of Proposition 5 is provided in Appendix C. It is clear from (32) and (33) that in the symmetric case, cooperative advertising can partially coordinate the channel, and that the manufacturer as well as the retailers are better off with cooperative advertising than without it.

We now return to the general case where explicit analytical relationships between various value functions are difficult to establish. We, therefore, resort to numerical analysis, and report our findings based on the results obtained. We compare $V, V_c, \text{ and } V_n$ with varying values of the optimal subsidy rates. Since the value functions depend on the state $(x_1, x_2)$, the analysis was carried out for different values of $(x_1, x_2)$.

We would like to study $V, V_c, \text{ and } V_n$ with respect to the changes in the optimal subsidy rates brought about by changes in the model parameters; for this we consider varying retailer
1’s margin. As $m_1$ increases, we know from Fig. 4, that the subsidy rates for both retailers decrease. As a result, we can compare the various value functions as $m_1$ increases or, roughly speaking, as subsidy rates decrease. Fig. 7 depicts the values of $V, V^c,$ and $V^n$ for $x_1 = x_2 = 0.3$. The data range for the calculations shown are the same as those used for the results shown in Fig. 4. Thus, for any point in Fig. 7, the values of the optimal subsidy rates are the same as the corresponding values in Fig. 4. Recall that as $m_1$ increases, the overall cooperation by the manufacturer decreases. It is found that under all instances, $V$ is greater than $V^c$ as well as $V^n$. This is understandable as we expect the channel value function in the integrated case to be higher than in the decentralized case, with or without cooperative advertising. In the scenario where both retailers get advertising support, we find that $V^c > V^n$, indicating that the channel attains partial coordination. Moreover, it is found that the difference $V - V^c$ is minimum at the point when both retailers receive equal positive subsidy rates, and it increases as the difference between the two subsidy rates increases. These results indicate that the level of coordination achieved is maximum when both retailers receive equal positive subsidy rates.

An interesting, perhaps even counter-intuitive, observation is that in the case when it is optimal for the manufacturer to support only one retailer, we find that the overall value function of the channel in the non-cooperative scenario is slightly higher than in the cooperative case. Thus, from the channel’s perspective, it is better in this case not to support any retailer than to support only one of the two.

This analysis was carried out for changes in parameters $m_1, \rho_1$ and $\delta_1$ with the corresponding changes in the optimal subsidy rates as shown in Figs. 4, 5 and 6, respectively, as well as for different values of $(x_1, x_2)$. However, we find that the nature of changes in $V, V^c$, and $V^n$ with respect to varying optimal subsidy rates does not change. Fig. 8 shows the difference in the value functions between the cooperative and non-cooperative settings for the manufacturer and the two retailers, respectively. As expected, the manufacturer always benefits from cooperative advertising. One of the retailers, however and surprisingly, do not
always seem to benefit from cooperative advertising. Furthermore, the manufacturer’s benefit from cooperative advertising increases as his subsidy increases. For the two retailers, it is found that as the difference between the two subsidy rates increases, the retailer getting the lower subsidy rate has a lower value function than in the non-cooperative setting. Moreover, as this difference in the subsidy rates increases further, even the combined value function of the retailers is lower than in the non-cooperative setting. Fig. 8 also shows that the region in which both retailers benefit from cooperative advertising is a small "window" around the point where the retailers get the same positive subsidy rate from the manufacturer. In other words, for both retailers to benefit from cooperative advertising, their subsidy rates should not differ significantly. Thus, the retailer getting a significantly lower subsidy rate might not be happy with the cooperative advertising program.

These observations raise an interesting issue relating to the differential treatment of the retailers by the manufacturer, i.e., when only one retailer is favored, then in some situations the channel makes less profit with cooperative advertising than without it. Indeed, this might be viewed as an additional argument for the non-discriminating practice legislated
under such acts as the Robinson-Patman Act of 1936 in the context of price discrimination, for reasons to enhance competition. In view of these results, we study next the case of non-discrimination in the context of cooperative advertising.

7 Equal subsidy rate for both retailers

We consider the case when the manufacturer is restricted to offer the same subsidy rate to both retailers. This case is motivated by possible legal issues that may arise when the manufacturer discriminates between the two retailers in terms of subsidy rates. Note that discrimination in terms of price, promotions, discounts, etc. are prohibited by the Robinson-Patman Act of 1936. Specifically, the Act proscribes discrimination in price between two or more competing buyers in the sale of commodities of like grade and quality. This and other anti-discrimination acts, such as the Sherman Antitrust Act of 1890, the Clayton Act of 1914, and the Celler-Kefauver Act of 1950, prevent discriminatory policies which might lead to reduced competition and create monopolies in the market.

In our model, different optimal subsidy rates for the two retailers arise from factors such as the manufacturer’s margins relative to the retailers’ margins, which affect the manufacturer’s profit. However, if the manufacturer is not allowed to offer different subsidy rates to the retailers, we need to reformulate our problem so that the manufacturer’s optimization problem has only one subsidy rate decision. In this case, let $V_m^{RP}(x_1, x_2), V_1^{RP}(x_1, x_2), V_2^{RP}(x_1, x_2)$ and $V^{RP}$ denote the value functions of the manufacturer, retailer 1, retailer 2, and the total channel, respectively, with the superscript RP standing for Robinson and Patman. These value functions solve the control problems defined by (1)-(4) with $\theta_1 = \theta_2 = \theta$. Once again, we expect these value functions to be linear in the market share vector, and express them as in (12)-(13), except that the coefficients will now be denoted as $\tilde{\alpha}, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{B}_1,$ and $\tilde{B}_2$. These coefficients will satisfy the system of equations obtained by setting $\theta_1^* = \theta_2^* = \theta^*$
in (15)-(19). Thus, we have the following equation system: For $i = 1, 2$,

\[
4r\tilde{\alpha}_i = -\frac{\tilde{\beta}_i^2\rho_i^2}{(1 + \theta^*)} + \frac{2\tilde{\beta}_3\tilde{\gamma}_i\rho_{3-i}^2}{(1 + \theta^*)},
\]

\[
4r\tilde{\beta}_i = 4m_i - 4\tilde{\beta}_i\delta_i + \frac{2\tilde{\beta}_3\tilde{\gamma}_i\rho_{3-i}^2}{(1 + \theta^*)} + \frac{\tilde{\beta}_i^2\rho_i^2}{(1 + \theta^*)},
\]

\[
4r\tilde{\gamma}_i = -4\tilde{\gamma}_i\delta_{3-i} + \frac{2\tilde{\beta}_3\tilde{\gamma}_i\rho_{3-i}^2}{(1 + \theta^*)} + \frac{\tilde{\beta}_i^2\rho_i^2}{(1 + \theta^*)},
\]

\[
r\tilde{\alpha} = -\frac{\tilde{\beta}_1^2\rho_1^2\theta^*}{4(1 + \theta^*)^2} + \frac{\tilde{\beta}_1\tilde{\beta}_2\rho_2^2}{2(1 + \theta^*)} + \frac{\tilde{\beta}_3\tilde{\beta}_3\rho_3^2\theta^*}{2(1 + \theta^*)^2} + \frac{\tilde{\beta}_3\tilde{\beta}_3\rho_3^2}{2(1 + \theta^*)},
\]

\[
r\tilde{\beta}_i = M_i - \tilde{\beta}_i\delta_i + \frac{\tilde{\beta}_1^2\rho_1^2\theta^*}{4(1 + \theta^*)^2} + \frac{\tilde{\beta}_1\tilde{\beta}_2\rho_2^2}{2(1 + \theta^*)} + \frac{\tilde{\beta}_3\tilde{\beta}_3\rho_3^2\theta^*}{4(1 + \theta^*)^2} + \frac{\tilde{\beta}_3\tilde{\beta}_3\rho_3^2}{2(1 + \theta^*)},
\]

\[
\theta^* = \max\left\{\frac{\tilde{\beta}_1^2(2\tilde{\beta}_1 - \tilde{\beta}_1)\rho_1^2 + \tilde{\beta}_2^2(2\tilde{\beta}_2 - \tilde{\beta}_2)\rho_2^2}{\tilde{\beta}_1^2(2\tilde{\beta}_2 + \tilde{\beta}_1)\rho_1^2 + \tilde{\beta}_2^2(2\tilde{\beta}_2 + \tilde{\beta}_2)\rho_2^2}, 0\right\}.
\]

The common threshold condition for no cooperation to be optimal is that

\[
P = \tilde{\beta}_1^2(2\tilde{\beta}_1 - \tilde{\beta}_1)\rho_1^2 + \tilde{\beta}_2^2(2\tilde{\beta}_2 - \tilde{\beta}_2)\rho_2^2 \leq 0.
\]

In the case of identical retailers, i.e., $m_1 = m_2 = m, \rho_1 = \rho_2 = \rho,$ and $\delta_1 = \delta_2 = \delta$, we can solve equations (34)-(38) explicitly when $\theta^* = 0$. The condition for no support by the manufacturer, with this simplification, reduces to

\[
9(r + \delta)^2\rho^2(M_1 + M_2) + \frac{(2(r + \delta)^2 - m^2\rho^2 - \sqrt{3(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^2})((r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^2})}{3(r + \delta)} \leq 0.
\]

In the general case, we performed numerical analysis to see the behavior of $\theta^*$ with respect to different model parameters. Figures 9, 10, 11 and 12 show the dependence of $\theta^*$ on $M_1, m_1, \rho_1$ and $\delta_1$, respectively, and compare the $\theta^*$ with the the optimal subsidy rates for two retailers without any legislation, i.e., $\theta^*_1$ and $\theta^*_2$. We find that as $M_1$ increases, the manufacturer has higher incentive to support retailer 1 and thus $\theta^*$ increases, but with a decreasing rate. The rate of increase of $\theta^*$ is lower than the rate of increase of $\theta^*_1$, which could be attributed to the fact that the subsidy rates have to be equal for both the retailers even though retailer
2 does not offer any incentive for higher support. The overall impact of parameters $m_1, \rho_1$ and $\delta_1$ on manufacturer’s support is similar to the general model, i.e., the subsidy rate $\theta^*$ decreases with $m_1$, decreases with $\rho_1$, and increases with $\delta_1$ because of reasons similar to those in the unrestricted model. Moreover, as is the case in the unrestricted model, the rate of change of $\theta^*$ (in absolute sense) decreases as any of the parameter values increases. It is noticeable though that $\theta^*$ lies between $\theta^*_1$ and $\theta^*_2$, and the retailer who is less profitable for the manufacturer and would have received a lower subsidy rate in the unrestricted model, benefits from this legislation.

![Figure 9: subsidy rate vs $M_1$](image9)

![Figure 10: subsidy rate vs $m_1$](image10)

![Figure 11: subsidy rate vs $\rho_1$](image11)

![Figure 12: subsidy rate vs $\delta_1$](image12)

Next, we study the impact of anti-discriminatory act on the profits of all the parties in
the supply chain and on the total channel profit. We compare the value functions in three cases: a channel without any cooperative advertising, a channel with no anti-discriminatory act and optimal subsidy rates, and a channel with an anti-discriminatory act and optimal common subsidy rate for the two retailers. Figure 13 shows the difference between the value functions in the cases of with and without the non-discriminatory legislation, with changes in subsidy rates brought about by the changes in $m_1$. As anticipated, the manufacturer earns less profit because of the added constraint to his optimization problem. The manufacturer’s loss is higher when $m_1$ is low, i.e., when subsidy rates $\theta^*, \theta_1^*$ and $\theta_2^*$ are high. We find that the retailer who would have obtained a higher subsidy rate in the absence of such legislation also earns less profit, whereas a less efficient retailer who would have earned a lower subsidy rate in an unconstrained problem benefits from this act. When $m_1$ is very low, the gain of the less efficient retailer is not able to offset the losses of the other two parties, and the supply chain in the whole loses. However, for a larger set of values of $m_1$, the inefficient retailer’s gain is more than the losses of the other two parties and the the total channel profit is higher. This result indicates that better supply chain coordination can be obtained with the anti-discriminatory act in effect. Figure 14 compares the the total profit of an integrated channel ($V$) with the total channel profit in three cases: no advertising cooperation ($V^n$), cooperation with no legislation ($V^c$), and cooperation with equal subsidy rates ($V^{RP}$). Here again, we see that with the non-discriminatory act, we are able to achieve a higher level of channel coordination in most cases, except when $m_1$ is too low.

8 Extension to $N$ identical retailers

We now consider an extension where the manufacturer sells through $N$ identical retailers and compute the subsidy threshold (21). Our objective is to find out how does an increase in retail level competition impact the willingness of the manufacturer to offer subsidy to the retailers and in this regard, we will extend some of the results presented in section 4.
Following similar notation, we define $x_i(t)$ as the market share of retailer $i$, $i = 1, 2 \cdots N$, $m$ as the margin of all the retailers, $\rho$ as the advertising effectiveness coefficient of all the retailers, $\delta$ as the sales decay coefficient of all the retailers, and $M_i$ as the margin of the manufacturer from retailer $i$, $i = 1 \cdots N$. In this case, the state dynamics (1) can be re-written as follows. For $i = 1, 2 \cdots N$,

$$
\dot{x}_i(t) = \rho_i u_i \sqrt{1 - x_i(t)} - \delta_i x_i(t), \quad X_i(0) = x_i \in [0, 1] , \quad (42)
$$

where $\bar{x}(t) = \sum_{j=1}^{N} x_j(t)$ is the combined market share of the $N$ retailers. Furthermore, we define the state vector, i.e., the market share vector of $N$ retailers at time $t$ as $X(t) = (x_1(t), x_2(t) \cdots x_N(t))$, and the subsidy rate vector n feedback form at time $t$ as $\Theta(X(t)) = (\theta_1(X(t)), \theta_2(X(t)) \cdots \theta_N(X(t)))$, which, for simplicity, will also be be written as $X$, and $\Theta(X)$, respectively. Now retailer $i$'s optimization problem (2) can be rewritten as

$$
V_i(X) = \max_{u_i(t) \geq 0, t \geq 0} \int_{0}^{\infty} e^{-rt}(mx_i(t) - (1 - \theta_i(X))u_i^2(t)) dt, \quad i = 1 \cdots N. \quad (43)
$$
The manufacturer anticipates the retailers’ optimal responses and incorporates these into his optimal control problem, which can now be written as

\[ V_m(X) = \max_{0 \leq \theta_i(t) \leq 1, i=1, \ldots, N, t \geq 0} \int_0^{\infty} e^{-rt} \sum_{j=1}^N \left[ M_j x_j(t) - \theta_i(t) [u_1(X(t) | \Theta(t))]^2 \right] dt. \quad (44) \]

subject to

\[ \dot{x}_i(t) = \rho_i u_i(X(t) | \Theta(t)) \sqrt{1 - \bar{x}_i - \delta_i x_i(t)}, \quad x_i(0) = x_i \in [0, 1]. \quad (45) \]

We solve the HJB equations for the value functions of the retailers and the manufacturer and by using an approach similar to the one used in analysis for two retailers, we can write the condition for no advertising support for retailer \( i \). As proved in Appendix B we can compute the threshold function \( P_i \) for retailer \( i, i=1 \cdots N \), as

\[ P_i = \frac{M_i}{(r + \delta)} \left[ \frac{m(N-1)\rho^2 + \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N-1)^2\rho^4}}{(r + \delta)^2 + mN\rho^2} \right] - \frac{M_{-i}}{(r + \delta)} \left[ \frac{2(r + \delta)^2 + m(N+1)\rho^2 - \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N-1)^2\rho^4}}{(r + \delta)^2 + mN\rho^2} \right] - \frac{m(N-1)}{(r + \delta)(2N-1)} + \frac{2(r + \delta)^2 - \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N-1)^2\rho^4}}{(r + \delta)(2N-1)\rho^2}, \quad (46) \]

where, \( M_{-i} = \sum_{j \neq i} M_j \). We can now conclude the following result

**Corollary 1:** When \( P_i \leq 0, \forall i, i=1 \cdots N \), where \( P_i, i=1 \cdots N \), is given by (46), it is optimal for the manufacturer to not support any retailer. Furthermore, if \( P_k > 0 \), for some \( k \in 1 \cdots N \) and \( P_j \leq 0, \forall j, j=1 \cdots N, j \neq k \), then only retailer \( k \) is supported and all others are not. We can also observe that \( P_i \) increases with the manufacturer’s margin from retailers \( i \) and decreases as the manufacturer’s margin from any other retailers decreases.

If we assume \( r \approx 0 \) and \( \delta \approx 0 \), then the condition of no support for any retailer simplifies. After taking the appropriate limits, we get

\[ \text{sgn}(P_i) = \text{sgn} \left[ M_i - \frac{M_{-i}}{(N-1)} - \frac{mN}{(2N-1)} \right], \quad (47) \]
where $sgn(y)$ represents the sign of $y$. Equation (47) shows that if $M_i < M_{-i}/(N - 1)$, then retailer $i$ does not receive any advertising support, where $M_{-i} = \sum_{j \neq i} M_j$. In addition, retailer $i$ will be supported only when the difference between his margin to the manufacturer and the average margin the other retailers offer to the manufacturer, i.e., is greater than $N/(2N - 1)$ times the retailers’ margin.

This observation once again clearly highlights the effect of retail level competition on the willingness of the manufacturer to offer subsidy to his retailers. If we assume that only one retailer, say retailer 1, sells the manufacturer’s product and all the other $N - 1$ retailers buy from another manufacturer and are plain competitors to retailer 1, then $M_2 = M_3 = \cdots M_N = 0$, hence $M_{-1} = 0$, and manufacturer supports retailer 1 when $M_1 > mN/(2N - 1)$. Thus, the threshold condition for support $N = 2$ is $M_1 > 2m/3$, (which is consistent with the result in previous section), for $N = 3$ is $M_1 > 3m/5$ and so on. If the number of competing retailers is very large, i.e., $N \rightarrow \infty$, then the manufacturer supports his retailer when his margin is at least half of the retailer’s margin.

An interesting question is that how will the manufacturer’s subsidy rate decision change as the number of retailers selling his product changes? While it is analytically difficult to answer this in the general case, the following segment does it for a case of symmetric retailers.

### 8.1 $N$ Symmetric Retailers ($M_1 = M_2 \cdots M_N = M$)

In this case, (46) $\forall i = 1 \cdots N$ reduces to

$$P_i = \frac{M}{(r + \delta)} \left[ \frac{N\sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N-1)^2\rho^4} - (N - 1)(Nm\rho^2 + 2(r + \delta)^2)}{(r + \delta)^2 + mN\rho^2} \right]$$

$$\quad - \frac{m(N - 1)}{(r + \delta)(2N - 1)} + \frac{2(r + \delta)^2 - \sqrt{4(r + \delta)^4 + 4mN(r + \delta)^2\rho^2 + m^2(N-1)^2\rho^4}}{(r + \delta)(2N - 1)\rho^2}. \quad (48)$$

After a simple algebraic analysis, it can be shown that $P_i$ in (48) decreases as $N$ increases for $N \geq 2$. This leads to the following result.

**Corollary 2:** When all the retailers are symmetric in nature, the manufacturer’s support
will be equal for all the retailers and his tendency to support his retailers decreases as the number of retailers increases.

A possible explanation of this result is that as $N$ increases, the competition among retailers for the market share increases and the retailers themselves have an incentive to advertise on their own accord to increase their respective sales. Then, the manufacturer need not provide as much advertising support. Furthermore, when $(r + \delta) \approx 0$, then $P_i < 0 \forall i, i = 1 \cdots N$; and no retailer gets any support, which can also be verified by (47).

9 An Extension: Model with Pricing Decisions

Thus far, we have solved the problem with given margins $M_i$ and $m_i$, $i = 1, 2$, for the manufacturer from the two retailers and for the retailers, respectively. We now consider an extension which includes optimal wholesale and retail price decisions for the manufacturer and the retailers, respectively. We assume that the manufacturer complies with the legislation forbidding price discrimination and sells to the two retailers at the same wholesale price. The manufacturer now decides his wholesale price, defined as $w(x_1, x_2)$, and the subsidy rates $\theta_i(x_1, x_2)$, $i = 1, 2$, in feedback form, and the retailers in response choose their optimal retail prices, denoted as $p_i(x_1, x_2)$ and their advertising efforts $u_i(x_1, x_2)$, $i = 1, 2$.

We assume that the manufacturer incurs a per unit constant cost of production, denoted by $c$. Furthermore, $D(p_1, p_2)$ denotes the total demand of the product, and we assume that

$$\frac{\partial D(p_1, p_2)}{\partial p_i} < 0, \; i = 1, 2.$$ 

The retailer $i$’s optimization problem is now given by

$$V_i(x_1, x_2) = \max_{p_i(t), u_i(t) \geq 0, t \geq 0} \int_0^\infty e^{-rt} \left\{ (p_i(t) - w(x_1(t), x_2(t)))D(p_1, p_2)x_i(t) 
- (1 - \theta_i(x_1(t), x_2(t)))u_i^2(t) \right\} dt, \quad i = 1, 2, \quad (49)$$

subject to (1). The manufacturer’s problem is
\[ V_m(x_1, x_2) = \max_{0 \leq w(t), 0 \leq \theta(t) \leq 1, i = 1, 2, t \geq 0} \int_0^\infty e^{-rt} \sum_{i=1}^2 \left\{ (w(t) - c)D(p_1, p_2)x_i(t) - \theta_i(t) [u_i(x_1(t), x_2(t) | w(t), \theta_1(t), \theta_2(t))] \right\} dt \quad i = 1, 2, \]  

subject to

\[ \dot{x}_i(t) = \rho_i u_i(x_1(t), x_2(t) | w(t), \theta_1(t), \theta_2(t)) \sqrt{1 - x_1(t) - x_2(t) - \delta_i x_i(t)}, \]

\[ x_i(0) = x_i \in [0, 1], \quad i = 1, 2. \]  

We can see that in the problems defined by (1), (49)-(51), the wholesale and the retail prices appear only in the profit functions and not in the state equations. Thus, we can compute the optimal retail and wholesale prices by maximizing the retailers’ and the manufacturer’s profit functions, respectively. Here again, we first compute the optimal retail price and incorporate the retailers’ response to solve for the wholesale price. By writing the first-order conditions w.r.t. \( p_i, i = 1, 2, \) and \( w \) we get the following result.

**Proposition 6:** For a given wholesale price \( w(x_1, x_2) \), the equilibrium retail prices \( p_i^\ast = p_i^\ast(x_1, x_2 | w, \theta_1, \theta_2), i = 1, 2, \) are given as the solution of the equations

\[ D(p_1^\ast, p_2^\ast) + (p_i^\ast - w) \left. \frac{\partial D(p_1, p_2)}{\partial p_i} \right|_{p_1 = p_1^\ast, p_2 = p_2^\ast} = 0, \quad i = 1, 2. \]  

Furthermore, equilibrium wholesale price \( w^\ast = w^\ast(x_1, x_2) \) solves

\[ (w^\ast - c) \left. \frac{\partial D(p_1^\ast, p_2^\ast)}{\partial w} \right|_{w = w^\ast} + D(p_1^\ast, p_2^\ast) = 0. \]  

The optimal margins for the retailers and the manufacturer can be written as \( m_i^\ast = (p_i^\ast - w^\ast)D(p_1^\ast, p_2^\ast), i = 1, 2, \) and \( M^\ast = (w^\ast - c)D(p_1^\ast, p_2^\ast) \). Using these margins the optimization problems for the retailers and the manufacturer reduces to (2) and (3), respectively, by
simply using \( m_i = m_i^\ast, M_1 = M_2 = M^\ast \).

**Remark:** With wholesale price as a decision of the manufacturer the application of the price non-discrimination legislation yields \( M_1 = M_2 \) whereas we have permitted \( M_1 \neq M_2 \) in our original formulation in Section 2 where these margins are given. This is because the additional generality does not present any difficulty in the analysis of the model when the margins are not decision variables and the possibility that these margins result of not only just the wholesale prices, but also some other considerations allowed in the law. However, if we were to allow different wholesale prices for the retailers, then the optimal wholesale prices and consequently the margins become functions of the state, and the problem becomes far too complex to permit a solution in explicit form, as achieved in the preceding sections for our original model with given margins.

## 10 Concluding Remarks

We consider a cooperative advertising model with a manufacturer supplying to one or both of two retailers, and formulate it as a Stackelberg differential game. We obtain the Stackelberg feedback equilibrium and derive the conditions under which there will or will not be any cooperative advertising. We also provide the sensitiveness of the optimal subsidy rates with respect to the various problem parameters. We examine the issue of channel coordination with cooperative advertising and find that partial coordination can be achieved when both retailers are supported. When only one of the retailers is supported, there are cases when the manufacturer’s gain from cooperative advertising does not offset the loss incurred by the retailers. This leads us also to examine the model when the manufacturer is required to offer the same subsidy rates to both retailers, in the spirit of non-discriminating legislations such as the Robinson-Patman Act of 1936. We find that the optimal common subsidy rate lies between the two optimal subsidy rates that would prevail in the absence of any such legislation. We find that the legislation benefits the less efficient retailer, and takes away
some profits from the manufacturer and the other retailer. We also see evidence of a higher channel profit and better supply chain coordination with the legislation. Furthermore, the presence of a second retailer allows us to study the effect of the retail level competition absent in He et al. (2009). In the case when the second retailer does not sell the manufacturer’s product but competes with the retailer selling the manufacturer’s product, we find that the manufacturer supports his retailer under a larger set of conditions than those in He et al. (2009).

We also study an extension with any number of retailers and an extension in which wholesale prices and retail prices are also decision variables for the manufacturer and retailers, respectively. In the extension with $N$ identical retailers when only one retailer sells the manufacturer’s product and the remaining retailers are competitors, we show that the manufacturer’s threshold to start supporting his retailers eases as the number of competing retailers increases. Furthermore, when all of the retailers sell the manufacturer’s product, we show that the manufacturer’s tendency to provide support to each retailer decreases as the number of retailers increases.

Appendices

A Derivation of $P_i$, $i = 1, 2$, for two identical retailers

For two identical retailers, the system (15)-(20) can be reduced to the following equations in four coefficients $\beta_i, B_i, i = 1, 2$, only:

$$
\beta_i = \frac{-(r + \delta)(r + \delta + \beta_{3-i}\rho^2) + \sqrt{(r + \delta)(r + \delta + \beta_{3-i}(r + \delta))(r + \delta)^2 + (2m + \beta_{3-i}(r + \delta))\rho^2}}{(r + \delta)\rho^2},
$$

$$
B_i = \frac{M_i - B_i\delta - (\beta_i B_i + \beta_{3-i}B_{3-i})\rho^2}{r}, \quad i = 1, 2.
$$

These equations can be solved explicitly to get $\beta_i$ and $B_i$ for $i = 1, 2$, as follows:
\begin{align}
\beta_1 = \beta_2 &= \frac{-2(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2}, \quad (A.1) \\
B_i &= \frac{2(2M_i + M_{3-i})(r + \delta)^2 + (M_i - M_{3-i})(m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4})}{2(r + \delta)((r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4})}. \quad (A.2)
\end{align}

Using (A.1) and (A.2) in (21), we get the values of $P_1$ and $P_2$ in (22).

**B Derivation of $P_i$, $i = 1 \cdots N$, for $N$ identical retailers**

Using (43), the HJB equation for the retailer $i$, $i = 1 \cdots N$, given the manufacturer’s subsidy rate policy $\Theta(X)$ can be written as follows. For $i = 1 \cdots N$,

\begin{equation}
rv_i(X) = \max_{u_i \geq 0} \left[ mx_i - (1 - \theta_i(X))u_i^2 + \sum_{j=1}^{N} V_{ix_j}(\rho u_j \sqrt{1 - \bar{x} - \delta x_j}) \right]. \quad (B.1)
\end{equation}

First-order conditions w.r.t. $u_i$ in (B.1), yield the optimal advertising policy of retailer $i$ as

\begin{equation}
u_i^* = u_i^*(X \mid \Theta) = \frac{V_{ix_i} \rho \sqrt{1 - \bar{x}}}{2(1 - \theta_i(X))}, \quad i = 1 \cdots N. \quad (B.2)\end{equation}

Using (B.2) in (B.1), the value function of retailer $i$, $i = 1 \cdots N$, can be rewritten as

\begin{equation}
rv_i(X) = mx_i - \sum_{j=1}^{N} \delta x_j V_{ix_j} + \frac{V_{ix_i}^2(-1 + \bar{x})\rho^2}{4(-1 + \theta_i(X))} + \sum_{j \neq i} \left[ \frac{V_{ix_j} V_{ix_j}(-1 + \bar{x})\rho^2}{2(-1 + \theta_j(X))} \right]. \quad (B.3)
\end{equation}

Now, using (B.2), the manufacturer’s HJB equation (44) can be written as

\begin{equation}
rv_m(X) = \max_{\theta_i \geq 0, i = 1 \cdots N} \sum_{j=1}^{N} \left[ M_i x_i - \delta x_i V_{mx_i} + \frac{V_{jx_j}(1 - \bar{x})(2V_{mx_j}(1 - \theta_j) - V_{jx_j} \theta_j)\rho^2}{4(-1 + \theta_j)^2} \right]. \quad (B.4)
\end{equation}

Using first-order conditions w.r.t. $\theta_i$ in (B.4), we get, for $i = 1 \cdots N$,

\begin{equation}
\theta_i^*(X) = \max\{\hat{\theta}_i(X), 0\}, \quad \text{where, } \hat{\theta}_i(X) = \frac{2V_{mx_i} - V_{ix_i}}{2V_{mx_i} + V_{ix_i}}. \quad (B.5)
\end{equation}
We now use the optimal subsidy rate policy of the manufacturer given by (B.5) in the HJB equation (B.4) to rewrite the manufacturer’s value function as

\[ rV_m(X) = \sum_{j=1}^{N} \left[ M_j x_j - \delta x_j V_{mx_j} + \frac{V_{jx_j}(1-\bar{x})(2V_{mx_j}(1-\theta^*_j(X))-V_{jx_j}\theta^*_j(X))\rho^2}{4(-1+\theta^*_j(X))^2} \right] \]  

(B.6)

Once again, we conjecture linear value functions of the following form,

\[ V_i(X) = \alpha_i + \beta_i x_i + \sum_{j \neq i} \gamma_{ij} x_j, \quad i = 1 \cdots N, \quad j = 1 \cdots N, \quad j \neq i \]  

(B.7)

\[ V_m(X) = \alpha + \sum_{j=1}^{N} B_j x_j \]  

(B.8)

and try to solve for the coefficients \( \alpha_i, \beta_i, \gamma_{ij}, \alpha \) and \( B_i, \ i = 1 \cdots N, \ j = 1 \cdots N, \ j \neq i \). We can see that \( \beta_i = V_{ix_i}, \gamma_{ij} = V_{ij}, \ B_i = V_{mx_i}, \ i = 1 \cdots N, \ j = 1 \cdots N, \ j \neq i \). We compare the terms of \( x_i, \ i = 1 \cdots N \), and the constant terms of the value functions in the equations (B.3) and (B.6) with the corresponding terms in (B.7)-(B.8) and we get the following system of equations to be solved in the coefficients \( \beta_i, \gamma_{ij}, B_i, \ i = 1 \cdots N, \ i = 1 \cdots N, \ j \neq i \).

\[
\begin{align*}
4r\alpha_i &= -\frac{\beta_i^2\rho^2}{(-1+\theta^*_i)} + \sum_{j \neq i} \frac{2\beta_j \gamma_{ij} \rho^2}{(-1+\theta^*_j)} , \\
4r\beta_i &= 4m - 4\delta \beta_i + \sum_{j \neq i} \frac{2\beta_j \gamma_{ij} \theta^*_j \rho^2}{(-1+\theta^*_j)} + \frac{\beta_i^2 \rho^2}{(-1+\theta^*_i)} , \\
4r\gamma_{ij} &= -4\gamma_{ij} \delta + \sum_{j \neq i} \frac{2\beta_j \gamma_{ij} \rho^2}{(-1+\theta^*_j)} + \frac{\beta_i^2 \rho^2}{(-1+\theta^*_i)} , \\
r\alpha &= -\sum_{j=1}^{N} \frac{\beta_j (2B_j(-1+\theta^*_j(X)) + \beta_j \theta^*_j j(X))\rho^2}{4(-1+\theta^*_j)^2} , \\
rB_i &= M_i - B_i \delta + \sum_{j=1}^{N} \frac{\beta_j (2B_j(-1+\theta^*_j(X)) + \beta_j \theta^*_j j(X))\rho^2}{4(-1+\theta^*_j)^2} , \\
\theta^*_i &= \max\{ \frac{2B_i - \beta_i}{2B_i + \beta_i}, 0 \} .
\end{align*}
\]

(B.9) (B.10) (B.11) (B.12) (B.13) (B.14)

To compute the condition under which the manufacturer will support a retailer, we let
\[ P_i = 2V_{mx_i} - V_{ix_i} = B_i - \beta_i, \; i = 1, 2, \cdots N. \] (B.15)

Thus, when \( P_i > 0 \), the manufacturer supports retailers \( i \), otherwise he does not. When \( P_i < 0, \forall i, i = 1, 2, \cdots n \), \( \theta^*_i = 0, \forall i, i = 1, 2, \cdots n \), and equations (B.9)-(B.11) can be solved independently of (B.12)-(B.13). Using \( \theta^*_i = 0, \forall i, i = 1, 2, \cdots n \), in (B.9)-(B.13) we can solve for the coefficients \( \beta_i, \alpha_i, \gamma_{ij}, \alpha, \) and \( B_i, \forall i, i = 1, 2, \cdots n \), and using these values in (B.15), we can get the values of \( P_i, \forall i = 1, 2, \cdots n \), as shown in (46).

C Proof of Proposition 5

As defined in section 7, \( V^c_r \) is the combined value function of the two retailers in the cooperative scenario and \( V^n_r \) is the same in the non-cooperative scenario. We can write \( \gamma_i \) and \( \alpha_i \) in terms of \( \beta_i \) from (D.1) and (D.3), respectively. Recall that in the case of symmetric retailers, \( \alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, \) and \( \gamma_1 = \gamma_2 = \gamma \). Furthermore, when there is no cooperation, \( \beta = \eta \), and its value is given by (A.1). Using \( \beta = \eta \), (D.1), and (D.3), we can find \( V^n_r \) by adding computing (12) for \( i = 1, 2 \), and adding the two. After a few steps of algebra, we get

\[
V^n_r = \frac{2(x_1 + x_2)(m(r + 2\delta) - 2\delta(r + \delta)\eta_1) + (-1 + x_1 + x_2)\eta_1(2m - 3(r + \delta)\eta_1)\rho^2}{2r(r + \delta)}. \tag{C.1}
\]

In the case of symmetric retailers with cooperation, using (D.1) and (D.3) from Appendix D, and using the fact that \( \alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, \) and \( \gamma_1 = \gamma_2 = \gamma \), we get

\[
V^c_r = \frac{4(x_1 + x_2)(m(r + 2\delta) - 2\beta\delta(r + \delta)) - (\beta + 2B)(1 - x_1 - x_2)(2m - 3\beta(r + \delta))\rho^2}{4r(r + \delta)}. \tag{C.2}
\]

Clearly, \( V^n_r \) and \( V^c_r \) depend only on the sum \( (x_1 + x_2) \), and this proves the first statement of Proposition 5. Next, we define \( \Delta V_r = V^c_r - V^n_r \), which can be computed as follows

\[
\Delta V_r = \frac{-8x\delta(r + \delta)(\beta - \eta_1) + (-1 + x)(2m(\beta + 2B - 2\eta_1) - 3(r + \delta)(\beta^2 + 2B\beta - 2\eta_1^2))\rho^2}{4r(r + \delta)},
\]
where $x = x_1 + x_2$. $\Delta V_r$ is linear in $x$, and we will write it as $\Delta V_r(x)$. We can see that $\Delta V_r(1) = 2\delta(\eta_1 - \beta)/r > 0$. This is because we know from (B.14) that to sustain a cooperative equilibrium, the parameters $(m, M, r, \delta, \rho)$ should be such that $\beta < \eta_1$.

Now consider

$$\Delta V_r(0) = \frac{(-2m(\beta + 2B - 2\eta_1) + 3(r + \delta)(\beta^2 + 2B\beta_2 - 2\eta_1^2))\rho^2}{4r(r + \delta)}.$$ 

Substituting the value of $B$ in terms of $\beta$ (from (D.10)) in the above expression, we can write

$$\Delta V_r(0) = \frac{4(r + \delta)(m - \beta(r + \delta)) + \eta(2m - 3(r + \delta)\eta)}{2r(r + \delta)}.$$ \hspace{1cm} (C.3)

It is clear by (C.3) that a decrease in the value of $\beta$ (caused by changes in parameters) also decreases the value of $\Delta V_r(0)$. We know that for a cooperative equilibrium, $\beta < \eta_1$, and so a lower bound for $\Delta V_r(0)$ can be obtained by using $\beta = \eta_1$ in (C.3). This lower bound is

$$\Delta V_r(0) > 0.$$ 

By using the value of $\eta_1$ from Appendix D, and after a few steps of algebra, we can see that the above expression reduces to zero. Therefore, $\Delta V_r(0) > 0$.

Because $\Delta V_r(x)$ is linear in $x$, $\Delta V_r(0) > 0$, and $\Delta V_r(1) > 0$, we can say that $\Delta V_r(x) > 0, \forall x \in [0, 1]$. Thus, $V_r^c(x) > V_r^n(x), \forall x \in [0, 1]$. The equality holds when $\beta = \eta_1$, i.e., when non-cooperation is optimal for the manufacturer.

D Uniqueness of optimal solution in the case of two symmetric retailers

The uniqueness of an optimal solution to the problem defined by (6), (7), (9), (10), and (11) is guaranteed by a unique solution of the system of equations (15)-(20). It appears
to be difficult to prove the uniqueness in the general case. However, in the special case of symmetric retailers \((M_1 = M_2 = M, m_1 = m_2 = m, \delta_1 = \delta_2 = \delta \text{ and } \rho_1 = \rho_2)\), we can establish the result as follows.

We first look at the signs of \(\alpha_i, \beta_i\) and \(\gamma_i, i = 1, 2\). It is expected that \(\beta_i > 0\). Now consider \(\gamma_i\), which can be expressed in terms of \(\beta_1\) and \(\beta_2\) by using equation (17):

\[
\gamma_i = \frac{\beta_i^2(-1 + \theta_i^*)\rho_i^2}{2(-1 + \theta_i^*)(r + \delta_{3-i})(-1 + \theta_i^* - \beta_i\rho_i^2)} , \; i = 1, 2.
\]

Since \(\beta_i > 0\) and \(\theta_i^* < 1, i = 1, 2\), we have \(\gamma_i = V_{ix3-i} < 0\), as intuition would suggest on account of the competition between the retailers. We can also use (16) and (17) to write

\[
\gamma_i = \frac{-m_i + \beta_i(r + \delta_i)}{(r + \delta_{3-i})} . \tag{D.1}
\]

Since \(\gamma_i < 0\), we must have

\[
\beta_i < \frac{m_i}{(r + \delta_i)}. \tag{D.2}
\]

Now consider \(\alpha_i\), which is retailer \(i\)'s value function when the initial market is zero for both retailers. We now show that this value is positive. By adding (15) and (17), we can conclude that \(\alpha_i = -\gamma_i(r + \delta_{3-i})/r\), which is positive since \(\gamma_i < 0\). Thus,

\[
\alpha_i = \frac{m_i - \beta_i(r + \delta_i)}{r} > 0. \tag{D.3}
\]

Moreover, using (D.1) in equation (15), we can write \(\alpha_i\) in terms of \(\beta_1\) and \(\beta_2\), and then rewrite (D.3) as

\[
\alpha_i = \frac{1}{4r} \left[ \frac{2\beta_{3-i}(m_i - \beta_i(r + \delta_i))\rho_i^2}{(r + \delta_{3-i})(-1 + \theta_i^*)} - \frac{\beta_i^2\rho_i^2}{(-1 + \theta_i)} \right] > 0, \; i = 1, 2. \tag{D.4}
\]

Clearly, in the symmetric case, we will have \(\alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, \gamma_1 = \gamma_2 = \ldots\)
\[ \gamma, B_1 = B_2 = B, \text{ and hence } \theta^*_1 = \theta^*_2 = \theta^*. \] We can thus rewrite (D.4) as

\[ \frac{\beta(2m - 3\beta(r + \delta))\rho^2}{2r(r + \delta)(-1 + \theta)} > 0. \]

This, along with \( \beta > 0 \) and \( \theta < 1 \), gives us

\[ \beta > \frac{2m}{3(r + \delta)}. \] (D.5)

From (D.2) and (D.5), we have

\[ \frac{2m}{3(r + \delta)} < \beta < \frac{m}{r + \delta}. \] (D.6)

To prove a unique solution to equations (15)-(20), we reduce them into one equation of a single variable \( \beta \), and then aim for the unique solution of \( \beta \). We will separately consider the cases of a cooperative equilibrium where \( \theta^* > 0 \) and a non-cooperative equilibrium where \( \theta^* = 0 \).

**Case I: Cooperative equilibrium \((\theta^* > 0)\)**

Since \( \beta_1 = \beta_2 = \beta \) and \( B_1 = B_2 = B \) in the symmetric case, (20) reduces to \( \theta^* = (2B - \beta)/(2B + \beta) \). Using this, (D.1), and (D.3), we can reduce equations (15)-(20) to two equations in variables \( \beta \) and \( B \), i.e.,

\[ 4r(r + \delta)\beta = 4(r + \delta)(m - \beta \delta) + (\beta + 2B)(2m - 3\beta(r + \delta))\rho^2 \] (D.7)

\[ 4rB = 4M - 4B\delta - (\beta + 2B)^2 \rho^2. \] (D.8)

Using (D.7), (D.8) and (14) in (21) and setting \( P_1 = P_2 = P \) on account of the case being
symmetric, we obtain the participation threshold function

\[ P = 2B - \beta = \frac{8(r + \delta)(-m + 2M + (\beta - 2B)\delta) - (\beta + 2B)(2m - (\beta - 4B)(r + \delta))\rho^2}{8r(r + \delta)}. \]  

(D.9)

Using (D.7), we can write \( B \) in terms of \( \beta \) as follows:

\[ B = \frac{-2m(4(r + \delta) + \beta\rho^2) + \beta(r + \delta)(8(r + \delta) + 3\beta\rho^2)}{2(2m - 3\beta(r + \delta))\rho^2}. \]  

(D.10)

Now using (D.9) and (D.10), we can rewrite \( P \) in terms of \( \beta \) only, and then write the condition of the cooperative equilibrium as

\[ P = \frac{-4m((r + \delta)^2 + \beta\rho^2) + 2\beta(r + \delta)(4(r + \delta) + 3\beta\rho^2)}{(2m - 3\beta(r + \delta))\rho^2} > 0. \]  

(D.11)

Next, we find the values of \( \beta \) for which the inequality in (D.11) holds. In order to see how \( P \) varies with \( \beta \), we first find the roots of the equation \( P = 0 \). The numerator is quadratic in \( \beta \) with the roots denoted as \( \eta_1 \) and \( \eta_2 \):

\[ \eta_1 = \frac{-2(r + \delta)^2 + m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2} \]

\[ \eta_2 = \frac{-2(r + \delta)^2 + m\rho^2 - \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2}. \]

Clearly, \( \eta_1 > 0 \) and \( \eta_2 < 0 \). Also, the denominator \((2m - 3\beta(r + \delta))\rho^2\) of (D.11) changes sign at \( \beta = 2m/3(r + \delta) \); the denominator is strictly positive when \( \beta < 2m/3(r + \delta) \) and strictly negative when \( \beta > 2m/3(r + \delta) \).

We will now compare the value of \( \eta_1 \) with \( m/(r + \delta) \) and \( 2m/3(r + \delta) \). We can see that the difference

\[ \eta_1 - \frac{2m}{3(r + \delta)} = \frac{-2(r + \delta)^2 - m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2} > 0, \]  

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and thus \( \eta_1 > 2m/3(r + \delta) \). Furthermore,

\[
\eta_1 - \frac{m}{(r + \delta)} = \left(\frac{-2(r + \delta)^2 - 2m\rho^2 + \sqrt{4(r + \delta)^4 + 8m(r + \delta)^2\rho^2 + m^2\rho^4}}{3(r + \delta)\rho^2}\right) < 0,
\]

and thus we have

\[
\frac{2m}{3(r + \delta)} < \eta_1 < \frac{m}{(r + \delta)}.
\]

(D.12)

We can then conclude from (D.11) that \( P > 0 \) is satisfied when

\[
\beta \in (-\infty, -\eta_2) \text{ or } \beta \in \left(\frac{2m}{3(r + \delta)}, \eta_1\right).
\]

(D.13)

Therefore, the conditions (D.6), (D.12) and (D.13) along with the fact that \( \beta > 0 \) give us the desirable range of the solution for \( \beta \), i.e.,

\[
\beta \in \left(\frac{2m}{3(r + \delta)}, \eta_1\right).
\]

(D.14)

Now using (D.10) in (D.8), we can write a single equation in \( \beta \). After some steps of algebra, for the symmetric retailer case with positive cooperation, this single equation for \( \beta \) can be written as

\[
F(\beta) = \frac{8\beta(r + \delta)^3(m - \beta(r + \delta)) - (2m - 3\beta(r + \delta))^2(2M + \beta(r + \delta))\rho^2}{(2m - 3\beta(r + \delta))^2\rho^2} = 0.
\]

Thus, a unique cooperative solution in the symmetric retailer case is guaranteed when exactly one root of the equation \( F(\beta) = 0 \) lies in the range given by (D.14). The numerator of the above expression, denoted as \( N(\beta) \), is cubic in \( \beta \). Thus, we can rewrite the equation for \( \beta \) as

\[
N(\beta) = a\beta^3 + b\beta^2 + c\beta + d = 0,
\]

(D.15)
where
\[
\begin{align*}
a &= -9(r + \delta)^3 \rho^2, \\
b &= -8(r + \delta)^4 + 6(2m - 3M)(r + \delta)^2 \rho^2, \\
c &= 4m(r + \delta)(2(r + \delta)^2 - (m - 6M)\rho^2) \quad \text{and} \\
d &= -8m^2 M \rho^2.
\end{align*}
\]

Since the denominator of \( F(\beta) \) is positive for all values of \( \beta \) except \( 2m / 3(r + \delta) \), the sign of \( F(\beta) \) is the same as that of \( N(\beta) \). In what follows, we perform a simple sign analysis of \( N(\beta) \) to draw inference about the roots of (D.15). After a few steps of algebra with the help of Mathematica, the following observations can be made:

\[
\begin{align*}
N(\beta) &\to \infty \quad \text{as} \ \beta \to -\infty; \\
N(\beta) &= -8m^2 M \rho^2 < 0 \quad \text{when} \ \beta = 0; \\
N(\beta) &= \frac{16}{9} m^2(r + \delta)^2 > 0 \quad \text{when} \ \beta = \frac{2m}{3(r + \delta)}; \\
N(\beta) &= -m^2(m + 2M)\rho^2 < 0 \quad \text{when} \ \beta = \frac{m}{(r + \delta)}; \\
N(\beta) &\to -\infty \quad \text{as} \ \beta \to \infty.
\end{align*}
\]

These observations make it clear that the equation \( N(\beta) = 0 \) has three real roots in the following intervals:

\[
(-\infty, 0), \ (0, \frac{2m}{3(r + \delta)}) \quad \text{and} \quad \left(\frac{2m}{3(r + \delta)}, \frac{m}{(r + \delta)}\right).
\]

Moreover, from (D.12) and (D.14), we see that there should be exactly one root in the desired interval \( \left(\frac{2m}{3(r + \delta)}, \eta_1\right) \) for there to be cooperation in the equilibrium solution. In fact, the location of the third root in the interval \( \left(\frac{2m}{3(r + \delta)}, \frac{m}{(r + \delta)}\right) \) w.r.t \( \eta_1 \) determines whether we will have a cooperative or non-cooperative equilibrium. Figure 15 shows the curve \( N(\beta) \) when \( \beta > 0 \). This curve gives us an idea of when exactly one of the two positive roots of the equation \( N(\beta) = 0 \) would be in the interval \( \left(\frac{2m}{3(r + \delta)}, \eta_1\right) \). Note that one root of this equation
is negative and is not shown in the figure. It can be easily seen that to attain exactly one

root in the interval \( \left( \frac{2m}{3(r+\delta)}, \eta_1 \right) \) and thereby to have a cooperative equilibrium, we must have

\[
F(\beta)|_{\beta=\eta_1} < 0, \quad (D.16)
\]

which, when using \( \beta = \eta_1 \) in \( F(\beta) \) gives us

\[
\frac{1}{5} \left( \frac{2m^2 \rho^2}{(r+\delta)^2} \frac{(r+\delta)^2 - 18M \rho^2 - \sqrt{4(r+\delta)^4 + 8m(r+\delta)^2 \rho^2 + m^2 \rho^4}}{\rho^2} \right) + m \left( \frac{2r^2 r^2 + 2(r+\delta)^2}{(r+\delta)^2} \right) < 0. \quad (D.17)
\]

Figure 15: subsidy rates vs \( M_1 \)
After a few steps of algebra, one can see that the condition (D.17) has just the opposite sign to the one that ensures a non-cooperative solution in the case of symmetric retailers, which can be obtained by simply using \( M_1 = M_2 = M \) in Proposition 3. In other words, when the parameters \( m, M, r, \delta, \) and \( \rho \) are such that the inequality (D.17) is not satisfied, then the third root of the equation \( N(\beta) = 0 \) will be greater than or equal to \( \eta_1 \), and the optimal solution will be a non-cooperative one. Thus, a unique cooperative equilibrium is guaranteed when (D.17) is satisfied.

Case II: Non-Cooperative equilibrium

We now consider the non cooperative equilibrium \( (\theta^* = 0) \) in the symmetric retailer case. As illustrated in Appendix A, the system of equations (15)-(20) can be solved explicitly in the non-cooperative case to get a unique positive solution of \( \beta_1 \) and \( \beta_2 \), given by (A.1). Since the symmetric retailer case is a further simplification of the case of identical retailers, i.e., with \( M_1 = M_2 = M \), the solution of \( \beta \) is unique and is given by (A.1). Note that this value of \( \beta \) equals \( \eta_1 \).

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