Budget and Effort Choice in Sequential
Colonel Blotto Campaigns

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Abstract

We consider dynamic campaigns. Two players fight in a series of battles. The player who is first to accumulate a given number of battle victories wins the campaign. A showdown effect and a discouragement effect have been identified in such campaigns if players choose their battle effort independently for each given battle once the state in which this battle takes place has been reached. We assume that players must choose their overall military budget prior to the first battle, but then allocate this chosen budget time-consistently during the course of the campaign. We find neither a discouragement effect nor showdown effect: the players tend to allocate their own budget symmetrically across the sequence of all battle states that are reached. Also, we find that the Colonel Blotto structure mobilises higher expected effort.

Keywords: Colonel Blotto, campaign, multi-battle conflict, best-of-n contest, discouragement effect, decisiveness, sequential battles.

JEL: D72, D74

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1 Introduction

It has been noted both in economic and political science research that military conflict is a dynamic phenomenon. The campaigns by Alexander or Attila, Napoleon’s campaign against Russia and Hitler’s attack on the Soviet Union, and many other instances, illustrate the dynamic nature of military campaigns. Campaigns typically consist of a multiplicity of sub-contests or battles. The army leaders face limitations as regards the stock of military resources that are at their disposal. This stock may be a fixed amount and may be used up during the campaign, or it may be possible to replenish the stock during the campaign.

Napoleon’s campaign against Russia 1812/1813 is an important example. Napoleon started this campaign with the Great Army. The precise size is difficult to number, but it may have consisted of more than 500,000 infantry, almost 100,000 cavalry, and 1,240 guns when crossing the Russian border (Dodge 2008, p.21ff.). According to Chandler (1967, p.853), from the central army group, „which in its heyday numbered 450,000 combatants; of this vast armament, only 25,000 bedraggled survivors recrossed the Niemen“ McNeill (1982, p.204) briefly describes Napoleon’s efforts and difficulties to „feed the Grande Armée from the rear“ and mentions the breakdown of these logistics when the army retreated from Moscow. His campaign had to rely largely on a given military endowment, with limited options to replenish the troops during the campaign. Napoleon had to allocate the resources in a series of battles. The situation on the Russian side may have been slightly different, but given the short duration of the total campaign the Russian side most likely did not have much discretion and had to draw on a given stock of military forces. Other campaigns may be characterised by very different patterns regarding the replenishment of troops and military equipment. For instance, at the beginning of Alexander’s campaign he departed from the Hellespont toward Asia with an army of about 7,000 Greek allied infantry plus approximately 5,000 mercenary infantry, plus cavalry soldiers. During the campaign that took more than 10 years he sacrificed troops, but his army
also received a "steady flow of reinforcements" (Sheppard 2008, p.86). For different reasons, Nazi Germany also did not only have to rely on a given initial military endowment. Hitler exploited millions of foreign manpower as slave labor for the production of war machinery, extracted resources from the conquered territory and reorganised German industrial production. As a result, war production peaked only in July 1944 (see McNeill, 1982, p. 353).

The three examples may illustrate that the military budget for a campaign may be exogenously given ex ante or subject to endogenous choices during the campaign. The degree of endogeneity of the military budget during the campaign is at the core of our analysis. We ask: How does this endogeneity affect the dynamics of fighting effort in the course of the campaign? We study the properties of the equilibrium in a generic set-up: a sequential multi-battle conflict with a finite horizon – the best-of-n contest.

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The abstract structure of the best-of-n contest is discussed in different contexts by Harris and Vickers (1987), Klumpp and Polborn (2006), Konrad and Kovenock (2009) and Gelder (2014). They study campaign equilibrium if the players have opportunities to simultaneously and independently choose the battle efforts in each and every battle during the various stages of the campaign. We contrast this setup with a campaign in which the military leaders choose their overall budgets but then have to draw on these given aggregate military budgets in the sequence of campaign battles of a best-of-n campaign. We find that whether conflict escalates or de-escalates from one battle to the other depends qualitatively on the setup. The first set-up is characterised by de-escalation and escalation that is governed by the respective state in the best-of-n contest. The second setup is characterised by constant efforts in all states that are potentially reached.

The literature on the dynamics of conflict is very large, even within the economic theory of contests. The best-of-n contest which is at the centre of the analysis in this paper is one of these and it will be described in more detail in the next section. Before turning to it, we briefly describe four other types. For more details on some of them see Konrad’s (2012) recent survey. One of these is labelled the "tug-of-war". While the best-of-n contest ends
as soon as one of the players has attained a predetermined number of battle victories, the tug-of-war has a potentially infinite number of sequential battles. The tug-of-war ends as soon as one of the players has accumulated a number of battle victories that exceed the number of victories of the other player by a sufficiently large number.\textsuperscript{1} A second dynamic structure considers repeated incumbency fights. This structure may appropriately map a situation in which the incumbent government is regularly challenged by revolt or coups or civil war. The incumbent then has to fight against a challenger. The player who is victorious in this fight assumes the role of incumbent in the next period and may face a challenger then. Challengers may come from the pool of players who lost the incumbency fights, or may newly enter into this competition.\textsuperscript{2} A third class of dynamic contest is the class of elimination tournaments. Conflicts of this type start with a number of rival players larger than two. These players compete against each other in one big group or in smaller subgroups. Players in this grand contest or members inside the respective groups compete in sub-battles internally. As a function of their efforts, group structure and group assignments some players drop out and others advance to more progressed stages of the tournament.\textsuperscript{3} Finally, a considerable literature studies dynamic contests between a given set of players in which players can expend efforts in several stages, and where these different types of expenditure aggregate or interact in various ways.\textsuperscript{4}


\textsuperscript{3}Early analyses of this structure are in Rosen (1986). The role of the number of rounds, the composition of players in subgroups, the rules that govern which player is dropped and which player advances to the next round have been studied in work by Gradstein and Konrad (1999), Amegashie (1999), Konrad (2004), Groh, Moldovanu, Sela and Sunde (2012) and Fu and Lu (2012).

\textsuperscript{4}An early contribution is Leininger and Yang (1994). See also Segev and Sela (2014), Beviá and Corchón (2013) and Kahana and Klunover (2016). This literature is also related to contests in which players choose their efforts sequentially (see, e.g., Dixit, 1987, Linster 1993, Baik and Shogren 1992, Leininger 1993).
In all these structures the players have no other alternative than to fight in any given battle contest. This abstracts from options such as peaceful bargaining. Players may attempt to avoid costly fighting and find Coasian bargaining solutions. The conditions for when peace prevails and when conflict is inevitable have been carefully studied. Limited commitment, time-consistency problems, information asymmetries and other reasons may make a peaceful Coasian bargaining outcome infeasible. Fighting may occur sooner or later, and with higher or lower probability. Other researchers have drawn attention to the fact that aspects such as the timing or the likelihood of negotiations, as well as the division of the „peace rent“ between two players may depend on whether negotiations take place during a violent war or in a non-violent status quo (Filson and Werner, 2002). These are only impressions on what can happen and what can be analysed in a dynamic context. Once we consider situations in which the state of war becomes endogenous, we open the door to a large literature which we cannot adequately survey here.

In what follows we start with a formal definition of the best-of-n contest. We study two cases in a more general framework. In one case players can freely choose the amounts of resources they would like to expend in any given battle fight once they reach this battle state. In this framework players are free to arbitrarily choose high efforts, but effort is costly to them and players have to pay their own effort whenever they expend it in any given battle. We briefly discuss the robustness of the equilibrium results and report on existing experimental results. We then turn to the case in which the campaign is also described by a best-of-n battle, but in which each player first chooses the given military budget at the beginning of the campaign. This budget is

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5Analyses that address the question of when conflict actually arises are Polborn (2006), Bester and Konrad (2004) and Bester and Konrad (2005). An important insight that draws on the problem of commitment problems which may make an early, violent resolution of conflict superior can be found in Garfinkel and Skaperdas (2000). Experimental work that considers the decision as to whether to fight if players can also bargain is Herbst et al. (2016).
costly when chosen and can only be used in the battle contests. Players can then choose how much of this budget to allocate to a given battle state when they reach this state, but they cannot later increase their chosen budget. We find that the assumption about a non-replenishable budget with a scrap value of zero changes the qualitative nature of the equilibrium.

2 The generic best-of-n contest

Our baseline model of dynamic contest is very similar to Klumpp and Polborn (2006) and Konrad and Kovenock (2009). We consider two players, A and B. The two players interact repeatedly in a finite sequence of battles that emerge during a campaign. We denote player A as the attacker who initiated a campaign to conquer or defeat player B. Player B is referred to as the defender. In each battle both players expend effort. One of them wins the respective battle. Final victory of the campaign is a function of the numbers of battle victories. In its most parsimonious version the sequence of battles comes to an end if one of the two players has accumulated two battle victories. If A has two battle victories first, then A successfully conquers and takes over the empire/territory of B and the game ends. This gives A a prize of final victory of size $W > 0$. Player B receives a loser prize of $-L < 0$ in this case. Otherwise, if B succeeds and is first to accumulate two battle victories, then this marks the final failure of A’s attack. In this case A and B receive zero. This version of the multi-battle contest has at least two, and at most three battles before the game ends. It ends after two battles if one player wins the first two battles in a row. It ends after three battles if each player wins one of the first two battles, in which case the third battle becomes decisive. This structure is referred to as a „best-of-three“ contest and the process can be mapped by Figure 1.

Figure 1 shows a grid with eight possible states, characterised by their coordinates $(i, j)$. Four of these states are terminal states. The campaign ends once one of these states is reached. Player A wins if one of the terminal states $(0, 1)$ and $(0, 2)$ is reached, player B wins if one of the terminal
Figure 1: Interior and terminal states in the best-of-three campaign

states (2, 0) or (1, 0) is reached. The other four states are characterised by min\{i, j\} \geq 0 and are called interior states. The coordinate i measures the number of battle victories which player A needs to reach one of the terminal states in which A wins, and j measures the analogous number for player B. The dynamic contest starts in an interior state \((i, j)\). In this state players A and B independently choose non-negative efforts denoted by \(x_{(i,j)}\) and \(y_{(i,j)}\). A function \(p^A_{(i,j)}(x, y)\) of these efforts determines the probability by which A wins this battle, and the complementary probability \(p^B_{(i,j)} = 1 - p^A_{(i,j)}\) by which player B wins the battle. If player A wins the battle at \((i, j)\), the process moves from thereon to state \((i_1, j)\), meaning that player A needs to win \(i_1\) further battles to reach final victory, whereas player B needs to win \(j\) battles from there. Similarly, if player B wins at \((i, j)\), the process moves to state \((i, j_1 - 1)\).

The battle-win probabilities at a given battle state \((i, j)\) are

\[
p^A_{(i,j)}(x_{(i,j)}, y_{(i,j)}) = \begin{cases} 
\frac{x_{(i,j)}}{x_{(i,j)} + y_{(i,j)}} & \text{if } x_{(i,j)} + y_{(i,j)} > 0 \\
\frac{1}{2} & \text{if } x_{(i,j)} + y_{(i,j)} = 0
\end{cases}
\]

and \(p^B_{(i,j)} = 1 - p^A_{(i,j)}\). The function (1) is often referred to as the lottery-contest-success function. The function has some natural properties. For
instance, the win probability of a player increases in own military effort and decreases in the military effort of the other player. We cannot be sure if the outcome of military combat is, in general, well described by this function. Clearly the nature of military battle confrontation depends on the state of technology and other aspects. The function (1) may also disregard the complexities of a single battle and abstract from many important aspects that matter in a given battle (see Clausewitz 1832/2003 for a discussion of a number of these aspects). However, this function has been suggested in many contexts, has been axiomatised and has received a number of microeconomic underpinnings.\(^6\)

Implicitly (1) assumes that the single battles are technologically independent of each other. In particular, effort choices made at state \((i, j)\) determine the win probabilities at this state, but the efforts have no direct impact on the function that determines the win probabilities in future battle encounters \((i', j')\) with \(i' < i\) and/or \(j' < j\). At any new state \((i', j')\) both players \(A\) and \(B\) independently and simultaneously choose efforts \(x_{(i', j')}\) and \(y_{(i', j')}\). The efforts \(x_{(i', j')}\) and \(y_{(i', j')}\) map into the win probabilities according to the same mapping (1) as in state \((i, j)\). This implies that the same function (1) applies at all interior states \((i, j)\). One should note that this assumption is usually made, but that other assumptions could also be reasonably made. After a finite number of battle encounters the process reaches a state in which either \(i = 0\) or \(j = 0\). The game ends once such a terminal state is reached.

We study two variants of this campaign game. In section 3 we discuss the campaign game studied by Klumpp and Polborn (2006). In their setup players \(A\) and \(B\) can choose any non-negative efforts \(x_{(i,j)}\) and \(y_{(i,j)}\) and they pay the cost of this effort at the time of making this choice. In section 4 we change the assumptions about players’ effort choice options. There, we assume that each player chooses the size of his overall budget at the very

beginning of the campaign. Then the player can choose how to allocate this amount during the campaign. At each battle state that is reached the player is left with the amount that was not used up in previous battles. He can freely allocate this amount between this battle and future battles, and so on for all battle states reached. This means the player chooses an overall budget only once, but allocates this budget in a time-consistent fashion. Once the overall budgets have been chosen, the subgame emerging from there can be interpreted as a dynamic variant of the static, multi-front Colonel Blotto games that have received considerable attention since Borel (1921). We therefore call it a sequential Colonel Blotto campaign.

Before we solve for the subgame perfect equilibrium for the two variants of the best-of-three contest in the next two sections, we introduce two further pieces of notation. We denote \( v_A^{(i,j)} \) and \( v_B^{(i,j)} \) as the continuation values of players A and B at state \((i,j)\), provided that these continuation values are well-defined. For terminal states these continuation values are exogenously given by winner and loser prizes: they are \( v_A^{(0,j)} = W \) and \( v_B^{(0,j)} = -L \) and \( v_A^{(i,0)} = v_B^{(i,0)} = 0 \). For interior states \((i,j)\) the continuation values are the expected payoffs which players attain from taking part in the subgame starting in \((i,j)\). We will conclude further below that, using sequential rationality, these values are, indeed, unique and are well-defined in the different setups that we consider.

We will also define

\[
z_A^{(i,j)} = v_A^{(i-1,j)} - v_A^{(i,j-1)} \quad \text{and} \quad z_B^{(i,j)} = v_B^{(i,j-1)} - v_B^{(i-1,j)}
\]

for states \((i,j)\) with \( \min\{i,j\} > 0 \). These are the differences in players’ continuation values from winning and losing the battle contest at \((i,j)\). We call \( z_A^{(i,j)} \) and \( z_B^{(i,j)} \) player A’s stake and player B’s stake in the interior state \((i,j)\).

Finally, we note that there is no discounting in either variant of the best-of-three contest: players are indifferent about whether the final prize of victory or defeat is handed over at an earlier stage or at a later stage (see Gelder, 2014, for discounting). Players are also indifferent to their „score“ when they
eventually reach a terminal state. Players may obtain some 'status'-rents from winning 2 : 0 or negative 'ego'-rents from a devastating defeat in which they were unable to score in a single battle. Such an assumption has been analysed by Gelder (2014).

3 Sequential effort choices

The Tullock-Klumpp-Polborn case  We first describe the campaign game between players $A$ and $B$ if they choose and resort to the military effort produced/paid for in any given battle state which might be reached. Players $A$ and $B$ are unconstrained with regards to the choices of their efforts. At a given battle state $(i, j)$ they may expend any finite non-negative amount of resources $x_{(i,j)}$ and $y_{(i,j)}$, and this choice does not constrain them as regards future battles. Efforts $x_{(i,j)}$ and $y_{(i,j)}$ translate into battle-win probabilities as in (1). The battle effort is sunk at the end of the battle, and the player who expended this effort needs to pay the cost of it, irrespective of the fighting outcome at a particular battle state. As discussed in the introduction, military capacity/effort may, in some instances, be chosen and produced in parallel with an enduring war.

Klumpp and Polborn (2006) solve a symmetric version of this game. They assume that prizes are $W$ and 0 for $A$ and $B$ in the terminal states $(0, 2)$ and $(0, 1)$ and prizes are 0 and $W$ for $A$ and $B$ in the terminal states $(2, 0)$ and $(1, 0)$. The assumption that the attacker values winning as much as the defender does makes the dynamic contest fully symmetric both at $(2, 2)$ and at $(1, 1)$. Moreover, we can normalise this prize to one: $W = L = 1$. While the symmetry assumption is not innocent, it can highlight two key effects that play a major role in determining escalation or de-escalation.\footnote{Mehlum and Moene (2004) exploit the property that there is a payoff asymmetry between attacker and defender and a countervailing asymmetry in their fighting technologies which leads to different, interesting results.} The subgame perfect equilibrium can be characterised as follows.

In state $(1, 1)$ the two contestants fight for a payoff difference between win-
ning and losing that is equal to \( z^A_{(1,1)} = z^B_{(1,1)} = 1 \). This contest is symmetric and they expend equilibrium efforts \( x_{(1,1)} = y_{(1,1)} = \frac{1}{4} \). This determines their continuation values at \((1,1)\) as \( v^B_{(1,1)} = -\frac{3}{4} \) and \( v^A_{(1,1)} = \frac{1}{4} \).

In state \((2,1)\), player \(B\) attributes a value \( z^B_{(2,1)} = \frac{3}{4} \) to reaching state \((2,0)\) rather than state \((1,1)\). Player \(A\) attributes a value \( z^A_{(2,1)} = \frac{1}{4} \) to reaching state \((1,1)\) rather than state \((2,0)\). Anticipating subgame perfect play in later stages, the battle contest at \((2,1)\) is described by an asymmetric lottery contest with prizes \( z^B_{(2,1)} = \frac{3}{4} \) and \( z^A_{(2,1)} = \frac{1}{4} \). The equilibrium of a static Tullock lottery contest with these prizes is well-known and given by \( x_{(2,1)} = \frac{3}{64} \) and \( y_{(2,1)} = \frac{9}{64} \). This implies that \( p^A_{(2,1)} = \frac{1}{4} \) and \( p^B_{(2,1)} = \frac{3}{4} \) at \((2,1)\). In turn, this implies that the continuation values of reaching \((2,1)\) are \( v^A_{(2,1)} = \frac{1}{64} \) and \( v^B_{(2,1)} = -\frac{21}{64} \).

In state \((1,2)\), player \(B\) attributes a value equal to \( z^B_{(1,2)} = \frac{1}{4} \) to reaching \((1,1)\) rather than \((0,2)\). Player \(A\) attributes a value of \( z^A_{(1,2)} = \frac{3}{4} \) to reaching \((0,2)\) rather than \((1,1)\). Anticipating subgame perfect play in later rounds, the equilibrium of a static Tullock lottery contest with these prizes is well-known and given by \( x_{(1,2)} = \frac{9}{64} \) and \( y_{(1,2)} = \frac{3}{64} \). This implies that \( p^A_{(1,2)} = \frac{3}{4} \) and \( p^B_{(1,2)} = \frac{1}{4} \) at \((1,2)\). In turn, this implies that the continuation values of reaching \((1,2)\) are \( v^A_{(1,2)} = \frac{43}{64} \) and \( v^B_{(1,2)} = -\frac{63}{64} \).

In state \((2,2)\) the two players compete about whether to move to \((1,2)\), which is the preferred state for \(A\), or to \((2,1)\), which is the preferred state for \(B\). The prizes at stake are \( z^A_{(2,2)} = v^A_{(1,2)} - v^A_{(2,1)} = \frac{43}{64} - \frac{1}{64} = \frac{21}{32} \) and \( z^B_{(2,2)} = v^B_{(2,1)} - v^B_{(1,2)} = -\frac{21}{64} - (-\frac{63}{64}) = \frac{21}{32} \). This shows that the battle contest at \((2,2)\) is symmetric and that \(A\) and \(B\) are fighting for prizes of size \(\frac{21}{32}\).

It is straightforward from here to calculate that \( x_{(2,2)} = \frac{21}{128} = y_{(2,2)} \) in the subgame perfect equilibrium.

The expected effort for each of the players that emerges in the equilibrium follows from these, taking into consideration that the campaign may, but need not reach battle state \((1,1)\). This expected effort is

\[
x_{(2,2)} + \frac{1}{2} x_{(1,2)} + \frac{1}{2} x_{(2,1)} + \frac{1}{4} x_{(1,1)} = \frac{41}{128}.
\]

We summarise these results in a proposition:
**Proposition 1** (Klumpp and Polborn 2006): Consider the campaign with unconstrained and independent effort choices at each battle, symmetric prizes from winning the campaign, and symmetric Tullock fighting technology (1). Fighting effort is symmetric at (1, 1) and at (2, 2). Fighting is higher at (1, 1) than at (2, 2). Fighting efforts are smaller in the asymmetric states (2, 1) and (1, 2) than in the symmetric states (2, 2) or (1, 1). Moreover, effort is lower for the disadvantaged (lagging) player in (2, 1) and (1, 2) than for the advantaged (leading) player. The sum of expected efforts exceeds the efforts in a single battle for the same prize.

The result highlights two important effects. One effect has been referred to as the *discouragement effect*. Players who fall behind become discouraged. Consider player A when the process moves from (2, 2) to (2, 1). The player could try to catch up and return to a strategically symmetric situation (1, 1). If successful, the player will then have to fight a strategically symmetric battle at (1, 1). Efforts at (1, 1) are high. The continuation value at (1, 1) is therefore low. This low continuation value reduces A’s incentive to choose much effort at (1, 2). The advantaged player B also chooses low effort in this state, but still three times as much as A, and B’s low effort can be interpreted as an equilibrium reaction to the low anticipated effort of the disadvantaged player. The difference in the efforts of the disadvantaged and the advantaged player causes a momentum effect: the advantaged player expends three times as much effort as the disadvantaged player in the best-of-three contest at (1, 2) or (2, 1). The advantaged player then wins the battle at this stage in three out of four cases. This is in line with the sometimes articulated hypothesis that players who have just won a battle get a strategic momentum and are more likely to win the subsequent battle. If such a momentum exists it leads to the prediction that in a sequence of battles the winning of players is systematically correlated.

The equilibrium also describes that players expend higher effort at (1, 1) than at (2, 2). We refer to this as the *showdown effect*. Intuitively, the players are symmetric in each of these states. However, the fight in (1, 1) is
decisive and resolves the conflict. The fight in \((2, 2)\) leads to an advantage of one player, but not to final victory. Accordingly, the stakes of both players are higher at \((1, 1)\) than at \((2, 2)\). When they reach \((1, 1)\) they choose their fighting efforts according to these higher stakes.

**Empirical and experimental evidence** Malueg and Yates (2010) find evidence from a tennis application of best-of-three, multi-battle contests that is in line with such strategic momentum. Gauriot and Page (2014) carefully survey the theory and the existing evidence on this phenomenon and address the phenomenon in an empirical paper on tennis. Using an elegant identification strategy that allows them to separate effects of genuine strength of players from strategic momentum, they do find momentum effects in line with the theory, particularly for male tennis players.

Mago, Sheremeta and Yates (2013) offer experimental results on a best-of-three contest for a prize of size 100 units experimental currency. The results that emerge from the theoretical model in Klumpp and Polborn (2006) are compared to the average efforts chosen in the experiment in the following table (p.292):

<table>
<thead>
<tr>
<th></th>
<th>Equilibrium</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort at ((2, 2))</td>
<td>16.4</td>
<td>22.2</td>
</tr>
<tr>
<td>Effort at ((2, 1)) or ((1, 2)) by the winner at ((2, 2))</td>
<td>14.1</td>
<td>28.5</td>
</tr>
<tr>
<td>Effort at ((2, 1)) or ((1, 2)) by the loser at ((2, 2))</td>
<td>4.7</td>
<td>23.5</td>
</tr>
<tr>
<td>Effort at ((1, 1)) by the winner at ((2, 1)) or ((1, 2))</td>
<td>25.0</td>
<td>33.2</td>
</tr>
<tr>
<td>Effort at ((1, 1)) by the loser at ((2, 1)) or ((1, 2))</td>
<td>25.0</td>
<td>31.7</td>
</tr>
<tr>
<td>Probability of not reaching ((1, 1))</td>
<td>0.75</td>
<td>0.61</td>
</tr>
<tr>
<td>Players’ average total effort</td>
<td>32.0</td>
<td>60.8</td>
</tr>
<tr>
<td>Total monetary payoff</td>
<td>18.0</td>
<td>-10.9</td>
</tr>
</tbody>
</table>

**Table 1**: Results on effort choices and win probabilities in a best-of-three contest for a prize of 100 and linear effort costs. The first column shows the outcomes that are predicted from the subgame perfect equilibrium of the complete information game in Klumpp
and Polborn (2006). The second column has the average values from the experiment by Mago et al. (2013).

The table shows that the average effort follows a pattern that is only partially consistent with the behaviour predicted by the model framework in Klumpp and Polborn (2006). Players generally expend more effort than what would be optimal for them if they both maximised their monetary payoffs under the assumptions of common knowledge and rationality. As is well-known, non-monetary aspects matter and subjects tend to expend higher efforts in the laboratory than what would be the equilibrium outcome under these circumstances. There are several possible reasons for this higher spending. For instance, some, but not all subjects, may attribute a non-monetary value to winning.

If players have such non-monetary payoff components, and if the size of these components is not publicly observable, the game is no longer one of complete-information. Effort choices at \((1, 1)\) may then be guided by observable monetary as well as non-monetary goals that are the private information of single players. The effort choices at \((2, 2)\), \((2, 1)\) and \((1, 2)\) are then potentially informative of the players’ unobserved types. Furthermore, players may understand that their choices are potentially informative. Players must form beliefs – about how to interpret their co-players’ choices, how their possible effort choices are interpreted by their co-players, and so on. Moreover, this preference heterogeneity affects their effort choices and leads to selection. The distribution of types of players who arrive at \((1, 2)\), \((2, 1)\), or \((1, 1)\) differs from the type distribution of players at \((2, 2)\). This selection may, for instance, explain the high probability by which players reach \((1, 1)\) in Mago et al. (2013), which is almost 40 percent (rather than 25 percent – see Table 1). Work by Herbst et al. (2014) suggests that subjects are not homogeneous with respect to non-monetary components of their effort costs or valuations of winning. Their results are in line with self-selection.

This discussion suggests that, if one allows for unobserved preference heterogeneity of the contestants, then the best-of-n contest becomes a much more
complicated game. For this game we should not expect that the straightforward equilibrium predictions of the complete information best-of-n game adequately describe the outcomes in an experimental setup.

**Further generalisations and departures** Klumpp and Polborn (2006) characterise how the best-of-three case considered above can be extended to a larger number of states. However, simple analytical closed-form characterisations of the equilibrium are not available.

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Konrad and Kovenock (2009) make the generalisation to more states and asymmetric valuations tractable by replacing the battle-success function (1) by what is called the all-pay-battle-success function without noise, by which efforts map into win probabilities such that $p_{(i,j)}^A(x_{(i,j)}, y_{(i,j)}) = 1$ if $x_{(i,j)} > y_{(i,j)}$ and $p_{(i,j)}^A(x_{(i,j)}, y_{(i,j)}) = 0$ if $x_{(i,j)} < y_{(i,j)}$ and for suitably chosen tie-breaking rules for the case $x_{(i,j)} = y_{(i,j)}$.\(^8\) This structure has an even stronger discouragement effect. Players $A$ and $B$ expend positive efforts only along symmetric states $(j, j)$. Battle fighting completely slacks off for all states $(i, j)$ with $i \neq j$. As a consequence, once the process has moved away from a symmetric state $(j, j)$, the process has moved straight and without further effort to the nearest terminal state, without ever returning to a symmetric state $(j - t, j - t)$.

Intuitively, this can be illustrated for $W = L$. The all-pay-no-noise battle-success function in Konrad and Kovenock (2009) leads to equilibrium efforts

\(^8\)This battle-success function has been adopted, for instance, by Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996) and variants of it have been analysed more systematically in a series of contributions. It has been argued that this battle-success function may seem extreme: it does not matter for the win probability if one player’s effort exceeds the effort of the other player by a very small or a very large amount. However, the function is only the limit case of a rich class of battle-success functions which are less extreme and allow for some, but not too much, noise in the determination of battle victory. Work by Alcalde and Dahm (2010) has shown that this whole class of all-pay battle-success functions with little noise leads to equilibria that are payoff-equivalent to the equilibrium of the static all-pay auction that emerges assuming this function.
that are so high in expectation that the sum of the players’ expected efforts at 
\((1, 1)\) is equal to the prize of winning. Accordingly, this makes it completely 
unattractive for a player who is lagging behind at \((2, 1)\) or \((1, 2)\) to expend 
effort to return to a strategically symmetric state. By backward induction, 
this argument extends to other asymmetric states.

Konrad and Kovenock (2009) offer a complete and explicit characteri-
sation of the subgame perfect equilibrium for the cases of asymmetry (i.e. 
\(L \neq W\) in our framework here), for intermediate prizes that are awarded to 
the winners of single battles, and for arbitrary starting points \((j, i)\). Their 
paper suggests that the discouragement effect is a more general phenomenon. 
Sela (2011) widens the set of admissible prize valuations. Gelder (2014) looks 
at discounting between periods. As a result, the discouragement effect may 
become weaker. Beviá and Corchón (2013) study a sequence of two bat-
tles by which efforts in a first round affect players’ cost of effort in a second 
round. This can be understood as a two-stage game in which players invest in 
their contest technology in the first period and use this potentially improved 
contest technology in the actual contest. Clark and Nilsen (2013) analyse a 
related problem, allowing the contestants to become more cost-effective in 
a future contest by expendig effort in a previous contest. They explore the 
implications of this ‘learning-by-doing’ for the contest organizer’s choice of 
how to allocate the prize money between the two contests. Clark, Nilssen 
and Sand (2014) study a problem in which high scores of battle-victories 
give players an technical advantage in future battles. The player who has 
a higher score of battle victories has ‘headstart advantage’ in the respective 
battle. Such a headstart advantage makes it easier for the leading player 
to win, and the player lagging behind must essentially expend more effort 
to compete with the leading player. In addition, the player’s higher score 
gives the player also a higher continuation value in future battles, hence, 
causing a two-fold advantage.\footnote{Headstart advantages have been considered in a static framework in the context of the all-pay auction without noise by Konrad (2002).} A countervailing effect to the discouragement 
effect emerges for the player who is lagging behind, compared to the standard
best-of-n contest.

4 Sequential Colonel Blotto campaigns

We now turn to the sequential Colonel Blotto campaign. As illustrated in the introduction, in campaigns like Napoleon’s attempt to conquer Russia, Napoleon had to choose the size of his army when he started the campaign, but the means to mobilise further military resources during the campaign may have been rather limited. More generally, logistics as well as the countries’ demography may limit the amount of replenishment of troops while the campaign is running. Military leaders may then have to choose their overall budget and to allocate their resource budgets in a series of battles that take place sequentially.

The allocation of a given budget among multiple fronts is an old topic in the context of military strategy. Its game theory origins are often attributed to Borel (1921). In this game the two players have a stock of military resources and choose how to allocate these among several simultaneous battlefields. Much of the literature on this Colonel Blotto game attributes an independent value of winning to each of the single battles. The framework here differs along two important dimensions. First, each player’s objective is not to win as many battles as possible, but to win a dynamic best-of-three contest as in Figure 1. Second, the different battles occur sequentially. Whether resources that were saved for later battles are still useful depends on the outcomes of earlier battles.

The rules of the dynamic Colonel Blotto campaign We start the analysis by describing and recalling key features of the sequential best-of-three Colonel Blotto campaign.

Before players $A$ and $B$ enter into the campaign, they both choose their

\footnote{Major steps have been made in the analysis of this game more recently by Kvasov (2007), Roberson (2006) and Roberson and Kvasov (2012).}
own total budget. This choice is denoted by \( a_{(2,2)} \geq 0 \) and \( b_{(2,2)} \geq 0 \). Endowed with these budgets they enter into the battle at \((2,2)\) in the best-of-three campaign. From there they move to \((1,2)\) if \(A\) wins and to \((2,1)\) if \(B\) wins, and so on along the grid described in Figure 1. If they are at a particular interior battle state \((i,j)\), they have to decide how much effort to allocate to the battle that is taking place there. They have budgets \(a_{(i,j)}\) and \(b_{(i,j)}\) at state \((i,j)\). These consist of the initial budgets \(a_{(2,2)}\) and \(b_{(2,2)}\) net of efforts they have expended in previous battle states. At \((i,j)\) they can expend any non-negative amounts \(x_{(i,j)}\) and \(y_{(i,j)}\) that do not exceed their budgets \(a_{(i,j)}\) and \(b_{(i,j)}\). What they do not use at \((i,j)\) is \(a_{(i,j)} - x_{(i,j)}\) and \(b_{(i,j)} - y_{(i,j)}\) and this is what remains for future battles and defines their budgets at the next battle stage.

In a given battle stage the winner is determined probabilistically. The players’ efforts \(x_{(i,j)}\) and \(y_{(i,j)}\) determine the win probabilities at \((i,j)\) according to the function (1).

The campaign proceeds according to these rules until it reaches one of the terminal states. If \((0,1)\) or \((0,2)\) is reached, then player \(A\) receives \(W\) and \(B\) receives \((-L)\). If \((2,0)\) or \((1,0)\) is reached, then both players receive zero. Also, both players have to pay their budgets \(a_{(2,2)}\) and \(b_{(2,2)}\) respectively. Accordingly, the payoff for \(A\) is \(W - a_{(2,2)}\) and the payoff for \(B\) is \(-L - b_{(2,2)}\) if terminal state \((0,2)\) or \((0,1)\) is reached, and the payoffs are \(-a_{(2,2)}\) for \(A\) and \(-b_{(2,2)}\) for \(B\) if the process ends in \((2,0)\) or \((1,0)\). These payoffs highlight that each player has to pay his overall budget, whether it is used up or not. When they reach a terminal state, any sum remaining from their budgets has a scrap value of zero.

**Equilibrium for given budgets** In this subsection we solve for the equilibrium of the best-of-three Colonel Blotto campaign for given budgets \(a_{(2,2)}\) and \(b_{(2,2)}\). We use backward induction.

Suppose players reach \((1,1)\). The efforts they can choose at \((1,1)\) cannot exceed their overall resources at this state. If player \(A\) was the winner of the battle at \((2,2)\) and \(B\) was the winner of the battle at \((1,2)\), this
means that \( x_{(1,1)} \leq a_{(2,2)} - x_{(2,2)} - x_{(1,2)} \) where \( a_{(2,2)} \) is \( A \)'s initial budget and \( x_{(2,2)} \) and \( x_{(1,2)} \) are the resources that \( A \) has already used up. Analogously, 
\[ y_{(1,1)} \leq b_{(2,2)} - y_{(2,2)} - y_{(1,2)}. \]
Similarly, if player \( B \) was the winner of the battle at \((2,2)\) and \( A \) was the winner of the battle at \((2,1)\), then their constraints at \((1,1)\) are \( x_{(1,1)} \leq a_{(2,2)} - x_{(2,2)} - x_{(1,2)} \) and \( y_{(1,1)} \leq b_{(2,2)} - y_{(2,2)} - y_{(1,2)} \). Players will use all the resources they have left at \((1,1)\): a player’s win probability is (weakly) increasing in own effort at \((1,1)\), the process moves from \((1,1)\) to a terminal state with probability 1 and the scrap value of unused resources is zero. The resources available are given and the Tullock contest success probability \((1)\) determines the win probabilities at \((1,1)\). Accordingly, the expected continuation payoffs at \((1,1)\) are
\[
p^A_{(1,1)}(x_{(1,1)}, y_{(1,1)})W \text{ and } p^A_{(1,1)}(x_{(1,1)}, y_{(1,1)})(-L) \tag{3}
\]
and are fully determined by choices that are made earlier.

Consider next the state \((1,2)\). Players start at this battle state with budgets \( a_{(1,2)} = a_{(2,2)} - x_{(2,2)} \geq 0 \) and \( b_{(1,2)} = a_{(2,2)} - y_{(2,2)} \geq 0 \). Their effort choices \( x_{(1,2)} \in [0, a_{(1,2)}] \) and \( y_{(1,2)} \in [0, b_{(1,2)}] \) determine their win probabilities at \((1,2)\) by \((1)\). Also, these choices determine \( x_{(1,1)} = a_{(1,2)} - x_{(1,2)} \), and \( y_{(1,1)} = a_{(1,2)} - y_{(1,2)} \). Let the efforts chosen be \( x_{(1,2)} \) and \( y_{(1,2)} \). If \( A \) wins, the payoffs are \( W \) and \(-L\) and the game ends. If \( B \) wins, both use their remaining resources at \((1,1)\). If \( A \) wins at \((1,1)\) then the payoffs are \( W \) and \(-L\). If \( B \) wins then the payoffs are zero for both players. Assuming that the effort choices are positive, for a given anticipated \( y_{(1,2)} \) the objective function for player \( A \) at \((1,2)\) is
\[
\pi^A_{(1,2)} = \frac{x_{(1,2)}}{x_{(1,2)} + y_{(1,2)}} W + \frac{y_{(1,2)}}{x_{(1,2)} + y_{(1,2)}} \left( \frac{a_{(1,2)} - x_{(1,2)}}{a_{(1,2)} - x_{(1,2)} + b_{(1,2)} - y_{(1,2)}} \right) W. \tag{4}
\]
The objective function of player \( B \) is
\[
\pi^B_{(1,2)} = \frac{x_{(1,2)}}{x_{(1,2)} + y_{(1,2)}} (-L) + \frac{y_{(1,2)}}{x_{(1,2)} + y_{(1,2)}} \left( \frac{a_{(1,2)} - x_{(1,2)}}{a_{(1,2)} - x_{(1,2)} + b_{(1,2)} - y_{(1,2)}} \right) (-L). \tag{5}
\]
We find
Lemma 1: The sequential conquest game at \((1, 2)\) with given total budgets \(a_{(1,2)}\) and \(b_{(1,2)}\) has an equilibrium with

\[
x_{(1,2)} = x_{(1,1)} = \frac{a_{(1,2)}}{2} \quad \text{and} \quad y_{(1,2)} = y_{(1,1)} = \frac{b_{(1,2)}}{2}.
\]

(6)

The continuation values at \((1, 2)\) are

\[
v_{(1,2)}^A = a_{(1,2)} W \frac{a_{(1,2)} + 2b_{(1,2)}}{(a_{(1,2)} + b_{(1,2)})^2} \quad \text{and} \quad v_{(1,2)}^B = -a_{(1,2)} L \frac{a_{(1,2)} + 2b_{(1,2)}}{(a_{(1,2)} + b_{(1,2)})^2}.
\]

(7)

The proof is conceptually straightforward. It shows that the values \(x_{(1,2)}\) and \(y_{(1,2)}\) in the Lemma 1 are mutually optimal replies. These equilibrium efforts can then be used to calculate the continuation values that result from these choices. The formal proof is in the Appendix.

Next we solve the analogous problem for battle state \((2, 1)\). It is evident that the situation is structurally very similar to the one considered before. The defender \(B\) is now a best-shot player who needs only one additional battle. The defender has a winner prize of 0 and a loser prize of \(-L\). Player \(A\) is a weakest-link player who needs to win two battles in a row in order to win the campaign. A formal analysis leads to the results in the following lemma.

Lemma 2: The sequential conquest game at \((2, 1)\) with budgets \(a_{(2,1)}\) and \(b_{(2,1)}\) has a Nash equilibrium with

\[
x_{(2,1)} = x_{(1,1)} = \frac{a_{(2,1)}}{2} \quad \text{and} \quad y_{(2,1)} = y_{(1,1)} = \frac{b_{(2,1)}}{2}.
\]

(8)

The continuation values at \((2, 1)\) are

\[
v_{(2,1)}^A = \frac{a_{(2,1)}^2 W}{(b_{(2,1)} + a_{(2,1)})^2} \quad \text{and} \quad v_{(2,1)}^B = -\frac{a_{(2,1)}^2 L}{(b_{(2,1)} + a_{(2,1)})^2}.
\]

(9)

The proof is in the Appendix. Now we turn to the battle state \((2, 2)\). Both players \(A\) and \(B\) anticipate the results from Lemma 1 and Lemma 2: they will allocate the remainder of their resources \((a_{(2,2)} - x_{(2,2)}) = a_{(1,2)} = a_{(2,1)} \equiv a\)
and \((b_{(2,2)} - y_{(2,2)}) = b_{(1,2)} = b_{(2,1)} \equiv b\) evenly in future battles. Consider the stakes at \((2, 2)\). These are

\[v^A_{(1,2)} - v^A_{(2,1)} = aW \frac{a + 2b}{(a + b)^2} - \frac{a^2 W}{(b + a)^2} = \frac{2ab}{(a + b)^2} W\]  

(10)

and

\[v^B_{(2,1)} - v^B_{(1,2)} = -\frac{a^2 L}{(b + a)^2} + aL \frac{a + 2b}{(a + b)^2} = \frac{2ab}{(a + b)^2} L\]  

(11)

Let us assume that \(B\) chooses \(b = \frac{2}{3} b_{(2,2)}\). Let \(A\) anticipate this choice and maximise \(A\)'s own payoff. We find

\[
\frac{\partial}{\partial a} \left( -\frac{a_{(2,2)} - a}{a_{(2,2)} + \frac{3}{2} b_{(2,2)}} aW \frac{a + 2b_{(2,2)}}{(a + \frac{3}{2} b_{(2,2)})^2} + \frac{b_{(2,2)}}{a_{(2,2)} - a + \frac{3}{2} b_{(2,2)}} \left( \frac{a^2}{a_{(2,2)} - a + \frac{3}{2} b_{(2,2)}} \right) W \right)
\]

\[= -36W b_{(2,2)}^2 \frac{-6a_{(2,2)}^2 - 2a_{(2,2)} b_{(2,2)} + 3ab_{(2,2)} + 9aa_{(2,2)} + (-3a_{(2,2)} + 3a - b_{(2,2)})^2 (3a + 2b_{(2,2)})^3}{(-3a_{(2,2)} + 3a - b_{(2,2)})^2 (3a + 2b_{(2,2)})^3}.
\]

The first-order condition has only one meaningful solution and this solution is

\[a = \frac{2}{3} a_{(2,2)}\]  

(13)

The problem for \(B\) is analogous and makes

\[b = \frac{2}{3} b_{(2,2)}\]  

(14)

the optimal reply to \(a = \frac{2}{3} a_{(2,2)}\). We conclude:

**Proposition 2** A subgame perfect allocation of military resources in the sequential best-of-three Colonel Blotto game with initial total budgets \(a_{(2,2)}\) and \(b_{(2,2)}\) has \(x_{(i,j)} = a_{(2,2)}/3\) and \(y_{(i,j)} = b_{(2,2)}/3\) for all states \((i, j)\) with \(i, j \in \{1, 2\}\).

The proposition shows that the best-of-three, sequential Colonel Blotto contest shows neither a discouragement effect nor an encouragement effect if an asymmetric campaign state is reached. Players do not expend less or more effort in asymmetric states such as \((2, 1)\) or \((1, 2)\). This discrepancy in the
fighting behaviour between the campaign game in section 3 and the dynamic Colonel Blotto game can be explained in intuitive terms. We first consider the absence of the discouragement effect for the disadvantaged player at \((2, 1)\) or \((1, 2)\). In the standard best-of-n campaign this effect is driven by concerns about a player’s high future cost of effort from returning to the fully symmetric battle state \((1, 1)\). These efforts are chosen and the respective costs emerge if and only if the process moves to \((1, 1)\). No additional effort is required if the contest is resolved at \((2, 1)\) (or at \((1, 2)\)). In the dynamic Colonel Blotto game, additional effort does not directly appear in the continuation payoff functions of players in the subgames. Military resources have a scrap value of zero. The resources for \((1, 1)\) are already preserved for this battle at \((2, 1)\) or \((1, 2)\), and whether they are used or not at \((1, 1)\) does not affect the overall effort cost.

Furthermore, players do not change their effort levels in \((1, 1)\) compared to \((2, 2)\). The showdown effect that emerged in the standard best-of-n contest without an overall budget is absent. The victory at \((1, 1)\) is decisive and final. In comparison, the battle at \((2, 2)\) does not lead to complete resolve. It only changes the probability of final victory and makes final victory more likely for one of the players. This suggests that the effort at \((1, 1)\) should be higher, and drives the showdown effect in section 3. However, there is also a second, countervailing effect. The battle at \((2, 2)\) takes place with certainty. The battle at \((1, 1)\) takes place only with a probability of

\[
\frac{2a_{(2,2)}b_{(2,2)}}{(a_{(2,2)} + b_{(2,2)})^2} < 1.
\]

With some positive probability the state \((1, 1)\) is never reached. From the ex-ante perspective, and also from the perspective of \((2, 1)\) or \((1, 2)\), this reduces the expected value of fighting resources that are allocated to the battle at \((1, 1)\).

Evaluated at the equilibrium values, these two effects just net out. To see this, consider the expected impact of an additional unit of resource allocated at the two states \((2, 2)\) and \((1, 1)\), and consider player A. For state \((1, 1)\) the marginal impact is
\[
\frac{\partial E\pi_A}{\partial x(1,1)} = \frac{2a_{(2,2)}b_{(2,2)}}{(a_{(2,2)} + b_{(2,2)})^2}\frac{b_{(1,1)}}{(a_{(1,1)} + b_{(1,1)})^2} W.
\]

The expected impact of an additional resource allocated at (2, 2) is

\[
\frac{\partial E\pi_A}{\partial x(2,2)} = (v_A^{(1,2)} - v_A^{(2,1)}) \frac{\partial ((a_{(2,2)}/3) + (b_{(2,2)}/3))}{\partial (a_{(2,2)}/3)}. 
\]

Making use of \(a_{(2,2)}/3 = a_{(1,1)}\), \(b_{(2,2)}/3 = b_{(1,1)}\) and (10) shows that the expected impact of an additional unit of resource used has the same impact on the expected payoff at (1, 1) and at (2, 2). The higher decisiveness at (1, 1) and the lower probability of reaching (1, 1) just balance each other out.

Note that the non-escalation result in the comparison between (2, 2) and (1, 1) emerges even though the players choose their efforts sequentially and in a time-consistent manner. They reconsider and re-optimise the allocation of their remaining resources at (1, 2) or (2, 1), that is, when the probability of reaching (1, 1) has changed. However, the aggregate budget constraint causes an arbitrage link between states. There is a link between states \((i, j)\) with \(i + j = 1\) and states with \(i + j = 3\) and states with \(i + j = 4\), which generates an indirect arbitrage link between (1, 1) and (2, 2).

**Budget choice**  Colonel Blotto games start by definition with given stocks of military resources for A and B. However, the budgets may be chosen by the players before they arrive at the first battle state (2, 2). We ask what the equilibrium choice of \(a_{(2,2)}\) and \(b_{(2,2)}\) is. We address this question for the case in which \(W = L = 1\), because this makes it straightforward to compare the result with the effort choices in the best-of-three campaign analysed in section 3.

Consider the payoff of player A if players A and B follow the equilibrium allocation of their military budgets as described by Proposition 2. Using \(W = 1\) we can write A’s payoff as a function of \(\alpha = \frac{a_{(2,2)}}{3}\) for the candidate equilibrium share \(\beta = \frac{b_{(2,2)}}{3}\) for player B as

\[
\Pi_A = 2 - \frac{b_{(2,2)} a_{(2,2)} a_{(2,2)}}{3} + \frac{a_{(2,2)} a_{(2,2)} a_{(2,2)}}{3} - a_{(2,2)}.
\]

(16)
The first term is the product of the winner prize $W = 1$ times the equilibrium probability that the campaign will be resolved after three battles with a victory of $A$. The second term is the product of the winner prize $W = 1$ times the equilibrium probability that the campaign will be resolved after two battle victories by $A$. Calculating the first-order condition $\frac{\partial \Pi}{\partial a(2,2)} = 0$ and making use of symmetry of the solution ($a_{(2,2)} = b_{(2,2)}$) in this first-order condition for a maximum of (16) we find

$$a_{(2,2)} = b_{(2,2)} = \frac{3}{8}.$$  

(17)

We can also confirm that $\frac{\partial^2 \Pi}{\partial a(2,2)^2} < 0$. We conclude:

**Proposition 3** For $W = L = 1$, if $A$ and $B$ can choose the size of their military budgets prior to entering into the sequential best-of-3 Colonel Blotto campaign, the symmetric equilibrium budgets are $a_{(2,2)} = b_{(2,2)} = \frac{3}{8}$.

We can now compare players’ expected equilibrium efforts that emerge for the two different variants of the dynamic best-of-three contest. We note that $\frac{3}{8} = 0.375 > 0.32031 = \frac{41}{128}$. We conclude that the military budgets that players $A$ and $B$ choose in the sequential best-of-three Colonel Blotto campaign exceed the expected expenditure which emerge in the best-of-three multi-battle contest with unconstrained sequential effort choices.

## 5 Conclusion

The analysis of the sequential best-of-$n$ campaign with endogenous, unconstrained effort choices and the sequential best-of-$n$ Colonel Blotto campaign shows remarkable differences as regards the equilibrium allocation of military resources.

As is well-known from the literature, the first type of campaign is characterised by strong discouragement effects for players who fall behind in the score of battle victories, and by escalation due to a showdown effect. Discouragement occurs if one player falls behind. The disadvantaged player
may expend effort and catch up. Expending these resources, the player may reach a state of strategic symmetry at which both players are highly motivated to expend effort. Precisely because of these high equilibrium efforts at such a state of symmetry, both players have low equilibrium payoffs there. This makes catching up less attractive, and reduces the player’s motivation to expend much effort in trying to catch up. The showdown effect emerges if we compare states in which both players are in strategically symmetric situations, but with one battle state closer to final resolve – i.e. closer to a state of decisive showdown. As the decisiveness of the battle increases when the campaign approaches the showdown state, the stakes are higher and this mobilises players to expend higher resources.

In contrast, the sequential Colonel Blotto campaign does not have a discouragement effect. It also does not have escalation when the campaign approaches the showdown state at which the battle is fully decisive of the overall campaign. The subgame perfect equilibrium in this structure has neither de-escalation nor escalation. The key to this result is that there are two countervailing effects in this type of campaign. Symmetric non-terminal states closer to the decisive state have higher stakes than states further away. This would make higher effort desirable at the states closer to the final showdown battle. But the arrival there is less likely, as the campaign may be resolved at an earlier stage. Any additional effort used in the showdown state needs to be taken from conflict resources available at earlier states. So the trade-off is to locate resources at a state at which resources have a high impact if the state is reached, compared to locating resources at a state that is more likely to be reached, but that is less decisive.

6 Appendix

This appendix collects the proofs of the two lemmata from section 4. It starts with the proof of Lemma 1, followed by the proof of Lemma 2.

**Proof of Lemma 1.** First we rewrite the win probability $p^A_{(1,2)} + (1 - p^A_{(1,2)})p^A_{(1,1)}$ of player $A$ at state $(1, 2)$ as $p^A_{(1,2)} + p^A_{(1,1)} - p^A_{(1,2)}p^A_{(1,1)}$. Furthermore,
we take into account that player $B$ chooses $y_{1,2} = \frac{b_{(1,2)}}{2}$. Player $A$ maximises

$$P_{(1,2)}^A + P_{(1,1)}^A - p_{(1,2)}^A p_{(1,1)}^A$$

subject to the budget constraint

$$x_{(1,2)} + x_{(1,1)} \leq a_{(1,2)}. \quad (19)$$

This problem has a unique maximum at $x_{(1,2)} = \frac{a_{(1,2)}}{2}$. To confirm this consider

$$\frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} - h + \frac{b_{(1,2)}}{2}} \right) + \frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} + h + \frac{b_{(1,2)}}{2}} \right) - \frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} + h} \right) - \frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} - h} \right)$$

$$= -8 \frac{ bb_{(2,2)}^2 }{ (a_{(1,2)} - 2h + b_{(1,2)})^2 (a_{(1,2)} + 2h + b_{(1,2)})^2}. \quad (20)$$

This term is negative for a positive $h$ and positive for a negative $h$ for any given $b_{(1,2)} > 0$. Now consider $B$. If $B$ thinks that $A$ chooses $x_{(1,2)} = \frac{a_{(1,2)}}{2}$, then $B$ maximises

$$1 - (P_{(1,2)}^A + P_{(1,1)}^A - p_{(1,2)}^A p_{(1,1)}^A)$$

subject to the budget constraint

$$y_{(1,2)} + y_{(1,1)} \leq b_{(1,2)}. \quad (22)$$

Again we consider the impact of deviations $y_{(1,2)} = \frac{b_{(1,2)}}{2} + h$ and $y_{(1,1)} = \frac{b_{(1,2)}}{2} - h$ from the candidate equilibrium. We find

$$\frac{\partial}{\partial h} \left( 1 - \frac{a_{(1,2)}}{2a_{(1,2)} + h + \frac{a_{(1,2)}}{2}} \right) + \frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} + h + \frac{a_{(1,2)}}{2}} \right) - \frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} - h} \right) - \frac{\partial}{\partial h} \left( \frac{a_{(1,2)}}{2a_{(1,2)} - h + \frac{a_{(1,2)}}{2}} \right)$$

$$= -8 a_{(1,2)} h \frac{2b_{(1,2)} + a_{(1,2)}}{(b_{(1,2)} + 2h + a_{(1,2)})^2 (b_{(1,2)} - 2h + a_{(1,2)})^2}. \quad (23)$$

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This term is negative for a positive \( h \) and positive for a negative \( h \), such that \( y_{(1,2)} = y_{(1,1)} = \frac{h_{(1,2)}}{2} \) is a globally optimal response to \( x_{(1,2)} = x_{(1,1)} = \frac{a_{(1,2)}}{2} \).

Inserting these optimal values into the payoff functions yields the equilibrium continuation values as

\[
v_A^{(1,2)} = \frac{a_{(1,2)}}{2} W + \frac{b_{(1,2)}}{2} \left( \frac{a_{(1,2)} - \frac{a_{(1,2)}}{2}}{a_{(1,2)} - \frac{a_{(1,2)}}{2} + b_{(1,2)} - \frac{b_{(1,2)}}{2}} \right) W \tag{24}
\]

and

\[
v_B^{(1,2)} = -\left( \frac{a_{(1,2)}}{2} + \frac{b_{(1,2)}}{2} \right) L + \frac{b_{(1,2)}}{2} \left( \frac{a_{(1,2)} - \frac{a_{(1,2)}}{2}}{a_{(1,2)} - \frac{a_{(1,2)}}{2} + b_{(1,2)} - \frac{b_{(1,2)}}{2}} \right) L \tag{25}
\]

as stated in Lemma 1.

**Proof of Lemma 2.** Player \( A \) maximises \( p_A^{(2,1)} p_A^{(1,1)} \). For given \( y_{(2,1)} = y_{(1,1)} = \frac{b_{(2,1)}}{2} \) player \( A \) allocates \( a_{(2,1)} \) symmetrically between the battles, such that the optimal reply is \( x_{(2,1)} = x_{(1,1)} = \frac{a_{(2,1)}}{2} \). To see this, consider

\[
p_A^{(2,1)} p_A^{(1,1)} = \frac{a_{(2,1)}}{2} - h \left( \frac{a_{(2,1)}}{2} + \frac{b_{(2,1)}}{2} \right) \left( \frac{a_{(2,1)}}{2} + h + \frac{b_{(2,1)}}{2} \right). \tag{26}
\]

It holds that

\[
\frac{\partial}{\partial h} \left( \frac{a_{(2,1)}}{2} - h \frac{a_{(2,1)}}{2} + h \frac{a_{(2,1)}}{2} + h + \frac{b_{(2,1)}}{2} \right) \tag{27}
\]

\[
= -8hb_{(2,1)} \left( a_{(2,1)} - 2h + b_{(2,1)} \right)^2 \left( a_{(2,1)} + 2h + b_{(2,1)} \right)^2
\]

This term is negative for positive \( h \), positive for negative \( h \) and zero for \( h = 0 \).
Similarly, $B$ minimises $p_A^{(2,1)}p_A^{(1,1)}$. We note that

$$\frac{\partial}{\partial h} \left( \frac{a_{(2,1)}}{\frac{a_{(2,1)}}{2} + \frac{b_{(2,1)}}{2}} - h \frac{a_{(2,1)}}{\frac{a_{(2,1)}}{2} + \frac{b_{(2,1)}}{2} + h} \right) =$$

$$= 8a^2 \frac{h}{(a_{(2,1)} + b_{(2,1)} - 2h)^2} \left( a_{(2,1)} + b_{(2,1)} + 2h \right)^2.$$

This term is positive for a positive $h$, negative for a negative $h$ and zero for $h = 0$. This implies that $p_A^{(2,1)}p_A^{(1,1)}$ has a global minimum for $h = 0$, and this makes $y_{(2,1)} = y_{(1,1)} = \frac{a_{(2,1)}}{2}$ the unique optimal reply to $x_{(2,1)} = x_{(1,1)} = \frac{a_{(2,1)}}{2}$.

Inserting (8) into $v_A^{(2,1)} = p_A^{(2,1)}p_A^{(1,1)}W$ and $v_B^{(2,1)} = -p_A^{(2,1)}p_A^{(1,1)}L$ yields the continuation values (9) as in Lemma 2.

**References**


[33] Kahana, Nava, and Doron Klunover, 2016, Sequential lottery contests, Department of Economics, Bar-Ilan University, unpublished manuscript.


