Contagious Mortgage Default

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PRELIMINARY

Abstract

I demonstrate how the default option in American mortgages prepares the ground for a strategic default crisis. Households borrow to buy durable housing, but find it advantageous to default on debt if house prices fall sufficiently. The key assumptions of the model is that households are relegated to the rental market upon default, and that there is a small pecuniary inefficiency (“iceberg cost”) in renting. This leads defaulters to substitute consumption of other goods for housing, implying lower demand for housing upon default. Consequently, when many households default, demand for housing is driven down, and so are prices, possibly inciting other households to default. This complementarity is a source of multiple equilibria. Using a specific case for which an analytical solution can be derived, I provide parameter conditions for the different equilibria. I show that contagion is possible: It may be that the default of a minority (interpretable as subprime borrowers) drags the majority (prime borrowers) into default.
1 Introduction

Over the last three years, house prices in the US have fallen sharply. According to the S&P/Case-Shiller U.S. National Home Price Index, house prices fell by about one third from the peak in the second quarter of 2006 to the second quarter of 2009\(^1\). The fall implies that millions of American households have mortgages that exceed the value of their homes, and for many of them, the discrepancy is large. Partly as a result of this, the rates of delinquent mortgages (one or more payments past due) and mortgages in foreclosure have exploded. According to the Mortgage Bankers Association’s National Delinquency Survey (NDS), 8.9 percent of all mortgage loans were delinquent, and another 4.3 percent were in the foreclosure process, in the second quarter of 2009, compared to 4.4 and 1.0 percent, respectively, in the second quarter of 2006\(^2\). While mortgage default (understood as the process starting with delinquency and ending in foreclosure) initially sparked within subprime loans, it has gradually spread to more and more conventional mortgages. In the second quarter of 2009, prime fixed-rate loans accounted for one in three foreclosure starts in the NDS. An important worry is that default \textit{in itself} puts downward pressure on house prices. A contagious chain of feedback effects may have been playing out, where default pushed down prices, causing more default and even lower prices, eventually leading to a large scale default crisis. Conceivably, the crisis could have been avoided if only the initial spark had been contained, and could indeed continue to unfold for the same reasons.

The purpose of this paper is to demonstrate, by means of a formal model, how the default option in American mortgage finance prepares the grounds for a large scale default crisis. The model is built on three crucial assumptions: 1) Mortgages are non-recourse, non-renegotiable debt contracts, 2) Mortgage default is strategic, and 3) House ownership is, all other things equal, less costly than tenancy.

Only few American states have mandatory non-recourse mortgages\(^3\). Nevertheless, it may be a good approximation, because defaulting homeowners rarely have significant wealth besides the home (at least net of the cost of legal procedures), and because federal bankruptcy law accords the right to obtain a ”fresh start”. Under Chapter 7 in the Bankruptcy Code, the worst

\(^2\)The National Delinquency Survey is based on a sample of more than 44 million mortgage loans serviced by mortgage companies, commercial banks, thrifts, credit unions and others. It provides quarterly delinquency and foreclosure statistics at the national, regional and state levels. Aggregate statistics are available on www.mbaa.org.
\(^3\)Capone (1996) includes a thorough discussion on foreclosure and bankruptcy law.
case scenario for households is that they are left with zero net assets and a poor credit rating. Given the limited legal punishment, it is important to investigate the economic incentives and consequences of strategic default.

As a matter of fact, most real world loan arrangements are uncontingent debt contracts. Recently, however, it has been debated why lenders do not renegotiate more mortgages in default. Widespread securitisation is the most widely believed reason for the rarity of renegotiation (see, for example, Eggert, 2007). Adelino et al. (2009), however, argue that lenders expect to recover more from foreclosure than from a modified loan, because many delinquent borrowers eventually resume payments without receiving a modification, and many borrowers receiving a modification eventually redefault, making loan modification a very inefficient strategy.

It is now widely acknowledged that negative home equity (a mortgage that exceeds the value of the home) is the primary driver of mortgage default (Deng et al., 2000). With positive home equity, households would be better off selling the home and prepaying the mortgage, than defaulting. The controversy is to what extent households with negative home equity actually choose default (strategic default), or are forced into default because exogenous events trigger an inability to meet scheduled payments. Foote et al. (2008) found that few people (6%) actually walked away from their mortgages when their home equity was negative, and has been interpreted as evidence that households do not default strategically. Guiso et al. (2009) challenge this view. Based on survey data, they estimate that approximately one of four recent defaults are strategic. The reason, they argue, is that the relationship between negative home equity and default is highly non-linear, and that this time home prices have fallen much more than in the episode studied by Foote et al. (2008).

Finally, I assume that renting is more costly than ownership, for any given house. There are several sources of additional costs involved in renting relative to ownership. One is due to moral hasard, as tenants may have insufficient incentives to take care of the property. Another is the implicit tax advantage of indebted owners, as mortgage interest payments are deductible for federal tax purposes. Moreover, there are overhead costs in the rental business that must be covered.

The key mechanism of my model is as follows. Households choose rationally whether to repay or default on their mortgage. When home equity is sufficiently negative, their incentive is to default, liberating resources to

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4The Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 (Pub.L. 109-8, 119 Stat. 23, enacted April 20, 2005), imposes a means-test to Chapter 7 filings when the debtor’s income is above the state median income, in order to prevent e.g. strategic default. However, this means test only applies when debt is primarily consumer debt.
consumption that they otherwise would have had to spend on debt service. Because defaulted households cannot obtain new credit, they are relegated to the rental market for housing, and because rental housing is more expensive than owned housing per unit of housing, their demand for housing decreases. Reduced demand for housing induces a further fall in house prices to clear the market, which may in turn create the incentive for other households to default. The private default decision does not take into account this negative externality.

My work is related to two strands of literature. First, it relates to a large literature modelling strategic consumer default. Deng et al. (2000) have shown that a simple option theoretic framework does a good job of explaining mortgage default, but it is not enough by itself. In particular, they find significant heterogeneity among borrowers. Guiso et al. (2009) provide evidence that relocation costs, as well as moral and social considerations, are important in predicting strategic mortgage default. The work by Chatterjee et al. (2007) on consumer credit suggests that the indirect cost of reduced access to credit after default is an important deterrent against actual default. Second, it relates to the literature on financial contagion. Though most of this literature is based on information asymmetries absent from my model, there are some interesting exceptions. For instance, Allen and Gale (2000) present a model with complete information where a small liquidity preference shock in one region can spread by contagion throughout the economy, through cross holdings of deposits between banks. Their notion of financial fragility, where an unforeseen shock (possibly small) can bring down the entire financial system, resembles my case of default equilibrium.

The theory that I present offers some significant advantages to the bulk of commonly given explanations to the current mortgage default crisis. In my model, both households and banks behave in an optimal manner with symmetric information and rational expectations. This shows that there is no need for grand scale fraud, misperceptions or irrationality in order to explain the crisis. Explanations that require key information, such as the financial fragility of US households and the possible consequences of a sharp fall in house prices, to be hidden from market participants in a large and very competitive markets such as the US mortgage market, are quite unsatisfactory. In fact, these issues were hotly debated, as evidenced by Case and Shiller (2003). Moreover, certain major international investment banks explicitly characterised housing "meltdown" scenarios in their own investment research reports on several occasions preceeding the crisis (see the evidence presented in Gerardi et al., 2009). Thus, the problem does not seem to be that a housing market collapse was unthinkable or misunderstood, but rather that it was considered too unlikely to be accorded any importance.
by any individual market participant. Perhaps precisely for this reason did they go on with the lenient lending that made the crisis possible.

2 Model

2.1 Households

There are two periods, $t = 1, 2$. In each period households obtain utility from two types of consumption: housing service $h$ (or simply "housing"), and the consumption of other goods $c$ ("consumption"). The latter are elastically supplied through an international market at the constant price of 1 (they act as numeraire). Housing service can be obtained in two ways: through house ownership or from renting. Housing stock can be bought in a domestic market. This stock is in fixed aggregate supply $H$. The relative price of a unit of housing stock, denoted $p_t$, will therefore depend on households’ aggregate demand. Housing stock is perfectly durable from the first to the last period, but after the last period, it no longer has any value, so it is as if it completely depreciated\(^5\). Owning one unit of the housing stock in period $t$ provides one unit of housing service in that period. Alternatively, households can rent, obtaining one unit of housing service in period $t$ for the relative price $r_t$. Both housing stock, housing service obtained from renting, and other consumption goods are perfectly divisible, and there are no transactions cost involved in buying or selling them.

There are two types of households denoted $i \in \{\text{sub, pri}\}$. Each individual household is small, but in aggregate the relative mass of $\text{sub}$ and $\text{pri}$ households is $\gamma$ and $1 - \gamma$ respectively. Households value housing $h_t > 0$ and goods $c_t > 0$ according to

$$u(h_1, c_1) + \beta E[u(h_2, c_2)],$$

where

$$u(h, c) = \left[\alpha c^\rho + (1 - \alpha) h^\rho\right]^{\frac{1-\theta}{\theta}} - 1$$

for $\rho \leq 1$ and $\theta \geq 0^6$. The parameter $\rho$ determines the substitutability between housing and other consumption, $\alpha \in (0, 1)$ is a weighing parameter for how much the household values other consumption, and $\theta$ governs risk aversion and intertemporal substitutability. $\beta \in (0, 1)$ is the time discount factor, and $E$ denotes the expectations operator. It can be verified

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5Other consumption goods perish at the end of each period, unless consumed.

6Except for $\rho = 0$ and/or $\theta = 1$, in which cases the Cobb-Douglas/logarithmic form applies.
that \( u(h,c) \) is a two times continuously differentiable, strictly increasing and concave function.

There is no explicit labor market. Instead, households receive wage endowments \( w_{t,i} \) in each period. For simplicity, I fix the period one wage endowment to \( w_{1,i} = w > 0 \) across households of all types \( i \), but I allow for type-specific and stochastic wages in period two. In particular, I am interested in a situation where one type of households have more uncertain income than the other\(^7\). Moreover, it is of interest to study the impact of wage growth. A simple set of assumptions that capture this is

\[
\begin{align*}
    w_{2,pri} &= Aw, \\
    w_{2,sub} &= (1 - \sigma) Aw,
\end{align*}
\]

where \( A \geq 1 \) is a constant and \( \sigma \in (0, 1) \) is a stochastic variable distributed according to \( G(\sigma) \). In other words, \( pri \) households have perfect income security (and income growth whenever \( A > 1 \)), while \( sub \) households may experience a relative income shortfall. Since all \( sub \) households are hit simultaneously, \( \sigma \) is an aggregate shock which will have indirect consequences also for \( pri \) households. I assume there is symmetric information about household type and the actual realisation of \( w_{2,sub} \).

### 2.2 Mortgage contracts with default

I assume that financial markets are incomplete. What I have in mind is household protection from creditors by a liquidation type bankruptcy law (à la chapter 7 in the American bankruptcy code), whereby future wage income is shielded from creditors upon default. Reflecting this institutional arrangement, I assume that households cannot credibly borrow against future wage income (as they will have no incentive to actually repay their debt in period two, being the last period). However, they can use housing stock as collateral since it is durable. Thus, I postulate a market for collateralised non-recourse mortgage contracts.

I model mortgage contracts as straight debt arrangements with a default option. A particular mortgage contract, denoted \((d, D, h_1)\), consists of an amount \( d \) to be paid out (in period one) to the household, i.e. the "principal", an amount \( D \) to be repaid (in period two) by the household, subsuming the total of principal, interest and any risk premium, and finally, a collateral

\(^7\)A motivating example could be when certain groups are more sensitive to business cycle downturns, e.g. "fragile" groups are hit harder by job loss.
I do not allow for renegotiation of the terms of the contract. In other words, the contract is *rigid* ex post (period two) in the sense that the borrower faces only two alternatives: Either pay \( D \) as specified by the contract, or default and give up the collateral. However, the mortgage contract is *flexible* ex ante (period one) in the sense that the terms of the contract are endogenously determined. The optimal mortgage contract will in general be type \( i \) dependent, because *sub* households may experience an income shortfall in period two, while *pri* households have no income uncertainty.

Mortgage financing is provided by small, risk neutral banks operating in a competitive financial market. Risk neutrality implies that banks require contracts to break even in expectation. Let \( \Delta_i \in \{0, 1\} \) denote a type dependent function mapping outcomes \( (p_2, r_2, \sigma) \) and contract characteristics \( (D, h_1) \) into default decisions, where \( \Delta_i = 0 \) means default. Assuming that banks discount time with the same factor as households, the type specific set of break even contracts is then\(^9\)

\[
\Omega_i = (d, D, h_1) : d \leq \beta E [\Delta_i (p_2, r_2, \sigma, D, h_1) D + (1 - \Delta_i (p_2, r_2, \sigma, D, h_1)) p_2 h_1] .
\]  

(3)

The set of break even contracts can be thought of as an menu of mortgage contracts available to households in the credit market, from which they may choose. To keep matters simple, I assume that banks are internationally owned and funded with abundant resources\(^{10,11}\).

Competition among banks lead them to provide the mortgage contracts that maximise households’ expected utility. Since for given \( (D, h_1) \), a household will always prefer more money (higher \( d \)), this implies that eq. (3) must hold with equality in equilibrium, establishing \( d \) as a function of \( (D, h_1) \).

Since only \( D \) and \( h_1 \) matter for the default decision (not \( d \)), it will often be convenient to refer to \( (D, h_1) \) itself as the ”mortgage contract”, where it is understood that \( d \) is determined from (3) evaluated at equality.

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8 Liquidation type bankruptcy procedures typically seize *all* the household’s assets, apart from certain exempted ones. Generally, then, there is no reason to posit less collateral than the entire house. Thus I assume that the collateral coincides with the amount of housing bought, \( h_1 \).

9 Here I abstract from losses in the foreclosure process, though this can easily be added by assuming that the net cash flow to the bank in case of default is \( \zeta p_2 h_1 \), with \( \zeta \in (0, 1] \).

10 To model the bank as owned by domestic households would necessitate handling feedback effects from banks’ cash flow onto households’ demand for housing, which would obscure the main point of the model (and, presumably, only aggravate the results).

11 With abundant funding, banks are able to provide any feasible mortgage contract.
2.3 Cost of default and the rental market

Conform to common perception, I assume that the direct (pecuniary) cost of default is negligible. This is not to say, however, that households face no indirect costs of default. Households will usually find themselves excluded from borrowing in a number of years following default. An important consequence of this is relegation to the house rental market\textsuperscript{12}. To the extent that the rental cost exceeds the cost of ownership for an equivalent property, this constitutes an indirect (pecuniary) punishment for default\textsuperscript{13}. I model this as an ”iceberg cost” $1 - \kappa$, so that for each dollar rent paid by a renter, the net proceeds to the owner equals $\kappa \in (0, 1)$. This cost can be thought of as due to unmodelled moral hazard issues, tax disadvantages or administrative costs that exist in rental markets but do not apply to home ownership. Because renting is inefficient relative to ownership, households \textit{ceteris paribus} prefer to own. This motivates a simplifying assumption (maintained throughout the paper) that all households start as owners (in $t = 1$)\textsuperscript{14}. Households only become renters because they have to, after default.

From these assumptions it follows that there exists a rental market (in $t = 2$) if and only if some households default. To implement this rental market, I let banks trade freely in housing stock, and propose rental service. Hence, banks can sell off any amount of housing stock they recoup upon default, and keep (or buy) any amount of housing stock to let out. Since period two is the last period, housing stock expires worthless at the end of the period. To preclude arbitrage opportunities a simple relationship between the rental price, denoted $r_2$ (the cost of renting one unit of housing in period two), and the price of one unit of housing stock $p_2$ must hold:

\[
p_2 = \kappa r_2 \iff r_2 = \frac{p_2}{\kappa} > p_2.
\]

\textsuperscript{12}Exclusion from borrowing may also hamper households’ ability to self insure against adverse shocks. See Chatterjee et al. (2007) for a demonstration of the quantitative importance of this issue in the context of consumer credit default. With only two periods, however, my model is not well-suited to analyse this issue.

\textsuperscript{13}Though without doubt, moral quelms and social stigma are important constraints on default (see e.g. Guiso et al, 2008, for recent survey evidence), it shall not be my business to theorise about them.

\textsuperscript{14}Presumably, in the real world, people stay tenants either because they value the flexibility of tenancy (which is outside the scope of this paper), or because they simply cannot obtain a mortgage. I discuss the consequences of excluding subprime borrowers at the end of the paper.
2.4 Problem definition

Consider a household of type \( i \) that enters period two with the mortgage contract \((D, h_1)\). He obtains a wage realisation \( w_{2,i} \) (type dependent only if \( \sigma = \bar{\sigma} \)), and observes market prices of housing stock \( p_2 \) and housing rental service \( r_2 \). Based on this he must decide whether to default. If he does not default, he owns a house worth \( p_2 h_1 \) and pays his debt worth \( D \), so available resources are \( w_{2,i} + p_2 h_1 - D \). He then decides how much of this to spend on housing stock \( h_2 > 0 \) (providing an equal amount of housing service), and other goods. \( c_2 > 0 \). He solves (denoted with superscript \( S \) for "solvent")

\[
v^S(p_2, m_i^S(p_2, \sigma, D, h_1)) = \max_{c_2, h_2} u(h_2, c_2) \tag{5}
\]

subject to

\[
c_2 + p_2 h_2 \leq m_i^S(p_2, \sigma, D, h_1) \equiv w_{2,i} + p_2 h_1 - D
\]

\[
w_{2,i} > 0, \ h_1 > 0, \ \text{and} \ D \ \text{given}.
\]

If he defaults, his debt \( D \) is cleared, but he looses his house, so his available resources are simply \( w_{2,i} \). He decides how much of this to spend on housing service \( h_2 > 0 \) (this time by buying rental service at price \( r_2 \)), and other goods \( c_2 > 0 \). He solves (denoted with superscript \( D \) for "default")

\[
v^D(r_2, m_i^D(\sigma)) = \max_{c_2, h_2} u(h_2, c_2) \tag{6}
\]

subject to

\[
c_2 + r_2 h_2 \leq m_i^D(\sigma) \equiv w_{2,i}
\]

\[
w_{2,i} > 0 \ \text{given}.
\]

The default decision, denoted \( \delta \in \{0, 1\} \) where \( \delta = 0 \) means default, solves

\[
v_i(p_2, r_2, \sigma, D, h_1) = \max_{\delta \in \{0, 1\}} \{ \delta v^S(p_2, m_i^S) + (1 - \delta) v^D(r_2, m_i^D) \} \tag{7}
\]

I will assume that households do not default if they are in fact indifferent\(^\text{15}\).

Consider a household of type \( i \) in period one. He receives a wage endowment \( w_1 = w \) (the same for all types), knows that the type dependent period two wages are realised according to (2), and that for a particular wage realisation \( w_{2,i} \), given a particular contract \((D, h_1)\), and prices \((p_2, r_2)\), he will act in a way that solves the period two problem (5)-(7). Based on this

\(^{15}\)Without this assumption, the model may feature additional equilibria at indifference points, where a share of identical households defaults and another share does not. I conjecture that such equilibria are unstable, however, because any one household changing action will move the market price infinitesimally, and this suffices to induce all other households to strictly prefer the same action.
he must decide how much housing stock $h_1 > 0$ to buy, how much to buy and consume of other goods $c_1 > 0$, and choose a mortgage contract $(d, D, h_1)$ from the menu of contracts $\Omega_i$ in available to households of his type in the credit market. The proceeds from the contract $(d_i)$ together with the wage endowment must suffice to finance the expenditures on housing stock and other goods. He takes the current price of housing stock $p_1$, and a distribution of future prices $(p_2, r_2)$ as given. It is convenient to condition future prices on the income shock $\sigma$ to sub households, so I write the distribution $F(p_2, r_2 | \sigma)$. His problem is thus

$$\max_{(d, D, h_1, c_1)} \left\{ u(h_1, c_1) + \beta \int \int \int v_i(p_2, r_2, \sigma, D, h_1) dF(p_2, r_2 | \sigma) dG(\sigma) \right\}$$

s.t.

$$c_1 + p_1 h_1 \leq w + d$$

$$(d, D, h_1) \in \Omega_i,$$

where $v_i(p_2, r_2, \sigma, D, h_1)$ solves the period two problem (5)-(7) for his type.

### 2.5 Equilibrium definition

**Definition 1** A competitive equilibrium is housing and consumption allocations $(c_{t,i}, h_{t,i})_{t=1,2}$, mortgage contracts $(d_i, D_i, h_{1,i})$ and default decisions $\delta_i$ for households $i \in \{\text{sub}, \text{pri}\}$, and corresponding default functions $\Delta_i(p_2, r_2, \sigma, D, h_1)$ and sets of contracts $\Omega_i$ offered to these households, and prices of housing stock and rental service $(p_1, F(p_2, r_2 | \sigma))$, such that

1. Households optimise:

   (a) $(d_i, D_i, h_{1,i}, c_{1,i})$ is optimal given the set $\Omega_i$ of contracts offered and the prices $(p_1, F(p_2, r_2 | \sigma))$, i.e. it solves problem (8), for each type of household $i \in \{\text{sub}, \text{pri}\}$.

   (b) $(\delta_i, c_{2,i}, h_{2,i})$ is optimal given the contract $(D_i, h_{1,i})$ and the prices $(p_2, r_2)$, i.e. it solves problem (5)-(7), for each type of household $i \in \{\text{sub}, \text{pri}\}$.

2. Mortgage contracts break even in expectation, and all mortgage contracts that would break even in expectation are offered, i.e.

$$\Omega_i = (d, D, h_1) : d \leq \beta \int \int \int \left[ \Delta_i(p_2, r_2, \sigma, D, h_1) D + (1 - \Delta_i(p_2, r_2, \sigma, D, h_1)) p_2 h_1 \right] dF(p_2, r_2 | \sigma) dG(\sigma),$$

for each type of household $i \in \{\text{sub}, \text{pri}\}$. 

3. Projected default behaviour and actual default decisions are consistent, i.e.
\[ \delta_i = \Delta_i (p_2, r_2, \sigma, D_i, h_{1,i}) , \]
for each type of household \( i \in \{ \text{sub}, \text{pri} \} \).

4. There are no arbitrage opportunities in renting, i.e.
\[ r_2 = \frac{p_2}{\kappa}. \]

5. Markets clear, i.e. the market for housing stock clears in each period
\[ \gamma h_{1,\text{sub}} + (1 - \gamma) h_{1,\text{pri}} = \gamma h_{2,\text{sub}} + (1 - \gamma) h_{2,\text{pri}} = H, \]
and the market for consumption goods clears.

3 Optimal choices

3.1 Default decision

In order to derive the optimal mortgage contract, we need to know under what conditions households will default. Under quite general circumstances, the default decision is characterised by a type specific threshold price \( p_i^* \).

**Proposition 2 (Threshold price)** Suppose the elasticity of substitution between housing and other consumption is smaller than one (i.e. suppose \( \rho \leq 0 \)). Then, for any given finite mortgage contract \( (D, h_1) \) and arbitrage free prices \( (p_2, r_2) \), there exists a type specific threshold price \( p_i^* \geq 0 \) such that default is strictly optimal if and only if the price of housing stock \( p_2 \) is strictly below \( p_i^* \), and strictly suboptimal if and only if the market price \( p_2 \) is strictly above \( p_i^* \), i.e.
\[ \exists p_i^* \geq 0 : v^S (p_2, m^S_i) < v^D (r_2, m^D_i) \Leftrightarrow p_2 < p_i^* \quad \text{and} \quad v^S (p_2, m^S_i) > v^D (r_2, m^D_i) \Leftrightarrow p_2 > p_i^*. \]

Whenever strictly positive, \( p_i^* = p_i^* (\sigma, D, h_1) \) is a continuous and differentiable function of \( (D, h_1) \) with
\[ \frac{\partial p_i^*}{\partial D} > 0 \quad \text{and} \quad \frac{\partial p_i^*}{\partial h_1} < 0 \]
for all \( (\sigma, D, h_1) \).
Proof. In the appendix. ■

Proposition 2 implies that households follow a cutoff strategy where they default if and only if the house price falls below a certain threshold level. In order to shed light on the underlying mechanism, it is useful to consider a specific example. When preferences are Cobb-Douglas, i.e. the limit case when \( \rho \) tends to zero, a simple analytical solution obtains for the threshold price:

\[
p^*_i = \max \left\{ \frac{D - (1 - \hat{\kappa}) w_{2,i}}{h_1}, 0 \right\}, \quad \text{where } \hat{\kappa} \equiv \kappa^{\frac{\alpha}{1+\alpha}}.
\]

(10)

We see that the threshold price is strictly increasing in debt \( D \) whenever \( p^*_i > 0 \). Intuitively, higher indebtedness makes default happen at higher prices (i.e. more easily) because the temptation to walk away is higher. This temptation is countered by the posited collateral \( h_1 \). Intuitively, the more collateral there is, the more the household has to lose from giving it up. Interestingly, for sufficiently low levels of debt, the household will never default (the threshold price falls to zero). This is because there is always a cost of default, reflected by \( (1 - \hat{\kappa}) w_{2,i} > 0 \). The cost of default is increasing in income \( w_{2,i} \), and decreasing in \( \hat{\kappa} \) (i.e. default happens less easily when income is high and \( \hat{\kappa} \) is low). Recall that a lower value of \( \kappa \) makes renting more expensive relative to owning, thus increasing the indirect cost of default. This effect is reflected in a lower threshold price. (Conversely, a higher value of \( \kappa \) decreases indirect cost of default, raising the threshold price.) The indirect cost of default also depends on income, because with higher income, households would like more housing. This makes it more painful to endure the higher unit cost of housing upon default that \( \kappa \) implies.

Since the threshold price depends on income, it follows that whenever income is stochastic, the threshold price is itself a stochastic variable. Proposition 2 suggests that, given a mortgage contract, default is more likely (occuring at higher house prices) when a household is more indebted and has less collateral, when the difference between renting and ownership is small (\( \kappa \) is high), and when the household’s income is low. Equilibrium default, however, depends on how households adjust to the possibility of default, through their choice of mortgage contract. Evidently, there are two distinct chan-

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16Recall that I assume households repay when they are in fact indifferent.

17It is worth pointing out that \( \rho = 0 \) is not a borderline case (see the proof of Proposition 1). The appendix contains an example illustrating what can go wrong when consumption and housing are close substitutes, using the case with perfect substitutability (\( \rho = 1 \)).

18Importantly, this is not about whether the household is able to repay the debt or not. To see this, suppose \( D > w_{2,i} \), so that the household will be unable to repay when the price falls below \( \bar{p}_{\text{min}} = \frac{D - w_{2,i}}{h_1} > 0 \). But from (10) we see that \( p^*_i > \bar{p}_{\text{min}} \). In words, households choose default before they are obliged to.
nels of default: One pertains to characteristics of the household in question and the mortgage contract he holds, i.e. \( w_2 \) and \((D,h_1)\). The second is the market price \( p_2 \).

### 3.2 Optimal mortgage contracts

Optimal mortgage contracts satisfy the following conditions:

**Proposition 3 (Optimal mortgage contracts)** Suppose \( \rho \leq 0 \), and consider any potential equilibrium outcome. Then a solution \((d,D,h_1,c_1)\) to the period one problem (8) satisfies

\[
\begin{align*}
    & u_c(c_1,h_1) \geq \int_{p_1^s(\sigma,D,h_1)}^{\infty} \frac{1}{F(p_2|\sigma)G(\sigma)} \int_{p_1^s(\sigma,D,h_1)}^{\infty} u_c(c_2(p_2),h_2(p_2))dF(p_2 | \sigma)dG(\sigma) \\
    & = E[u_c(c_2(p_2),h_2(p_2)) | p_2 \geq p_1^s]
\end{align*}
\]

\[
\begin{align*}
    p_1 & \geq \frac{u_h(c_1,h_1)}{u_c(c_1,h_1)} + \beta \left( \int p_1^s(\sigma,D,h_1) p_2^s(\sigma,D,h_1) u_c(c_2(p_2),h_2(p_2))dF(p_2 | \sigma)dG(\sigma) \\
    & \quad + \int 0 p_1^s(\sigma,D,h_1) p_2 dF(p_2 | \sigma)dG(\sigma) \right)
\end{align*}
\]

where \( p_1^s(\sigma,D,h_1) \) is the threshold price defined in Proposition 2.

**Proof.** In the appendix. ■

Condition (11) is the usual Euler equation, describing how households seek to smooth marginal utility of consumption over time. Because the terms \((D,h_1)\) of the mortgage contract are not state contingent per se, households smooth marginal utility in expected terms. Indeed, imposing \( p_1^s = 0 \), so that the household never defaults, we recover the result of the standard incomplete markets model. By construction, the terms \((D,h_1)\) cannot impact the household’s consumption in states where he defaults\(^\text{19}\). Therefore, the Euler equation considers marginal utility only across states where households do not default, i.e. when \( p_2 \geq p_1^s \). This is not to say that the mortgage contract has no state contingency at all. On the contrary, whenever default occurs, it is because it is optimal. In other words, given the contract \((D,h_1)\), utility is

\(^{19}\)One way to impact consumption in states of default could be to save outside the mortgage contract. To meet this goal, the savings would have to be protected from creditors upon default. In the US, when mortgages are not legally non recourse, investing in property exempt from bankruptcy could be a way to achieve it. My model abstracts from these issues, and can thus be interpreted as the case without exemptions.
higher than it otherwise would have been. The Euler equation only indirectly sheds light on this contingency, however, through the determination of $p^*_i$.

Condition (12) is an Euler equation for housing. To interpret it, impose $p^*_i = 0$, so that the household never defaults. This yields

$$p_1 \geq \frac{u_h(c_1, h_1)}{u_c(c_1, h_1)} + \beta E \left[ p_2 \frac{u_c(c_2(p_2), h_2(p_2))}{u_c(c_1, h_1)} \right].$$

The first part reflects the consumption value of housing, whereas the second reflects the uncertain investment value of housing. The latter part includes a stochastic discount factor $u_c(c_2(p_2), h_2(p_2))/u_c(c_1, h_1)$ reflecting the uncertain marginal value of funds in period two. This is also reflected in condition (12) for the cases when the household keeps the house (i.e. when he does not default). But in the cases with default, it is the bank that recouperates the value of the house. Since they are risk neutral, only the expected price matters in these cases.

4 Limit-symmetric equilibrium

The purpose of this paper is to investigate how mortgage default can occur in equilibrium, and how it can spread from some households to others. A key finding is that for given mortgage contracts solving the period one problem (8), there may be multiple period two equilibria. This multiplicity arises because the default choice of some households can have feedback effects on the default choices of other households. To enlighten this mechanism, much would be to gain from solving the model analytically. General analytical solutions to (11)-(12) are not available, but for particular cases they may be possible to find\textsuperscript{20}. Consider the following special, but interesting, case for the income shock $\sigma$ to sub households:

$$\sigma = \begin{cases} \bar{\sigma} \in (0, 1) & \text{with prob. } \nu \\ 0 & \text{with prob. } 1 - \nu \end{cases}. \quad (13)$$

Note that with this specification, in the limit as $\nu \to 0$, both types of households have the same expected income profile, as $\lim_{\nu \to 0} E[w_{2,sub}] = E[w_{2,pri}] = Aw$. Together with Cobb Douglas (logarithmic) preferences ($\rho = 0$), this limit case permits an analytical solution\textsuperscript{21}. Because the model is continuous at this point, the limit case is a reasonable approximation to cases

\textsuperscript{20} The key issue is the covariance between the price and the marginal utility of consumption in period two, when households do not default.

\textsuperscript{21} The essential benefit of assuming $\nu \to 0$ is that it yields (in the limit) $h_{1,sub} = h_{1,pri}$. Together with the analytical demand functions obtained with separable logarithmic preferences
when $\nu$ is strictly positive, but small\textsuperscript{22}. I set $A = 1$, but this is not essential (it gives first best allocations when $\sigma = 0$), and I set $\alpha = \frac{1}{2}$.

**Proposition 4 (First-best limit-symmetric equilibrium)** Suppose $\rho = 0$, $A = 1$, and suppose $\sigma$ is distributed according to (13). Let $(d_i, D_i, (c_{t,i}, h_{t,i})_{t=1,2})_{i \in \{\text{sub}, \text{pri}\}}$ denote limits as $\nu \to 0$. Then

1. A solution to problem (8) exists, is unique, is identical across household type, and corresponds to an interior pair 
   \[(D_i, h_{1,i}) = \left( \frac{w}{2}, H \right), \text{ for } i \in \{\text{sub, pri}\}.\]
   
   The period one consumption decision is
   \[c_{1,i} = \frac{w}{2}, \text{ for } i \in \{\text{sub, pri}\},\]
   and the borrowed amount $d_{\text{sub}} = d_{\text{pri}} = \beta \frac{w}{2}$.

2. Given that no income shortfall occurs to sub households ($\sigma = 0$), there is a unique period two equilibrium characterised by the unique solution to problem (5)-(7). The equilibrium outcomes are identical across different types of households, and implement the "first best" outcome to the period one problem, i.e. for $i \in \{\text{sub, pri}\}$
   - No default: $\delta_i = 1$
   - Smoothing: $h_{2,i} = h_{1,i}$
   $c_{2,i} = c_{1,i}$.

3. Given that an income shortfall occurs to sub households ($\sigma = \bar{\sigma} > 0$), there are one, two or three period two equilibria, depending on the parameters $(\kappa, \gamma, \bar{\sigma})$ of the problem. The equilibrium candidates are:

   (a) Solvent equilibrium (SS): Neither sub nor pri households default ($\delta_{\text{sub}} = \delta_{\text{pri}} = 1$) \(\iff p_{SS} \geq p^*_{\text{sub}} > p^*_{\text{pri}}\). This equilibrium exists if and only if
   \[\frac{\gamma \frac{\bar{\sigma}}{1 - \bar{\sigma}}}{\bar{\sigma}} \leq 1 - \hat{\kappa}, \quad \hat{\kappa} = \sqrt{\kappa}.
   
   preferences, this property admits simple expressions for the possible equilibrium prices. Comparing these to the threshold price in (10) admits analytical, manageable existence conditions for different equilibria.

\textsuperscript{22}Numerical experiments indicate that the goodness of the approximation is a quantitative matter, depending on the parameters of the problem.
(b) Partial default equilibrium (DS): sub households default, but pri households do not (δ_{sub} = 0 and δ_{pri} = 1) ⇐⇒ p_{sub}^* > p_{DS} ≥ p_{pri}^*.
This equilibrium exists if and only if
\[ \gamma (1 - \bar{\sigma}) \kappa - (1 + \gamma) (1 - \bar{\sigma}) \bar{k} + 1 - (1 + \gamma) \bar{\sigma} < 0 \]
and
\[ \gamma (1 - \bar{\sigma}) \kappa - (1 + \gamma) \bar{k} + 1 \geq 0. \]

(c) Default equilibrium (DD): Both sub and pri households default (δ_{sub} = δ_{pri} = 0) ⇐⇒ p_{sub}^* > p_{pri}^* > p_{DD}. This equilibrium exists if and only if
\[ (1 - \gamma \bar{\sigma}) \kappa - 2 \bar{k} + 1 < 0. \]

Proof. Under revision! (The proof of an earlier version of Proposition 4 is available.) □

4.1 Unexpected income shortfall
When there is no income shortfall (σ = 0), the equilibrium solution is first best: There is no default, hence no inefficiency, and both the consumption of housing service and other goods are perfectly smooth. The more interesting case to study is when an unexpected income shortfall occurs (σ = \bar{\sigma} > 0). From the derived threshold price (10), we see that \( w_{2,sub} < w_{2,pri} \Rightarrow p_{sub}^* > p_{pri}^* \), so there are only three possible equilibrium outcomes, as stated in Proposition 4, part 3: Either no household defaults, or only sub households default, or all households default. Figure 1 plots the existence conditions of these three alternatives for the case when \( \bar{\sigma} = \frac{1}{2} \).
Period two equilibrium existence conditions (illustr. for $\alpha = \frac{1}{2}$, $\bar{\sigma} = \frac{1}{2}$)

Along the horizontal axis is the share of sub households in the economy. Along the vertical axis is the square root of $\kappa$ (note that $\hat{\kappa} = \sqrt{\kappa}$ when $\alpha = \frac{1}{2}$). Recall that the larger $\kappa$ is, the smaller is the gap between the rental price of housing and the price of housing stock, thus the smaller is the indirect punishment of default. As $\kappa \in (0, 1) \Rightarrow \sqrt{\kappa} \in (0, 1)$, so the square root of $\kappa$ is an equivalent measure of the (inverse of the) cost of default.

The solvent (SS) equilibrium exists below the dotted line (separating the figure in two equal parts) which plots the condition $p_{SS} \geq p_{sub}^*$. For a given cost of default, this equilibrium exists if and only if the share of sub households is not too large. The reason is that a larger share of sub households imply a larger aggregate income shortfall in the economy. Because of the fixed supply of housing stock, a larger aggregate income shortfall translates into a lower price of housing stock. For moderate price decreases, sub households prefer to repay debt and change to a smaller house. But as the price falls sufficiently low, sub households find it best to default. The share of sub households that will trigger default depends on the size of the income short-
fall \( \tilde{\sigma} \). Clearly, the default decision critically depends on the cost of default. In particular, for \( \sqrt{\kappa} \) in the high range, even a small share of sub households suffices to bring about default. In other words, when the cost of default is low, mortgages to sub households are very sensitive to house prices and may easily end in default.

The two dashed lines plot the existence conditions for the partial default (DS) equilibrium. In this equilibrium, sub households default, but pri households do not. For this to be the case, the price of housing stock must lie between the two threshold prices, i.e. \( p^*_\text{sub} > p_{\text{DS}} \geq p^*_\text{pri} \). Consequently, there are two existence conditions. The lower of the two plots the condition \( p_{\text{DS}} < p^*_\text{sub} \). For the same reasons as described above, if the cost of default is high or the aggregate income shortfall in the economy small, the market clearing price of housing stock would in any case stay above \( p^*_\text{sub} \), hence there can be no default. Similarly, the upper line plots the condition \( p_{\text{DS}} \geq p^*_\text{pri} \).

Figure 1 illustrates the indeterminacy of equilibria. For the two cases just discussed, we see that there is an area where both \( p_{SS} \geq p^*_\text{sub} \) and \( p_{DS} < p^*_\text{sub} \). The reason for this indeterminacy is that the act of defaulting moves the market price. Consider parameter conditions such that both conditions are satisfied. Suppose sub households believe no sub households will default. Then the market clearing price would be \( p_{SS} \) and their beliefs confirmed. But suppose to the contrary that all sub households do default. Then, because the unit price of rental housing is higher than the unit price of housing stock, sub households reduce their demand for housing service. Consequently, for a fixed aggregate stock of housing, prices must be lower in order to clear the market. This is a price externality: The individual household does not take into account the impact of his choice to default or not, through market prices, onto other households’ default decision.

Finally, the default (DD) equilibrium exists above the solid line, which plots the condition \( p_{DD} < p^*_\text{pri} \). It means that in these cases, if all households defaulted, the market price of housing stock would be sufficiently low to justify the default decision. Interestingly, it seems that even for small shares of sub households, this equilibrium may exist as long as the cost of default is relatively low.

These results suggest the following definition, and characterisation, of contagion.

**Definition 5** Default of sub households is "contagious" if it brings pri households into default even though they experience no income shortfall.

**Corollary 6** Default of sub households may be contagious if \( (1 - \gamma \sigma) \kappa - 2\tilde{\kappa} + 1 < 0 \) (DD equilibrium existence condition). Moreover, default of sub
households is always contagious if \( \gamma \frac{\sigma}{1-\sigma} > 1 - \hat{\kappa} \) and \( \gamma (1 - \sigma) \kappa - (1 + \gamma) \hat{\kappa} + 1 < 0 \) (neither the SS equilibrium nor the DS equilibrium may exist).

5 Conclusions

(Discussion of equilibrium selection mechanisms: sunspots, global games)

References


6 Appendix

Proof of proposition 1 (Threshold price). Consider the period two problem (5)-(7). For any given finite mortgage contract \((D, h_1)\) it is clear that for a sufficiently high price, \(p_2 h_1 > D\) must hold, so repaying debt implies strictly more resources than defaulting. Moreover, arbitrage free prices imply that defaulter must pay a strictly higher price \(r_2 = p_2/\kappa > p_2\) in order to obtain housing service. Hence, for sufficiently high prices, it must be that \(v^S > v^D\). Now observe that since \(u(c, h)\) is continuous and differentiable, both \(v^S(p_2)\) and \(v^D(r_2)\) are continuous and differentiable. With arbitrage free prices, we can write \(v^D(r_2) = v^D(r_2(p_2))\) with \(r_2 = \frac{p_2}{\kappa}\). So if we can show that, for any potential arbitrage free pair of prices \((p^*, r^* = p^*/\kappa)\) : \(v^S(p^*) = v^D(r_2(p^*))\) it is the case that

\[
\frac{dv^S(p^*, m^S_i(p^*))}{dp_2} > \frac{dv^D(r_2(p^*), m^D_i)}{dp_2},
\]

then there can be at most one such crossing point \(p^*\), proving the first part of the proposition. Suppose such a point exists. (If no crossing point exists, then the threshold price is trivially zero, i.e. households never default.) A minimum requirement for existence is that \(p^* > \max\{(D - w_{2,i})/h_1, 0\} \Rightarrow m^S_i(p^*) > 0\), so that problem (5) is well defined. Differentiating \(v^S(p^*)\) and \(v^D(r_2(p^*))\), and using Roy’s identity, gives

\[
\frac{dv^S(p^*, m^S_i(p^*))}{dp_2} = \frac{\partial v^S(p^*, m^S_i)}{\partial p_2} + \frac{\partial v^S(p^*, m^S_i(p^*))}{\partial m^S_i} \frac{dm^S_i(p^*)}{dp_2} = -\{h^S(p^*, m^S_i(p^*)) - h_1\} \frac{\partial v^S(p^*, m^S_i(p^*))}{\partial m^S_i}
\]

\[
\frac{dv^D(r_2(p^*), m^D_i)}{dp_2} = \frac{1}{\kappa} \frac{\partial v^D(r_2(p^*), m^D_i)}{\partial r_2} = -\frac{1}{\kappa} h^D(r_2(p^*), m^D_i) \frac{\partial v^D(r_2(p^*), m^D_i)}{\partial m^D_i} < 0,
\]

where \(h^S\) and \(h^D\) denote the demand for housing in the cases when the household repays debt and defaults, respectively. Since \(u(c, h)\) is homothetic, and \(v^S(p^*, m^S_i(p^*))\) equals \(v^D(r_2(p^*), m^D_i)\) by construction, we have

\[
\frac{\partial v_S(p^*, m^S_i(p^*))}{\partial m^S_i(p^*)} = \frac{\partial v^D(r_2(p^*), m^D_i)}{\partial m^D_i} = \frac{m^S_i(p^*)}{m^D_i} \frac{\partial v^S(p^*, m^S_i(p^*))}{\partial m^S_i(p^*)}.
\]

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Combining these two results shows that (14) holds if and only if

\[ h^S(p^*, m_i^S(p^*)) - h_1 < \frac{1}{\kappa} h^D(r_2(p^*), m_i^D) \frac{m_i^S(p^*)}{m_i^D}. \]

Multiplying both sides by \( p^*/m_i^S(p^*) \) reveals the expenditure shares in the cases when the household repays debt and defaults, respectively.

\[ \frac{p^* h^S(p^*, m_i^S(p^*))}{m_i^S(p^*)} - \frac{p^* h_1}{m_i^S(p^*)} < \frac{r_2(p^*) h^D(r_2(p^*), m_i^D)}{m_i^D}. \]

When the elasticity of substitution is smaller than unity (i.e. when \( \rho \leq 0 \)), the household spends a larger share on housing when housing is relatively more expensive, which is the case for renters since \( r_2(p^*) > p^* \). The presence of \( p^* h_1/m_i^S(p^*) \), which is strictly positive, ensures that the inequality holds strictly even when \( \rho = 0 \). (Moreover, it shows that \( \rho = 0 \) is not a borderline case.) Finally, note that since \( u(c, h) \) is continuous, differentiable and strictly increasing, \( v^S(p_2, m_i^S(\sigma, D, h_1)) \) is continuous, differentiable and strictly increasing in \( m_i^S \), hence in \(-D\) and \( h_1 \) for all \((\sigma, D, h_1)\). Continuity and differentiability of both \( v^S \) and \( v^D \) with respect to \( p_2 \) means that

\[ v^S(p^*, m_i^S(\sigma, D, h_1)) = v^D(r_2(p^*), m_i^D) \]

implicitly defines \( p_i^* = p_i^*(\sigma, D, h_1) \) as a continuous and differentiable function in \((D, h_1)\), whenever it exists. Since \( dv^S/dp_2 > dv^D/dp_2 \) at any such point, we have

\[ \frac{\partial p_i^*}{\partial D} > 0 \quad \text{and} \quad \frac{\partial p_i^*}{\partial h_1} < 0. \]

\[ \text{Example 7 (Perfect substitutes, no threshold price)} \]

Suppose consumption and housing are perfect substitute (i.e. \( \rho = 1 \)). Households then maximise utility by devoting the entire budget to the cheapest of the two, adjusting for the relative weight \( \alpha \). For concreteness, let \( \alpha \geq \frac{1}{2} \). The repayer’s problem can be represented by

\[ v^S(p_2, w_2, D, h_1) = \max \left\{ \alpha c^S, (1 - \alpha) h^S \right\} \]

where \( c^S \) and \( h^S \) denote the allocation if the entire budget is spent on consumption goods or housing, respectively

\[ c^S = w_2 + p_2 h_1 - D \]

\[ h^S = h_1 + \frac{w_2 - D}{p_2}, \]
and the defaulter’s problem can be represented by

$$v^D(p_2, w_2) = \max \left\{ \alpha c^D, (1 - \alpha) h^D \right\}$$

where, given $r_2 = p_2/\kappa$,

$$c_D = \frac{w_2}{\kappa w_2}, \quad h_D = \frac{p_2}{\kappa w_2}.$$ 

Consequently, the default decision is

$$\delta^* = \arg \max_{\delta \in \{0, 1\}} \left\{ \delta v^S + (1 - \delta) v^D \right\}.$$ 

Consider any finite mortgage contract $(D, h_1)$ satisfying the following conditions:

$$D - h_1 > 0 \quad \text{and} \quad w_2 (1 - \kappa) - D > 0.$$ 

I start by verifying that, as always, repaying debt is optimal when the price is sufficiently high. This holds because

$$\lim_{p_2 \to \infty} v^S(p_2) = \max \left\{ \lim_{p_2 \to \infty} \alpha c^S(p_2), \lim_{p_2 \to \infty} (1 - \alpha) h^S(p_2) \right\} = \max \{\infty, (1 - \alpha) h_1\} = \infty$$

$$\lim_{p_2 \to \infty} v^D(p_2) = \max \left\{ \alpha c^D, \lim_{p_2 \to \infty} (1 - \alpha) h^D(p_2) \right\} = \max \{\alpha w, 0\} = \alpha w.$$ 

Next, I consider the intermediate price $p_2 = 1$. For a household that repays debt, this means consumption and housing have the same price, but with $\alpha \geq \frac{1}{2}$, he prefers consumption:

$$v^S(1) = \max \{\alpha c^S(1), (1 - \alpha) h^S(1)\} = \max \{\alpha (w_2 + h_1 - D), (1 - \alpha) (h_1 + w_2 - D)\} = \alpha w - \alpha (D - h_1).$$

However, since $D - h_1 > 0$, the household would obtain even more consumption by defaulting:

$$v^D(1) = \max \{\alpha c^D, (1 - \alpha) h^D(1)\} = \max \{\alpha w_2, (1 - \alpha) \kappa w_2\} = \alpha w.$$ 

What I have described so far reflects the same mechanism that prevails in proposition 1. Roughly speaking, it is the ”income” effect that dominates: As
house prices fall, may come a point where debt outweighs the value of the house inducing households to default in order to increasing their resources, incidentally substituting goods consumption for housing because \( r_2 = p_2/\kappa > p_2 \). As the house price falls further, however, the household gradually revert back into housing. Take the defaulter in the above example. He would prefer housing to consumption when the price falls to imply

\[
(1 - \alpha) h^D (p_2) = (1 - \alpha) \frac{\kappa w_2}{p_2} \geq \alpha w_2 = \alpha c^D,
\]

which holds whenever

\[
r_2 = \frac{p_2}{\kappa} \leq \frac{1 - \alpha}{\alpha}.
\]

What can go wrong when \( h \) and \( c \) are close substitutes is that as the price becomes sufficiently low, households eventually regret their default, because repaying debt provides housing at an even lower price than defaulting. For this to hold, the price difference between renting and owning must be sufficiently large (i.e. \( \kappa \) must be sufficiently small) for this effect to more than compensate the capital loss households endure when they repay debt. To see this, let the price tend to zero in the above example. This gives infinite utility both to the repayer and the defaulter

\[
\lim_{p_2 \to 0} v^S (p_2) = \max \left\{ \lim_{p_2 \to 0} \alpha c^S (p_2), \lim_{p_2 \to 0} (1 - \alpha) h^S (p_2) \right\} = \infty
\]

\[
\lim_{p_2 \to 0} v^D (p_2) = \max \left\{ \lim_{p_2 \to 0} \alpha c^D (p_2), \lim_{p_2 \to 0} (1 - \alpha) h^D (p_2) \right\} = \max \left\{ \alpha w_2, \lim_{p_2 \to 0} \frac{\kappa w_2}{p_2} \right\} = \infty,
\]

but repaying debt is in fact the optimal choice when \( w_2 (1 - \kappa) - D > 0 \), because

\[
\lim_{p_2 \to 0} h^S (p_2) = \lim_{p_2 \to 0} \frac{h_1 + \frac{w_2 - D}{p_2}}{\frac{\kappa w_2}{p_2}} = \frac{w_2 - D}{\kappa w_2} > 1.
\]

Note that the conditions in this example is consistent with the first best limit symmetric equilibrium allocations considered in proposition 3 if, for instance, \( w = 4, H = 1 \) and \( \kappa \in \left(0, \frac{1}{2}\right)\).

**Proof of Proposition 3 (Optimal mortgage contracts).** As \( u(c,h) \) is strictly increasing, households always fully exhaust their available resources
in equilibrium. Consequently, the budget constraint must hold with equality, and equilibrium mortgage contracts must satisfy eq. (9) with equality. This establishes \( d \) as a function of \((D, h_1)\), which can be used to substitute in the budget constraint. I define the Lagrangian function \( L \) as

\[
L = u(h_1, c_1) + \beta \left\{ \int_0 \int \int_{\mathcal{D}} v^D \left( \frac{p_2}{\kappa}, m^D_i(\sigma) \right) dF \left( p_2 \mid \sigma \right) dG(\sigma) \right\} + \int_0 \int \int_{\mathcal{D}} v^S \left( p_2, m^S_i(\sigma, D, h_1) \right) dF \left( p_2 \mid \sigma \right) dG(\sigma)
\]

\[
-\lambda \left[ c_1 + p_1 h_1 - \beta \left\{ \int_0 \int \int_{\mathcal{D}} p_2 h_1 dF \left( p_2 \mid \sigma \right) dG(\sigma) \right\} + \int_0 \int \int_{\mathcal{D}} D dF \left( p_2 \mid \sigma \right) dG(\sigma) \right]
\]

using the threshold price property established in Proposition 2. Note that in equilibrium, projected default behaviour and actual default decisions are consistent, i.e. \( \delta_i = \Delta_i \), so the threshold price applies to the equilibrium contract as well as to the indirect period two utility function. Note that as equilibrium prices must be arbitrage free, I can substitute for \( r_2 \) using \( p_2/\kappa \) throughout the problem, and write \( F \left( p_2 \mid \sigma \right) \) instead of \( F \left( p_2, r_2 \mid \sigma \right) \) without loss of generality. The standard procedure consists of differentiating \( L \) to derive first order conditions for optimality. These are valid necessary optimality conditions for interior solutions to the period one problem if the problem is differentiable around the solution. When \( F \left( p_2 \mid \sigma \right) \) is a continuous distribution (so that any particular value of \( p_2 \) has zero probability mass), this holds and yields the following conditions:

\[
u_c(h_1, c_1) = \int_{p^*_i(\sigma, D, h_1)}^\infty \frac{1}{dF(p_2 | \sigma) dG(\sigma)} \int_{p^*_i(\sigma, D, h_1)}^\infty u_c(c_2(p_2), h_2(p_2)) dF(p_2 | \sigma) dG(\sigma) + \Phi_i \]

\[
p_1 = \frac{u_h(h_1, c_1)}{u_c(h_1, c_1)} + \beta \left( \int_{p^*_i(\sigma, D, h_1)}^\infty \frac{p_2 u_c(c_2(p_2), h_2(p_2))}{u_c(h_1, c_1)} dF(p_2 | \sigma) dG(\sigma) \right) + \Phi_i,
\]

where

\[
\Psi_i = \Psi \left( p^*_i(\sigma, D, h_1), F \left( p_2 \mid \sigma \right), G(\sigma) \right)
\]

\[
= \int_{p^*_i(\sigma, D, h_1)}^\infty \frac{u_c(h_1, c_1)}{dF(p_2 | \sigma) dG(\sigma)} \int [D - p^*_i(\sigma, D, h_1) h_1] \frac{\partial p^*_i(\sigma, D, h_1)}{\partial D} F' \left( p^*_i(\sigma, D, h_1) \mid \sigma \right) dG(\sigma) \geq 0
\]
and
\[ \Phi_1 = \Phi_1(p^*_1(\sigma, D, h_1), F_p(2 | \sigma), G(\sigma)) \]
\[ = -\beta \int [D - p^*_1(\sigma, D, h_1) h_1] \frac{\partial p^*_1(\sigma, D, h_1)}{\partial h_1} F'(p^*_1(\sigma, D, h_1) | \sigma) dG(\sigma) \]
\[ \geq 0 \]
are non negative functions, strictly positive if and only if \( p_2 = p^*_1(\sigma, D, h_1) \)
can be an equilibrium outcome for possible realisations of \( \sigma \), given the optimal contract \((D, h_1)\), i.e. when the density \( F'(p^*_1(\sigma, D, h_1) | \sigma) \) is strictly positive. (Note that \( D > p^*_1(\sigma, D, h_1) h_1 \) holds for any \( p^*_1 \).) To derive these conditions, first condition on \( \sigma \) and apply Leibniz rule, noting that whether the integration limits are open or closed is irrelevant when no single values have positive probability mass, and using that by construction, \( v^S(p^*_1) = v^D(p^*_1) \).

Next, apply the envelope theorem to substitute \( \partial \) derivatives of the value function \( \partial \) and non degenerate when there is multiplicity.) In this case, the terms of the type
\[ \int_{0}^{p^*_1(\sigma, D, h_1)} p_2 h_1 dF(p_2 | \sigma) + \int_{p^*_1(\sigma, D, h_1)}^{\infty} DdF(p_2 | \sigma) \]
that appear in the last part in curly brackets in the Lagrangian will not be differentiable at any point where \( p^*_1(\sigma, D, h_1) \) exactly equals one of the values in \( F(p_2 | \sigma) \). To see this, consider a discrete distribution of \( p_2 \) conditional on \( \sigma \) where
\[ P(\sigma) = \left\{ p^1(\sigma), p^2(\sigma), ..., p^i(\sigma), ..., p(\sigma)^{N(\sigma)} \right\} \]
are the possible values of \( p_2 \) (with \( N(\sigma) \) finite). Note that both the values and the number of possible outcomes may be functions of \( \sigma \). The conditional probability of a particular price \( p^i \in P(\sigma) \) is denoted \( \pi(p^i | \sigma) > 0 \), with \( \sum_{i=1}^{N(\sigma)} \pi(p^i | \sigma) = 1 \). (All other prices \( \notin P(\sigma) \) have conditional probability zero.) Suppose we are evaluating a point \((D, h_1)\) such that \( p^*_1(\sigma, D, h_1) = p^n \in P(\sigma) \) for some \( \sigma \). The discrete version of the integral () is then, for \( p^i \in P(\sigma) \),
\[ \sum_{i=1}^{n-1} \pi(p^i | \sigma) p^i h_1 + \pi(p^n | \sigma) D + \sum_{i=n+1}^{N(\sigma)} \pi(p^i | \sigma) D. \]
Consider increasing the value of \( D \) to \( D + \varepsilon \), where \( \varepsilon > 0 \) is very small number. Since \( p^*_1(\sigma, D, h_1) \) is a differentiable and strictly increasing function
of $D$, the new value $p^*_i (\sigma, D + \varepsilon, h_1) > p^n$, and therefore $\notin P$. The value of the integral at $D + \varepsilon$ is therefore

$$\sum_{i=1}^{n-1} \pi (p^i | \sigma) p^i h_1 + \pi (p^n | \sigma) p^n h_1 + \sum_{i=n+1}^{N(\sigma)} \pi (p^i | \sigma) (D + \varepsilon),$$

implying that the absolute difference between the two as $\varepsilon$ tends to zero is finite

$$\lim_{\varepsilon \to 0} \left( \varepsilon \sum_{i=n+1}^{N(\sigma)} \pi (p^i | \sigma) - \pi (p^n | \sigma) [D - p^n h_1] \right) = -\pi (p^n | \sigma) [D - p^n h_1].$$

The derivative does not exist at this point, because

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{i=n+1}^{N(\sigma)} \pi (p^i | \sigma) - \pi (p^n | \sigma) [D - p^n h_1] \varepsilon = -\infty.$$

Intuitively, at this point the value of the integral jumps because there is one more case in which households will default. That case occurs with probability $\pi (p^n | \sigma)$, and the difference in value in that case is $- [D - p^*_i (\sigma, D, h_1) h_1]$, which is strictly negative for any $p^*_i$. (Following the same steps for $h_1$ verifies that the effect of marginally decreasing $h_1$ is the same.) In these cases, the conditions in Proposition 2 will hold with strict inequalities. Note, however, that if $G (\sigma)$ is continuous and $\Pr (p_2 = p^*_i (\sigma, D, h_1) | \sigma) > 0$ for any finite number of values $\sigma$, then these cases constitute a zero probability event. Since the absolute change (fall) in $L$ is finite, the term can be disregarded, and the conditions will hold with equality.

Proof of Proposition 4 (First best limit symmetric equilibrium).
(Under revision)