Stock Market Participation, Portfolio Choice and Pensions over the Life-Cycle*

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Abstract

The empirical evidence on stock market participation and portfolio choice defies the predictions of standard life-cycle theory. In this paper we develop and estimate a model of portfolio choice that can account for the limited stock market participation and substantial portfolio diversification seen in the data. We present three realistic extensions to the basic framework: per period fixed costs, public pension provision, and a small chance of a stock market crash. The estimated model is able to explain observed patterns at reasonable wealth levels, while keeping to a fairly simple framework. We demonstrate that it is no longer necessary to assume counterfactual asset holdings, heterogeneity in preferences, or implausible parameter values, in order to match key financial statistics.

Keywords: precautionary saving, portfolio choice, stock market participation and uninsurable labour income risk

JEL Classification: G11, H31

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1 Introduction

This paper presents a life-cycle portfolio choice model, with realistically calibrated stochastic labour income and reasonable risk aversion, explaining important stylised facts of household asset allocations. Empirical studies consistently find that approximately fifty percent of US households do not invest in the stock market (whether directly or indirectly), and those that do participate hold only a fraction of their wealth in risky assets. There is substantial turnover in the set of participants, with widespread movements both in and out of the stock market. Despite recent developments in financial markets, leading to greater levels of participation and higher shares of risky assets in household portfolios, the empirical evidence still presents a significant challenge to the life-cycle model. This paper demonstrates that the addition of a per period fixed cost to stock market participation, public pension provision, and a small possibility of a stock market crash enables us to explain the observed limited participation and substantial portfolio diversification, while matching assets.

A wide literature presents models of intertemporal choice incorporating precautionary and retirement motives for saving; many of the empirical patterns of wealth accumulation and consumption have been accounted for within this framework (Hubbard, Skinner and Zeldes, 1995; Carroll, 1997; Attanasio et al., 1999; Gourinchas and Parker, 2002). More recently, a literature has emerged which allows for the simultaneous determination of consumption and portfolio allocation within a life-cycle framework. However, it has proved difficult to explain asset allocations without assuming unrealistic wealth accumulation, extreme parameter values or heterogeneity in preferences. Cocco, Gomes and Maenhout (2005) present a thorough analysis of the standard household portfolio choice model without any fixed cost considerations. They are able to force portfolio shares down to reasonable levels, in addition to obtaining significant age effects. However, this is achieved by accepting unrealistically high levels of saving.\footnote{A combination of using a very high coefficient of risk aversion ($\gamma = 10$) and assuming a small probability} Further, their model predicts
one hundred percent participation at all ages, which we do not see in the data.

It is argued that some form of information cost is required to move away from the prediction that all households should participate in the stock market at all times. This is corroborated by the empirical work of Paiella (2001) and Vissing-Jorgensen (2002), both of whom have shown that fixed costs to stock market participation can empirically rationalise the observed limited participation. These fixed costs can be one-off entry costs or per period costs, and the estimates of these costs are extremely low.\textsuperscript{2} There has been significant progress with incorporating entry costs into household portfolio choice models. Alan (2006) calibrates the level of this cost to match moments from the PSID, and with this she is able to match the participation rate very precisely. As a buffer stock saving model Alan’s framework does not match the wealth data, and the resulting low levels of saving means that her model cannot address the issue of portfolio diversification.\textsuperscript{3} Gomes and Michaelides (2005) set an exogenous entry cost, and attain reasonable age profiles of both participation and shares. However, they only achieve this by assuming preference heterogeneity, Epstein Zin utility functions, and very high levels of savings. While these entry costs are a convincing way to lower participation early in life, they cannot be the causal factor behind the large degree of turnover in the stock market, or the low levels of participation for older households. Despite the empirical evidence strongly in favour of per period fixed costs, until now, there has been little progress incorporating such considerations into a household portfolio choice model.

The striking result from research to date is that it appears as though the standard life-cycle model is restricted to explaining one key statistic at a time: matching wealth, or participation, or shares, but never more than one concurrently. The contribution of this paper is to illustrate how this inopportune prediction can be overcome with some relatively simple extensions to the standard framework. We introduce three distinct of a zero income event, which triggers the precautionary response too much.

\textsuperscript{2}Paiella (2001) finds a per period fixed costs of US$ 95-175, while Vissing-Jorgensen (2002) finds that costs of US$ 260 can explain the behaviour of 75% of nonparticipants.

\textsuperscript{3}Alan’s model predicts complete specialisation in stocks for all participants.
and novel features into a household model of portfolio choice and calibrate the model to match average moments from the US data. This is the first study to simultaneously match participation, shares and asset holdings with observables. In addition we obtain reasonable life-cycle profiles for each of these variables, despite our model not calibrating on any age characteristics. We achieve all this with a per period fixed cost of less than one per cent of the permanent component of labour income, a proportional tax rate of thirty percent and a small probability of a stock market crash.

We introduce our three innovations sequentially in order to separate out their effects. Firstly, we account for the time costs involved with determining and managing an optimal portfolio by incorporating per period fixed costs to stock market participation. We find that introducing such cost considerations into the standard household model has a remarkable influence on our simulated statistics, and in particular on the age profile of stock market participation. With this extension we can now account for low participation at both early and late stages of the life-cycle without resorting to unrealistic wealth levels. Secondly, we introduce a stylised pay-as-you-go public pension scheme, funded by a proportional tax on labour income. The retirement motive for saving is of great importance for wealth accumulation, and tends to be ignored or treated rather imprecisely in the literature on household portfolios.\textsuperscript{4} This approach opens up the possibility of examining the impact of changing demographics on optimal portfolio allocations, and offers a potential explanation for the low empirical estimates of fixed costs. Thirdly, we modify the standard assumption of lognormal risky returns to allow for a small possibility of a stock market crash. This simple extension enables the expected return to encompass large departures from the mean,\textsuperscript{5} which can be thought of as representing episodes such as the crash of 1987 or the collapse of Enron in 2001. By allowing for large negative movements we are able to match the household model to data with reasonable parameter values.

\textsuperscript{4}For example, Alan (2006) only considers asset allocations for households up to the age of sixty, completely disregarding participation and portfolio share decisions during retirement.

\textsuperscript{5}Notably absent when using lognormal returns, where all the action takes place in close proximity to the mean.
In the next section we provide some data on asset holdings, participation and portfolio shares by households in the US. Section 3 develops our household model of portfolio choice. Section 4 provides simulated life-cycle profiles of asset accumulation, stock market participation rates and risky asset shares; showing the effect of introducing per period fixed costs, varying the public pension provision, changing the level of risk aversion, and introducing the possibility of stock market crashes. Section 5 concludes.

2 Data on asset allocations

This section details some stylised facts on asset accumulation and allocation which are addressed by our life-cycle model. Asset accumulation profiles are well documented, where mean asset holdings are seen to rise sharply when households are in their 30s and 40s and then diminish gradually during retirement. Low (2005) shows that the mean asset holdings peak around retirement, at a magnitude of around seven times greater than mean income. Low calculates that the US median asset to income ratio across working life is 1.84, and it is to this statistic that we calibrate our simulated asset holdings.

When addressing household investment allocations it is important to distinguish between two distinct decisions: participation and portfolio shares. First, participation represents the binary choice of whether to participate in the stock market or not. Stock market participation in the US is currently just above fifty percent, Bertaut and Starr-McCluer (2002) report that 56.9% of American households hold risky assets. There is a clear age effect on participation and a hump shape in the decision of whether or not to hold risky assets has been much discussed (Alan, 2006; Ameriks and Zeldes, 2001; Poterba and Samwick, 1997; Gomes and Michaelides, 2005). The top panel in figure 1 uses data from

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6 Although this decline is not always present in median asset holdings, see Burnheim (1987).
7 He uses the 1995 PSID wealth supplement, defining wealth to include housing wealth and using PSID weights.
8 Using 1998 SCF weighted data. Risky assets are defined as including: directly held stock; stock held through mutual funds, retirement accounts, trusts and other managed assets; corporate, foreign and mortgage-backed bonds; business equity; and investment real estate.
Figure 1: Asset allocation profiles in the Survey of Consumer Finances. Graph (a) shows the share of households holding risky assets. Graph (b) shows the portfolio shares conditional on stock market participation. The results for both graphs are taken from Bertaut and Starr-McCluer (2002), using weighted data from 1998 SCF. Risky assets include directly held stock; stock held through mutual funds, retirement accounts, trusts and other managed assets; corporate, foreign and mortgage-backed bonds; business equity; and investment real estate.
the US Survey of Consumer Finances to demonstrate the hump shape in the age profile for stock market participation. In figure 1 we can clearly see limited participation at early and late stage of life, with a peak around ages forty-five. Participation is also increasing in wealth, with richer households much more likely to hold risky assets than poorer households. Using multivariate probit techniques Bertaut and Starr-McCluer (2002) find that both age and wealth effects are significant for the participation decision in the US.

The second important statistic is the portfolio share. This represents the share of risky assets to total assets, conditional on participating in the stock market. These portfolio shares are typically well below 100% (Ameriks and Zeldes, 2001; Bertaut and Starr-McCluer, 2002; Gomes and Michaelides, 2005; and others). Bertaut and Starr-McCluer (2002) find that the average portfolio share in the US is 54.4%. Gomes and Michaelides (2005) find similar results, with an average share of 54.8%. The bottom panel of figure 1 demonstrates that the age effect on portfolio shares is fairly weak. In reality, it is common practice for financial advisors to tell their clients to shift their portfolios away from risky assets as they age, especially as they enter retirement. This is evident in figure 1, where portfolio shares fall from just over 60% for households aged 55-64 to around 55% for the retired. However, Bertaut and Starr-McCluer (2002) find that these age effects are not robust to more sophisticated econometric analysis.

The model in this paper is calibrated to match these three average statistics (asset holdings, participation and shares) and aspires to explain some of the limited participation and strong portfolio diversification seen in the data.

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9 Using 1998 SCF weighted data. Defining risky assets as before.
10 So called “life-cycle” funds do this automatically for investors.
11 It is important to point out that cohort effects may be important in asset allocation decisions, our framework focuses on age effects and does not model differences between cohorts. See Alan and Ball (2007) for an example where cohort effects are analysed.
3 Life-cycle model

Individuals are assumed to maximise the sum of expected discounted lifetime utility in light of uncertain and uninsurable labour income and rate of return risk. Utility is defined over a single nondurable consumption good, and is additively separable across time. Let there be $T$ periods in the life-cycle, of which a given $K$ periods are spent in retirement and $T - K$ give the working life. This standard set-up gives the following constrained maximisation problem:

$$\max E_t \left[ \sum_{s=t}^{T} \beta^{s-t} u(C_s) \right]$$

subject to,

$$A_{s+1} = (A_s + (1 - h_t)Y^n_t - C_s - FP_s[partic.]) [\omega_s(1 + r^{eq}_s) + (1 - \omega_s)(1 + r)]$$

and a terminal condition, $A_{T+1} = 0$. In each period $\beta$ is the discount factor; $C_t$ is consumption; $A_t$ is the amount of assets held at the start of period $t$; $\omega_t$ is the portfolio share. $Y^n_t$ is exogenous real disposable labour income net of taxes. During working life, labour income is uncertain and non-diversifiable and is taxed at a proportional rate of $\tau$:

$$Y^n_{t+1} = (1 - \tau)Y_{t+1}$$

Following MaCurdy (1982), Abowd and Card (1989), and Mofftt and Gottschalk (2002), we shall assume that the labour income process can be broken down into a deterministic component and two stochastic components, a permanent and a transitory shock.\textsuperscript{13}

\textsuperscript{12}Following Gomes and Michaelides (2005) we account for the life-cycle pattern of housing expenditures, with an age varying fraction of income devoted to housing. This implicitly assumes that housing decisions are separable from consumption decisions and independent of wealth, while this is a strong assumption modelling explicit housing allocation decisions is beyond the scope of this paper.

\textsuperscript{13}Although this approach is commonly used in the literature, recent studies have pointed towards the weakness of empirical evidence for such restricted income profiles (see Guvenen, 2007).
\[ Y_{t+1} = G_{t+1} P_{t+1} v_{t+1} \]

where \( v_t \) is transitory income, \( P_t \) is permanent income, and \( G_t \) is a deterministic growth trend. The log of permanent income is assumed to follow a random walk.

\[ P_{t+1} = P_t \varepsilon_{t+1} \]

Both transitory and permanent shocks are assumed to be independently and identically distributed lognormally such that \( \ln(v_t) \sim N(-0.5\sigma_v^2, \sigma_v^2) \) and \( \ln(\varepsilon_t) \sim N(-0.5\sigma_{\varepsilon}^2, \sigma_{\varepsilon}^2) \). The income process is truncated such that a zero level of income cannot be realised.

For the final \( K \) periods of life individuals are retired, during which time they receive a constant and certain pension income, \( \Gamma \). We construct a pay-as-you-go pension scheme, somewhat akin to the arrangements in the US, with pension payments being funded out of contemporaneous tax revenues. Considering the tax collection requirement of public pension provision is new to the literature,\(^{14} \) and enables us to investigate the implications of changing demographics and public policy on optimal portfolio allocations. At any particular point in time, we assume that there exists an equal mass of individuals at each age; with working population share being \( \frac{T-K}{T} \), and the fraction of households in retirement given by \( \frac{K}{T} \). Tax revenue collected from individual \( i \) at age \( a \) is given by \( \tau Y_a^i \). Summing pension contributions over all individuals, \( H \), and all working ages, \( T-K \), and normalising by the number of retirees, \( H.K \), gives the following constant annual pension income for each retired household:

\[
\Gamma = \sum_{a=22}^{T-K} \frac{\tau Y_a}{H.K} 
\]

We assume the portfolio investment is comprised of two distinct saving tools, a riskless

\(^{14}\)Previous work either imposed exogenous retirement pension income, or ignored asset allocation decisions during retirement.
and a risky asset, with $\omega_t$ representing the share of assets held in risky forms. The riskless asset has a constant real return of $r$ and the risky asset has a stochastic real return of $r_{eq}^t$, which is assumed to be i.i.d. over time and distributed lognormally. We do not impose any correlation between stocks and labour income as the empirical evidence on the size and magnitude is mixed. Heaton and Lucas (2000) find insignificant estimates for all but the highest educational group, and Davis and Willen (2000), surprisingly, find negative correlations for low educational groups. Further, the effect of such a correlation has been shown to make very little difference to participation and portfolio shares in practice (Gomes and Michaelides, 2005), unless unrealistically high correlation coefficients are chosen.

In order to invest in risky assets, in any given period, we assume an individual has to pay some fixed cost. It is clear that gathering and processing financial information plays an important role in investment decisions: acquiring data on financial markets, monitoring brokers, and keeping up to date with trends all take considerable time. These financial decisions must be made at each and every point an individual invests in stocks, implying that any relevant information costs must be considered on a per period basis. In addition, econometric studies by both Paiella (2001) and Vissing-Jorgensen (2002) have asserted the empirical significance of fixed costs paid in each and every period of stock market participation. In the asset evolution equation above the per period fixed costs are given by $FP_t$; these are interacted with an indicator function, $\mathbb{I}\{\cdot\}$, which is equal to one for every period of participation and zero otherwise. We scale by the level of the permanent component of labour income in order to make $F$ represent a proportion of permanent income, following from the motivation of these costs representing opportunity costs of time.

Some limited borrowing is allowed, up to the discounted sum of the minimum income individuals will receive in each remaining period, with no borrowing permitted against pension income. A short sales constraint is imposed on equity such that the portfolio shares must always lie between zero and one (inclusive). Therefore, agents can only borrow
at the risk free rate and can only invest in risky assets if they hold positive balances.\textsuperscript{15}

This set-up gives us a system with three state variables \((A_t, Y_t, P_t)\) and two control variables \((C_t, \omega_t)\). The first order condition of the value function with respect to consumption is given by:

\[
\begin{equation}
\begin{gathered}
u'(C_t) = \beta E_t \left\{ [\omega_t (1 + r_{eq}^t) + (1 - \omega_t) (1 + r)] u'(C_{t+1}) \right\}
\end{gathered}
\end{equation}
\]

and the first order condition with respect to portfolio share is given by:

\[
\begin{equation}
\begin{gathered}
0 = E_t [(r - r_{eq}^t) u'(C_{t+1})]
\end{gathered}
\end{equation}
\]

These two conditions can be solved for the following policy functions in each period:

\[
\begin{equation}
\begin{gathered}
C_t(A_t, Y_t, P_t)
\end{gathered}
\end{equation}
\]

\[
\begin{equation}
\begin{gathered}
\omega_t(A_t, Y_t, P_t)
\end{gathered}
\end{equation}
\]

These two optimal decision rules are solved recursively from the final period for a discrete number of points in the state space. The details of the solution method are given in Appendix A.

\section{Simulation results}

Once the optimal policy functions have been determined, we can simulate the model to imitate the behaviour of a population of households and report average allocations to show the effect of introducing fixed costs and changing the public pension scheme. Initially, we detail the functional forms, parameter values and calibration statistics used. Subsequently,

\textsuperscript{15}While it is true that we do see households simultaneously borrowing and holding stocks in the data, this restriction is imposed on the model for simplification.
we show the results of simulating the model for 10,000 ex-ante identical households who differ in the realisation of shocks.

We then put forward the results in four subsections: first, we analyse the effect of introducing per period fixed costs to our baseline life-cycle model and show this modification changes the participation results substantially. Second, we illustrate the effects of altering the generosity of public pensions, and explain the link between pension generosity and the calibrated level of fixed costs. Third, we calibrate the coefficient of risk aversion and demonstrate how our model is able to match both participation and shares. Finally, we detail the effect of including a small possibility of a stock market crash, and argue that this enables us to explain all key financial statistics. This structure permits us to disentangle the different consequences of varying fixed costs, public insurance and return risk in order to better fit the data.

**Parameters and Calibration** The utility function is assumed to take the typical constant relative risk aversion (CRRA) form

\[ u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}. \]  

(10)

with a baseline value of 2 for the coefficient of risk aversion. This is in line with estimates based on consumption data: Attanasio and Weber (1995) estimate the coefficient of risk aversion to be around 1.5, Gourinchas and Parker (2002) estimate risk aversion to range from 0.28 to 2.29 and Alan and Browning (2003) find ranges from 1.2 to 1.95.

The riskless rate of return is set at a constant value of 1.6%, which represents the average annualised real rate of US 3-month treasury bills from 1960-2000. The risky return is drawn from a distribution with a mean of 5.6%, corresponding to a 4% equity premium,\(^{16}\) and a standard deviation of 0.2.\(^{17}\) Following Haliassos and Michaelides (2003)

\(^{16}\)As is common in this literature; see Gomes and Michaelides (2005), Haliassos and Michaelides (2002), Yao and Zhang (2005), Cocco (2000), Campbell et al. (2001)

\(^{17}\)See Alan (2006). Gomes and Michaelides (2005) and Haliassos and Michaelides (2002) use a similar value of 0.18
we set the deterministic trend of income, $G$, to 3%. Standard magnitudes for the variance of permanent and transitory shocks to income are used. We take the values directly from Gomes and Michaelides (2005), which are very similar to those used by Gourinchas and Parker (2002) and Carroll (1997). All exogenous parameters are shown in table 1.

The baseline tax rate of 30% gives a retirement replacement rate of 45% of final working period labour income. In reality the pension tax system in the US is more complicated than the set-up assumed here, but this proportionate rate is chosen as a reasonable proxy. The implied replacement ratio is in line with those used in the literature, for example, Low (2005) used a 55% replacement rate.

The values for the fixed cost, discount rate and (in the final three scenarios) risk aversion are chosen by calibration. The value of the fixed cost is chosen to match simulated mean participation with the observed mean in the data (0.57). We choose the discount rate such that simulated ratio of median assets to median incomes matches the PSID for the working households (1.84). In addition, in the final three scenarios, we choose the risk aversion parameter to match the mean portfolio share statistic of 0.54. Calibration parameters and statistics are shown in table 2.

\[ 18 \text{ Except for the scenarios where we explicitly fix the discount rate.} \]
At this point, it is important to point out that one of the main contributions of this paper is to emphasise the importance of matching asset holdings. In these household models both participation and portfolio shares are highly sensitive to the level of assets. Many of the studies claiming to explain certain features of asset allocations implicitly rely on counterfactual asset holdings for their results. To understand the importance of asset levels the solid line in figure 2 plots the portfolio share policy function for an individual of age 27\(^{19}\) under these baseline parameters.

Studies, such as Gomes and Michaelides (2005), are able to obtain reasonable portfolio shares by allowing asset holdings to be unrealistically high. This pushes households onto the non-linear sections of their portfolio share policy functions (positioned on the solid line towards the far right in figure 2), enabling movements away from complete specialisation in risky assets. In contrast, studies claiming to be able to match participation (Alan, 2006) achieve this by forcing assets down to levels where the borrowing constraint binds.

\(^{19}\)The choice of age 27 is completely arbitrary, analogous policy functions would be attained for any other age (see Appendix B).
At these low amounts of saving agents are operating on the horizontal section of their portfolio share policy functions (positioned on the solid line close to the y-axis in figure 2), hence these authors find very high shares, typically unity. In this paper we ensure a more genuine matching of asset allocations by explicitly calibrating our model to the actual level of assets observed in the US.

4.1 Introducing per period fixed costs

Standard model without fixed costs In our initial parameterisation we set the per period fixed costs to zero and calibrate the model to match the PSID median asset holding to income ratio (1.84). The dashed-line graphs in figure 3 show the cross-sectional mean profiles of normalised asset holdings, stock market participation and conditional portfolio shares.

Under the baseline parameters the discount rate necessary to match the level of assets is 0.5%, giving a discount factor \( \beta = 0.995 \). This high calibrated discount factor results from the high effective impatience introduced with our public pension scheme. The high tax, and subsequent high pension payments, increases the steepness of disposable income growth requiring only a modest \( \delta \) to keep savings at the required level. The dashed line in the top panel of figure 3, shows agents accumulating assets up until retirement, and then running their balances down during the last 15 years of non-working life, reaching levels that are in the same order as those seen in the data. Average assets are always positive, but some individuals do borrow, up to the endogenous borrowing limit, in the early stages of life.

The dashed line in figure 3(b) shows participation in the stock market is 100% for all but a few years in early and late life; only those individuals with negative savings do not participate. This is in conflict with the data, clearly illustrating the famous ‘portfolio participation puzzle’, discussed at length in the household finance literature. Moreover,

\[ 20 \text{This is asset holdings divided by the permanent component of annual labour income.} \]
in this model framework, without any transactions costs, there is no explicit participation decision; nonparticipants are simply those agents in debt, who are constrained not to hold risky assets by the bounds on $\omega_t$. Figure 3(c) details portfolio shares conditional on participation in the stock market. The mean portfolio share of stock market participants is 0.99 which is well above the true value of 0.54. These results exemplify the inability of the standard framework to explain household asset allocations at reasonable wealth levels: neither participation nor shares are close to being matched.

**Including per period fixed costs** Introducing these market frictions, and calibrating the model to match the asset to income ratio of 1.84, leads to a slightly lower discount rate.
Figure 3: The effect of introducing per period fixed costs. Graph (a) shows assets normalised by permanent income. Graph (b) shows stock market participation by age. Graph (c) shows conditional portfolio shares at each age. The discount rate is calibrated separately in each scenario such that median assets match the data. For the scenario with fixed costs (represented by the solid line in the graphs), the costs are calibrated to match mean participation.
rate (from 0.5% to 0.3%). These per period fixed costs increase the cost of saving, and so the discount rate must fall to keep the level of savings constant. The level of the per period fixed cost is calibrated to match the average participation rate of 0.57, giving a value of 4.6% of the permanent component of labour income. These results are detailed in the solid lines in figure 3. We can see that the increased cost of saving depresses asset accumulation early in life. This reduced precautionary saving is compensated by more retirement saving in mid-life, in order to keep the median level calibrated to 1.84.

The introduction of per period fixed costs has a marked effect on the stock market participation profile, with a clear hump shape now evident in figure 3(b). Few individuals accumulate sufficient wealth in the first ten years of working life to make it viable to pay the fixed costs of participation, and they hold only riskless assets in their portfolio. As individuals age they accumulate assets, risky investments become profitable and participation increases - peaking at age 60 with 87% participation. As assets are drawn down later in life, the number of individuals participating is seen to fall gradually back to very low levels. It is important to note that we are not calibrating the model to match the participation age-profile, but we obtain a clear hump shape in all simulations incorporating these per period fixed costs. These per period fixed costs ensure that stock market participants endogenously hold greater levels of savings, generating a more pronounced age profile in the conditional portfolio shares - as evident in figure 3(c). However, the model continues to predict complete specialisation in risky assets early in life with a mean portfolio share of 0.98, which is considerably higher than observed in the data.

4.2 Changing the generosity of the public pension scheme

In this section we analyse the effect of changing the public pension generosity in a model where fixed costs are calibrated to match the US average participation in the stock market.

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21 The results from keeping the discount rate the same and only calibrating the fixed cost are not significantly different.
22 This is marginally smaller than the mean share predicted by the model without fixed costs, as savings behaviour has only been altered modestly.
Varying the tax rate changes effective impatience through altering the income profile: lowering the tax leads to less pronounced income growth and lower effective impatience. This gives us the choice of either allowing this lower effective impatience to manifest in the form of higher savings, or increasing the discount rate in order to keep savings at the same level as prior to the tax change. Thus, we consider two different scenarios: first, we shall analyse the effect of varying the tax rate while keeping the level of asset holdings constant, and allowing the discount rate to change in order to achieve this. Second, we consider keeping the discount rate constant, and allow savings to vary.

**Keeping savings constant**  In this scenario we calibrate the model separately under each tax rate to match two empirical statistics: the discount rate is calibrated to match median asset holdings (1.84) and the per period fixed cost is calibrated to match mean participation of 0.57. The results of these simulations are shown in figure 4.

Reducing the tax rate from 30% to zero requires the discount rate to increase from 0.5% to 2.8% in order to keep savings constant. A discount rate of 2.8% is in line with similar calibration studies conducted without an explicit public pension scheme and is well within the range estimated by econometric studies, such as Alan and Browning (2003). Ensuring that asset holdings remain the same, leads to a redistribution of savings from earlier years in the working life to mid-age. This represents a shift from precautionary savings to retirement savings, which is clearly evident in the top panel of figure 4.

In order to explain the observed limited average participation, the (calibrated) per period fixed cost increases from 4.6% to 6.3% of annual labour income, as the tax rate is brought down from 30% to zero. The tax change has a modest effect on the participation profile, with lower participation during early and late life. From figure 4(c) we can see how lowering the level of public insurance leads to more diversified portfolios. Mean

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23 If we had kept the discount rate the same across the scenarios, assets would be higher under lower tax rates due to the lower effective impatience - see next subsection.

24 See, for example, Low (2005) who obtains a discount rate of 3% using a similar framework, without endogenising the return to saving.
Figure 4: The effect of varying the generosity of public pension provision, keeping savings constant. Graph (a) shows normalised assets. Graph (b) shows stock market participation by age. Graph (c) shows conditional portfolio shares at each age. The fixed costs are calibrated separately to match mean participation. Discount rates are calibrated separately to match asset holdings.
shares fall from 0.98 to 0.88 (as shown in table 2), as higher asset levels during mid-life push households onto the non-linear portions of their portfolio share policy functions.\textsuperscript{25} As taxes are reduced, the age effects become more pronounced, reaching the complete markets outcome at retirement when taxes are zero, and there is no pension income.\textsuperscript{26} This framework offers insights into the impact of demographics on portfolio allocations. An ageing population will lead to similar shifts from precautionary to retirement savings, potentially leading to increases in portfolio diversification and mid-life participation.

**Keeping the discount rate constant** In this scenario we hold the discount rate constant, at the level calibrated in section 4.1, while varying the tax rate. This scenario examines the effect of allowing average savings to increase in response to lower taxes. As before, we calibrate the per period fixed costs to match average participation.

We find that reducing the generosity of the public pensions scheme has a significant effect on the level of fixed costs: increasing from 4.6\% (when the tax rate is 30\%), to 7.1\% (when the tax rate is 10\%), to 10.7\% (when the tax rate is zero). In figure 5(a) the crowding out effect of public pensions is clearly evident. Reduced public pension provision is compensated for by greater private retirement savings, leading to a welfare loss being associated with public pension provision.\textsuperscript{27} The increased saving over the life-cycle, created by less generous pension provision, allows earlier initial entry and higher participation during mid-life. Again, smaller tax rates lead to more diversification of assets; however, shares remain unrealistically high at early stages of life, and mean shares are substantially above their true value (see table 2).

These scenarios demonstrate two important issues: firstly, more generous public pensions moderate the calibrated discount factor; and secondly, fixed costs are negatively

\textsuperscript{25}See figure 2.
\textsuperscript{26}The seminal work by Samuelson (1969) analysed portfolio choice in a complete markets situation with no background risk. He showed that the portfolio share should be independent of wealth and age and be given by $\omega = \frac{\sigma}{\mu - r}$, giving a value of 0.5 using our parameter values.
\textsuperscript{27}This loss stems from the higher average returns to self-insurance.
Figure 5: The effect of varying the generosity of public pension provision, keeping the discount rate constant (calibrated to the 20% tax rate). Graph (a) shows normalised assets. Graph (b) shows stock market participation by age. Graph (c) shows conditional portfolio shares at each age. The fixed costs are calibrated separately to match mean participation.
related to the intensity of public insurance. This latter point implies that the small empirical estimates of these fixed costs\textsuperscript{28} should not be surprising. Given the well developed social security system prevalent in the US, households hold modest retirement savings and only a small fixed cost is required to explain the observed limited participation.

4.3 Calibrating risk aversion

The above scenarios demonstrate how restricting ourselves to plausible participation and asset holdings lead to excessively high portfolio shares. This trade off between participation and shares has been well documented in the household finance literature. At the standard model specification, the structure of power utility and the ensuing policy functions dictate that large diversification can only take place if savings are substantial (see the policy function depicted in the solid line in figure 2). However, when savings are large households are highly motivated to participate, making it impossible to reconcile limited participation with small portfolio shares. Our per period fixed cost extension enables us to break this trade off, and match both features simultaneously.

The shapes of the portfolio share policy functions are determined to a large extent by the degree of risk aversion.\textsuperscript{29} Furthermore, the absolute lower bound of these functions is governed by this parameter through the Samuelson rule.\textsuperscript{30} The impediment to matching shares stems from the fact that low risk aversion, as advocated by econometric studies on consumption, leads to an intolerably high value for this lower bound. Under our baseline parameters we have a lower bound to the policy functions of 0.5, which is far too high if we hope to achieve mean shares of 0.54 at plausible levels of savings. By increasing the coefficient of risk aversion we can reduce this lower bound, shifting the policy functions down and inwards towards the origin. This can be seen by comparing the solid and dashed

\textsuperscript{28}See Paiella (2001) and Vissing-Jorgensen (2002).
\textsuperscript{29}Other parameters also determine the shape of the policy functions; see Alan and Ball (2007) for a discussion of the effect of changing the transitory income variance and moments of the expected return.
\textsuperscript{30}As savings go to infinity, labour income becomes of negligible importance, and we approach the complete markets outcome (see footnote in section 4.2).
lines in figure 2.\textsuperscript{31} This identifies a link between risk aversion and shares: for any given level of savings we obtain smaller shares with higher risk aversion. Our model set-up allows us to take advantage of this relationship, and calibrate the risk aversion parameter to match simulated average shares with the data. In this section we detail the results of this calibration in two different scenarios: firstly, keeping the discount rate at our baseline parameterisation; and secondly, re-calibrating the discount rate to match actual savings.

**Matching participation and shares** In this scenario we fix the discount rate to the value calibrated in section 4.1. Rather than exogenously set the coefficient of risk aversion, we choose it by calibration such that simulated mean shares match the 0.54 value seen in the data. We calibrate fixed costs to match mean participation as before. The calibrated parameters and simulated statistics are shown in table 2, and the simulated profiles are represented by the dashed line in figure 6.

We obtain a high risk aversion parameter of 9.5 and a per period fixed cost of 2.5% of the permanent component of labour income. Figure 6(b) shows the age profile for participation displaying the expected hump shape. The portfolio share profile in figure 6(c) is now much more realistic, with values well below unity for most of life and declining over mid-life and into retirement.\textsuperscript{32} We have moved away from the prediction of complete specialisation in stocks at young ages. This scenario demonstrates that we are able to match both mean participation and mean shares, within the standard life-cycle framework. However, asset holdings are very high. This outcome is comparable to that of Gomes and Michaelides (2005), without requiring preference heterogeneity or non-power utility.

**Matching participation, shares and wealth** In this scenario our model is shown to match all three key financial statistics. We calibrate the discount rate to match median

\begin{itemize}
  \item \textsuperscript{31} The dashed line in figure 2 shows the policy function for a risk aversion coefficient of 8.8 with all other parameters set to the baseline levels.
  \item \textsuperscript{32} The slight initial increase in portfolio shares is a result of the sharp rise in participation at these ages. At first only those who begin life with large asset holdings can participate, and these individuals have highly diversified portfolios; as more individuals participate, average wealth of the participants will fall leading to higher average shares.
\end{itemize}

24
Figure 6: Calibrating risk aversion. Graph (a) shows normalised assets. Graph (b) shows stock market participation by age. Graph (c) shows conditional portfolio shares at each age. The fixed costs, discount rates and risk aversion are calibrated separately to match mean participation, asset holdings and portfolio shares respectively.
asset holdings (1.84), risk aversion to match mean portfolio shares (0.54) and per period fixed costs to match mean participation (0.57); the results are detailed in the dotted lines in figure 6.

The age profiles are all reasonable, in particular we have portfolio diversification at all ages (figure 6(b)). Calibrated fixed costs now fall to 1.2% of the permanent component of labour income, which is close to the empirical findings discussed earlier. The calibrated coefficient of risk aversion is high at just over 12. In order to counter this risk aversion, and the entailing high precautionary motive, an uncomfortably large discount rate of 26.2% is required to match asset holdings. Even the high tax rate of 30% cannot moderate impatience at these parameter values.

4.4 Stock market crash possibility

The model’s failure to explain the stylised facts at reasonable preference parameters originates from the nature of the policy functions. We believe the problem stems from an incorrect specification of risk. Solving such a dynamic programme involves specifying a subjective probability distribution for the stochastic structural parameters. The standard process for this involves applying the law of large numbers, and assuming we have sufficient time series of data to substitute sample moments of past growth rates for population moments of future growth rates. Situations where the law of the law of large numbers does not hold lead to two main causes for concern: firstly, past growth rates may not be good proxies for expected future growth rates; and secondly, uncertainty over true parameters may create heightened background risk, fattening the tails of the subjective probability distribution.

Here we focus on the second concern, challenging the widespread use of lognormal distributions for expected returns. Lognormality is adopted mainly for analytical con-

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33 See recent work by Alan and Ball (2007), who question the use of past realised returns to determine the moments of expected future returns, and propose a structural estimation techniques to elicit the implied moments of the subjective distribution for returns from observed financial wealth and consumption data.
venience, and there is now growing debate over the adequacy of such a distribution for approximating financial stochastic variables, most vocally by Mandelbrot (1997, 2001). While normal distributions may be considered appropriate for many variables, such as physical weight or height distributions, it is difficult to believe they accurately describe stocks. Stock returns are subject to many random leaps and jumps, pushing the realised value far from the mean. The stock market crash of 1987 and the collapse of Enron in 2001 are examples of such extreme movements. Solving dynamic programmes with discretised lognormal distributions does not allow for such large departures from the mean, and is likely to lead to an underestimation of the true risk of holding stocks.

We propose a simple extension to the standard lognormal framework, whereby we incorporate a small probability of a stock market crash. While we accept this is a somewhat ad hoc extension, it is a relatively intuitive way of incorporating fat tails into the life-cycle model. To the best of our knowledge, this is the first time fat tails have been applied to the formation of return expectations within a household portfolio choice model. This extension can be thought of as analogous to the Carroll (1992, 1997) zero income shock possibility. We exogenously set the probability of a stock market crash to 1%, and assume that during a crash households lose all wealth held in risky assets, that is \(1 + r_t^{eq} = 0\). This approach clearly demonstrates the effect of fat tailed distributions in a simplified manner, we leave more coherent analysis for future research.\(^{34}\)

This small probability of a stock market crash has a significant effect on the portfolio share policy functions. The dotted line in figure 2 shows how this extension shifts the policy function further down and inwards towards the origin.\(^{35}\) This enables significant portfolio diversification at low, realistic levels of savings. Under these assumptions, we again calibrate the discount rate, fixed cost and risk aversion parameters to match wealth, participation and shares respectively (as in the previous scenario). The results are detailed in the final row in table 2 and the simulated profiles are shown in the solid line in figure

\(^{34}\)Such as discretising a t-distribution in the solution to the dynamic programme.

\(^{35}\)The coefficient of risk aversion is 8.8 and all other parameters are at baseline values.
6.

The life-cycle profiles are all reasonable: with assets being accumulated at the correct pace, a hump shape in participation, and portfolio shares being diversified across all ages. The most striking result is on the calibrated parameters, we are now able to match the data at plausible parameter values. A risk aversion of 8.8 is required to enable modest share holdings. This estimate is somewhat higher than suggested by empirical studies using consumption data, however, it remains within the bounds considered reasonable by Mehra and Prescott (1985) and is similar to values commonly used in the household finance literature. The per period fixed cost is now calibrated to less than one percent of annual labour income, which is closely in line with the empirical estimates of Paiella (2001) and Vissing-Jorgensen (2002). The calibrated discount rate is 9%, which is well within the range estimated by Alan and Browning (2003).

5 Conclusion

In this paper we analyse household consumption and saving decisions in the presence of two distinct saving tools and public pension insurance. We are able to match the life-cycle averages of savings and asset allocations, while keeping to a fairly simple framework with realistic parameter values.

We find that a standard portfolio choice life-cycle model, calibrated to match the level of savings in the PSID, gives extreme rates of stock market participation and very high portfolio shares, which is at odds with empirical findings. Introducing per period fixed costs to the baseline model generates the desired hump shape in participation, resolving the issues of limited participation and high turnover in the stock market. We have presented a stylised pay-as-you-go pension scheme and shown how this acts as a moderator for impatience. In the absence of pensions we would need a higher discount rate to keep

\[^{36}\text{Haliassos and Michaelides (2003) use a risk aversion parameter of 8 and Cocco, Gomes and Maenhout (2005) use 10.}\]
asset accumulation at the required level. More generous public pensions are seen to crowd out self-insurance and lead to higher average conditional shares. Further, the size of fixed costs is found to be conditional on the level of taxes, providing some explanation for the small empirical estimates.

The conditional portfolio shares remain implausibly high for all scenarios using low risk aversion coefficients. Disregarding the level of wealth, we can calibrate the model to match the mean of both participation and conditional shares, with very reasonable age-profiles. Furthermore, if we are willing to accept slightly higher risk aversion and a very high discount rate, it is possible to match all three statistics. Adapting the subjective probability distribution of returns to allow for a fat lower tail, has a considerable effect on our results. We are now able to match all three statistics with reasonable parameters: a per period fixed cost of 0.8% of annual labour income, risk aversion of 8.8 and discount rate of 9%.

It is important to stress that the literature has tended to ignore the matching of wealth, and has focused on matching one asset allocation statistic at a time (participation or shares, but not both). In this paper, we are able to overcome the apparent trade off between participation and shares, matching both simulated statistics to data while maintaining appropriate wealth levels. We achieve this without resorting to heterogeneous preferences or moving away from power utility functions. Through our innovation of augmenting the life-cycle model to include per period fixed costs of stock market participation, public pension provision, and a small probability of a disastrous event in the stock market, we demonstrate the ability of this framework to account for asset holdings and allocations simultaneously.
Appendix A: Solution and simulation methods

The results presented in section 4 use standard techniques to solve the model by backwards induction; starting from a terminal condition, in order to obtain the optimal policy functions for each age, mapping the state variables into the controls. Using these functions the model is simulated forward from $t = 1$ with 10,000 ex-ante homogenous individuals who differ, ex-post, due to different shock realisations. Following Alan (2006), initial assets are drawn from a lognormal distribution, whose underlying normal distribution has a mean of -3.15 and standard deviation of 1.96. Mean life-cycle profiles of accumulated wealth and asset allocations are then computed.

Solving the Euler equations corresponds to the determination of a fixed point within an infinite dimension state space, involving expectations over a non-linear marginal utility function, where the unknown is a function over a continuum of points. Such complexity means that the model cannot be solved analytically, which entails the implementation of numerical techniques. The state space is discretised into a finite number of nodes and interpolated using local approximation methods.\textsuperscript{37}

The grids are defined so as to avoid the need for extrapolation outside the grid.\textsuperscript{38} The concavity of the consumption function leads us to choose a non-uniform spacing of the asset nodes. Extra points are positioned close to the lower bound, where the consumption policy function displays a significant amount of curvature. The nodes are more spread out at high asset levels, at which point the functions become approximately linear. The savings grid is also non-uniformly spaced as the portfolio share policy function is non-linear. More points are positioned around the kink in the policy function, where the short sales constraint ceases to bind, and fewer nodes at high levels of savings, where the policy function becomes horizontal as it approaches the complete markets outcome. The solution

\textsuperscript{37}Four hundred points are used for both the asset and savings grids. Linear splines are used for interpolation.

\textsuperscript{38}Extrapolation is much less reliable that interpolation, especially where the policy functions are non-linear.
is found using NAG routines,\textsuperscript{39} except for when these methods fail to converge, in which case the non-linear system is solved using a bisection method.\textsuperscript{40}

We perform all numerical integration using Gaussian quadrature to approximate the distributions of labour income and the risky asset. The income shocks are discretised into six values and the risky return uses three point quadrature. In the simulations, the permanent shock to labour income is approximated as a continuous random variable. Each time period is taken as a year of life. $T$ was taken to be 58 years and $K$ as 15 years, giving a working life of 43 years (from age 22 to 65) and life coming to an end at 80.

A check on the accuracy of the solution method is undertaken by computing the realised values of the Euler equations. When averaged across individuals, these results do not deviate significantly from their expected values.

**Appendix B: Portfolio share policy functions**

Figure B1 shows the portfolio share policy functions for three different ages, under parameters outlined in table 1. At each age the policy function follows a highly nonlinear pattern. At low levels of saving, the short sales constraint is binding and agents hold all of their wealth in risky assets; represented by the horizontal section of the function. This derives from non-tradable labour (or pension) income acting as a substitute for the risk free asset.\textsuperscript{41} At low levels of saving the agent is highly endowed with this implicit risk free asset, driving portfolio shares to 100%. As savings are increased, income becomes a relatively less important fraction of wealth, giving low implicit risk free asset holdings and allowing more diversification. At very high levels of saving, income becomes an insignificant determinant of wealth, and portfolio shares tend to the complete market solution,

\textsuperscript{39}Fortran code is available from the author on request.
\textsuperscript{40}Bisection is an iterative procedure that computes the root of a one-dimensional function on a bounded interval of the real line. It is one of the most robust procedures but it converges slowly, hence, it is only used when the NAG routine fails.
\textsuperscript{41}Income is risky but bounded below, and hence reduces effective risk tolerance.
given by $\omega = \frac{\mu}{\gamma\sigma^2}$. This depicts what has been termed the ‘portfolio specialisation puzzle’, whereby only wealthy agents are able to diversify away from complete specialisation in risky assets.

As agents age, the policy functions in figure B1 shift inwards, enabling more portfolio diversification for a given level of saving. The young have a high present discounted value of lifetime income, and this represents a large implicit holding of riskless assets, resulting in high portfolio shares. As they grow older, the proportion of total lifetime wealth accounted for by the present discount value of income declines, leading to lower implicit holdings of riskless assets and a tilting of portfolios away from equities.

The policy functions are sensitive to the degree of risk aversion. In the low risk aversion case in figure B1, very high levels of saving are required before individuals operate on the nonlinear sections of their policy functions, and move away from complete specialisation in stocks. At high levels of risk aversion the policy functions shift in towards the origin, as shown in panel (a) of figure B2, which plots the policy functions for a risk aversion

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\(^{42}\)Under the baseline parameter values the Samuelson rule gives a complete markets portfolio share of 0.5.
Figure B2: The effect of varying risk aversion and the risk of stocks on the portfolio share policy functions. Panel (a) shows the policy functions with high risk aversion. Panel (b) shows the policy functions with high risk aversion and a small probability of a stock market crash.
parameter of 8.8. As agents become more risk averse, the range over which the short-sales constraint binds becomes smaller, leading to less specialisation in stocks. In addition, the increase in risk aversion reduces the lower asymptote of the policy functions, as the complete markets share falls with $\gamma$, enabling more diversification of portfolios. Increasing the risk of investing in stocks, through introducing a small probability of a stock market crash, shifts the policy functions further in towards the origin. This can be seen in panel (b) of figure B2, which depicts the policy function under the same parameters are panel (a), but incorporating a zero return shock.

References


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