CLIMATE TIPPING AND ECONOMIC GROWTH:

PRECAUTIONARY CAPITAL AND THE PRICE OF CARBON*

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Abstract

The optimal reaction to a climate tipping point which becomes more imminent with global warming is to be precautionary in accumulating additional capital to curb the adverse effects of the calamity and to price carbon to make catastrophic change less imminent. However, if the mean lag for impact of the catastrophe is long enough, the additional saving response will be smaller and can turn negative. We also decompose the optimal carbon price into its catastrophe components and a conventional marginal damages component, and show the separate effects of relative intergenerational inequality aversion and relative risk aversion using Duffie-Epstein preferences. Focusing on a productivity catastrophe, we calibrate our model and show how sensitive the policy responses are to the degrees of intergenerational inequality aversion and risk aversion, the trend rate of economic growth, the hazard rates, and how long it takes for the catastrophe to have its full impact.

Key words: gradual climate tipping point, precautionary saving, optimal social cost of carbon, trend growth, Duffie-Epstein preferences, speed of impact, hazard functions.

JEL codes: D81, H20, O40, Q31, Q38.

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1. Introduction

One of the biggest challenges the planet faces is global warming. The standard remedy is to price carbon at the social cost of carbon (SCC) via a carbon tax or an emissions market, where the SCC is the present discounted value of all future production damages that result from emitting one ton of carbon today. Integrated assessment models of climate change and economic development allow for production damages that rise *gradually* with global warming (e.g., Nordhaus, 1991; Tol, 2002; Nordhaus, 2008, 2014; Stern, 2007; Golosov et al., 2014). However, it is increasingly recognised that another important concern of climate policy is to deal with the small risk of irreversible climate disasters at high temperatures, besides internalising smooth global warming damages at moderate temperatures (e.g., Lenton and Ciscar, 2013; Lemoine and Traeger, 2014, 2016ab; Lontzek et al., 2015; Cai et al., 2015, 2016b). It takes time before the full impact of tipping points has materialised. There is still discussion on the size of these time lags (e.g., Lenton and Ciscar, 2013), but to give an idea: about fifty years for the dieback of the boreal forests or Amazon rainforests, less than 100 years for the release of methane from melting permafrost[1], about 100 years for the reorganisation of the Atlantic Meridional Overturning Circulation, and over 300 years for the melting and collapse of the Western Antarctic and Greenland Ice Sheets[2]. The size of the impact of a tip can vary just as much.[3]

We analyse the effects of pending catastrophic shocks, also known as tipping points,[4] on optimal climate policy. We focus directly on the shock to economic productivity and do not explicitly model the relationship with temperature, except that the hazard of such shocks rises with global warming. Our main result is that in the face of a pending catastrophe, on the one hand, adjustments to saving are needed to smooth consumption[5], and, on the other hand, carbon has to be priced more vigorously to curb global warming and make a catastrophe less imminent as has been pointed out by Lemoine and Traeger (2014), Cai et al. (2015) and Lontzek et al.

[1] Rising sea temperatures and sea levels trigger this (e.g., Dutta et al., 2006).
[2] The reduction in cooling when such ice sheets collapse derives from the ice-albedo effect, which acts more quickly over oceans than land as sea ice melts faster than continental ice sheets (e.g., Oppenheimer, 1988). Also, the demise of rain forests curbs transpiration as plants have lower reflectivity than soil.
[3] One can distinguish between catastrophic shocks to total factor productivity that are too a large extent local and private in nature (e.g., flooding of cities, increased storm frequency, droughts and desertification) and those that are more global and public in nature. Our analysis applies to both types of shocks, but the former aspect of climate change is more positive whilst the latter more normative.
[4] Persistent changes in the climate system are called regime shifts in the ecological literature (e.g. Biggs et al., 2012). A point where such a regime shift occurs is called a tipping point.
[5] The need for precautionary saving to prepare for catastrophic shocks has been pointed out by Smulders et al. (2014) for when the hazard rate is constant instead of a temperature-dependent. Gjerde et al. (1999) consider carbon cycle and temperature modules with gradual and catastrophic damages and offer detailed simulation studies of the effects of pending catastrophes on temperature and the economy, but do not discuss the optimal price of carbon or the need for saving adjustments to deal with the tip.

(2015). We show how the optimal price of carbon and the additional precautionary saving response interact and decompose the effect of marginal damages and the risk of tipping on the optimal price of carbon. An additional precautionary saving response is called for if the full impact of the tip is felt immediately or the mean impact lag of the tip is not too large, but dissaving is required for larger mean impact lags of the tip. Dissaving might also occur more easily if the economy starts off in the early phases of economic convergence.

Using the stochastic differential utility framework of Duffie and Epstein (1992) based on the preferences proposed by Epstein and Zin (1989), we can separate the coefficients of intergenerational inequality aversion (the inverse of the elasticity of intertemporal substitution) and relative risk aversion in a continuous-time tipping framework.[6] This disentanglement in the context of climate change has also been done in a discrete-time tipping framework by Lemoine and Trager (2016b) who show that this does not change the optimal policy much. For the empirically relevant case of a greater dislike for risk than for intertemporal fluctuations, we show that the precautionary savings response is stronger.

To get an order of magnitude, we calibrate our model and show by how much the optimal SCC must be adjusted upwards compared with the conventional SCC based on only gradual damages and how big the adjustment to saving and capital has to be. We show the sensitivity of these adjustments with respect to intergenerational inequality aversion, the trend rate of growth, relative risk aversion, the mean speed of impact of the catastrophe, and the hazard rates.

To be fair, integrated assessment modellers have adjusted their damage functions to allow for the risk of catastrophic change. For example, Golosov et al. (2014) first recalibrate and simplify the damage function of Nordhaus (2008) to show what remains of output when global warming is given by an exponential function that depends negatively on the carbon stock. They then adjust the flow damage coefficient upwards to allow for a 6.8% risk of a catastrophic drop in aggregate output of 30% at 6º Celsius which boosts the optimal SCC.[7] [8] This procedure raises several questions. First, it ignores the risk of a catastrophic drop in aggregate output below or

[6] An alternative is to use a multiplicative choice model that displays risk aversion with respect to intertemporal utility. This alternative has been used to analyse how this makes climate policy more stringent in a simple model of climate change and growth with the catastrophe leading to an exogenous lower level of consumption (Bommier et al., 2015).

[7] They argue that at 2.5 ºC or 1035 GtC the aggregate output loss is 0.48% or a flow damage of 1.06% of world GDP /TtC from $\exp(-1.06\times10^{-5}\times(1035-581)) = 0.9952$ where 581 GtC is the pre-industrial carbon stock. At 6º Celsius or 2324 GtC the output loss is 30% or 20.46% of world GDP/TtC. Given a risk of 6.8%, the expected flow damage is $0.934\times1.06 + 0.068\times20.46 = 2.38\% > 1.06\%$ of world GDP/TtC.

[8] The flow damage coefficient does not vary much with global warming, since the mild convexity of the Nordhaus function linking marginal damages to temperature is offset by the concavity of the log function linking temperature to the stock of atmospheric carbon. With more convex functions linking damages to temperature, a higher expected damage coefficient results and thus a higher SCC results.

above 6^{o} Celsius. Second, it ignores that it takes many decades or even centuries for a catastrophe to have its full impact. Third, it is unclear by how much the adjustment changes if the hazard function is more convex. Fourth, the certainty-equivalent procedure is only valid under restrictive assumptions (logarithmic utility, Cobb-Douglas production, 100% depreciation of capital each period, linear carbon cycle) that eliminate the dynamics of the Ramsey growth model and the need for saving adjustments (cf. Engström and Gars, 2016). We establish that without these assumptions and with a continuous hazard function, the optimal SCC is targeted to delay the tipping point and saving need is adjusted to cope with the pending catastrophe. Fifth, adjusting the expected damage upwards does not allow one to investigate separately the effects of relative intergenerational inequality aversion and relative risk aversion.

Our illustrative model of growth and climate change uses these thought experiments to obtain an illustrative calibration of our catastrophic shock to aggregate output and hazard function. Our calibration of gradual damages of global warming is taken from Golosov et al. (2014). Our model has at its core a Ramsey growth model with capital, fossil fuel and renewable energy as production factors, steady labour-augmenting technical progress, and a decarbonisation trend so that carbon emissions per unit of fossil fuel use fall with time as a result of balanced technical progress. We determine the catastrophe-driven and smooth-damages components of the SCC and the adjustments that need to be made to saving to deal with the looming tipping point. We conclude that for a range of reasonable parameter specifications the required increase in the SCC are significant, but the additional saving adjustment to deal with the tip is relatively small unless the full impact of the catastrophe is felt relatively quickly.

The DICE integrated assessment model of climate change and economic growth developed by Nordhaus (2008) has been adopted by Keller et al. (2004), Lemoine and Traeger (2014, 2016ab), Cai et al. (2015, 2016b) and Lontzek et al. (2015) to numerically analyse the effects of tipping points on the optimal carbon tax. Keller et al. (2004) study the combined effects of a *given* climate threshold, a carefully calibrated potential ocean thermohaline circulation collapse, and learning. The impact of the tip in this classic study is felt immediately and it is interesting that this study always finds a precautionary saving response, as in our model, if the impact of the tip is felt immediately. Lemoine and Traeger (2014) add learning by formulating a hazard rate that is zero at temperatures that have proven to be safe. They have slow worsening of economic conditions after their carbon sink/release or positive feedback tipping point and find almost no precautionary saving response (see their Figure A.1), which is in line with what we find if we have a large enough mean impact delay of the shock. However, we show that one can also have negative precautionary saving and we spell out under what circumstances this occurs. Lontzek et al. (2015) allow for catastrophes whose impact is only felt gradually over decades or centuries and find that as a result of pending catastrophes the SCC has to be up to twice as large. Our contribution is inspired by these path-breaking studies, but aims to offer an improved understanding of what is driving the optimal response to a catastrophe and build a bridge between these numerical studies with detailed integrated assessment models and earlier simple analytical studies of how to respond to climate tipping (e.g., Clarke and Reed, 1994; Tsur and Zemel, 1996; van der Ploeg, 2014).

Others have focused on catastrophes before. Barro (2015a) studies the optimal investment needed to curb the risk of environmental disaster. Weitzman (2007) highlights uncertainty at the upper end of the probability distribution of possible increases in temperature and damages, and shows that the impact of a fat instead of a thin tail on climate policy can be dramatic. Martin and Pindyck (2015) study the 'strange' implications for cost-benefit analysis of a cascade of negative shocks in partial equilibrium. Bretschger and Vinogradova (2016) study these in presence of endogenous growth. Pindyck and Wang (2013) obtain the general equilibrium price of insurance against catastrophic risks.

Section 2 sets up the model. Section 3 derives the optimal after-tip climate policy. Section 4 derives the upwards biases of the optimal pre-tip SCC and required adjustments to saving. Section 5 gives geometric insights into how the post-tip and pre-tip dynamics interact with the additional pre-tip saving response needed to deal with the pending tip. Section 6 disentangles intergenerational inequality aversion and risk aversion. Section 7 discusses the decentralisation of the command optimum in the market economy. Section 8 discusses our calibration. Section 9 discusses the expected value approach to dealing with catastrophes and the post-tip outcomes. Section 10 presents the optimal policy simulations and derives estimates of the optimal price of carbon and required saving adjustments and discusses the sensitivity of these estimates with respect to intergenerational inequality aversion, growth, risk aversion, the hazard rates and the time it takes for the catastrophe to have its full impact. Section 11 concludes.

2. The model

Let $\bar{g}$ be the constant rate of labour-augmenting technical progress, so that $A(t) = e^{\bar{g}t}$ is the efficiency of labour at time *t*. Let *P* denote the stock of atmospheric carbon as an indicator of global warming. We follow Golosov et al. (2014) and use $e^{-\chi P}$ with $\chi > 0$ as the multiplying factor to output, which is equal to 1 in the absence of global warming and decreases with global warming. This implies that we make the simplifying assumption that the stock of atmospheric carbon immediately impacts global mean temperature, whereas it actually takes a few decades.

We use the current stock of atmospheric carbon as a proxy for global mean temperature, but we take into account that the impact of the catastrophic shock is not immediate. The unknown date of the catastrophe is $T > 0$ and another multiplying factor to output, $B(t),\, t \geq T$, indicates what is left of output after the catastrophe. The catastrophic shock follows an exponential lag:

$$(1)\qquad \begin{aligned} &B(t) \equiv 1 \text{ for } t < T, \\ &0 < B(t) \equiv 1 - \Delta\left[1 - e^{-\varphi(t-T)}\right] < 1 \text{ for } t \geq T \text{ with } 0 < \Delta < 1 \text{ and } \varphi > 0. \end{aligned}$$

The speed of impact of the catastrophe is given by φ. The long-run size of the catastrophe corresponds to a multiplicative drop of Δ to TFP and the average time it takes for this to materialise is $1/\varphi$. The size of the pending drop in output and the speed at which this drop occurs are known, but it is not known *when* the climate regime shift will take place. The hazard of the catastrophe, $H(P)$, is endogenous and increases in the carbon stock (a proxy for global mean temperature). With global warming the hazard rate, $H(P)$, increases over time, so failing climate policy makes the shock to productivity more imminent. [9]

Aggregate output is given by the concave and constant returns to scale production function $Q = e^{-\chi P} BG(K, F, X, AL)$, where K denotes the aggregate capital stock, F fossil fuel use, X use of the carbon-free alternative (renewable energy) and L labour use. The labour market clears and, for simplicity, we set labour supply to a constant (w.l.o.g. unity). Extracting one unit of fossil fuel use requires $d_F > 0$ units of output, where we assume that fossil fuel is abundantly available.[10] The production of one unit of renewable energy needs $d_X > 0$ units of output. Let C denote aggregate consumption and $\delta > 0$ the depreciation rate of capital. Net investment is

$$(2)\qquad \dot{K} = e^{-\chi P} BG(K, F, X, AL) - d_F F - d_X X - C - \delta K, \quad K(0) = K_0,$$

Burning fossil fuel leads to accumulation of carbon in the atmosphere which decays at the rate $\gamma > 0$ (cf. Nordhaus, 1991)[11]. The stock of atmospheric carbon evolves according to

$$(3)\qquad \dot{P} = \left(\frac{1}{A}\right) F - \gamma P, \quad P(0) = P_0.$$

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[9] $H\left(P(t)\right) = \lim_{\Delta t \to 0} \Pr[T \in (t, t + \Delta t) \mid T \notin (0, t)] / \Delta t \equiv h(t)$ is the conditional hazard of the tip occurring at time t, so $h(t)\Delta t$ is the probability that it takes place between t and $t+\Delta t$ given that it has not occurred before time t. The survival probability of the tip *not* occurring in the interval $[0,T]$ is $\exp\left(-\int_0^t h(s)ds\right)$.

[10] This is a reasonable assumption for coal but less for oil and natural gas.

[11] This one-box carbon cycle ignores that about 20% of carbon remains permanently or at least for thousands of years in the atmosphere and the remainder eventually returns to the oceans and the surface of the earth (e.g., Golosov, et al., 2014). It turns out that this one-box approximation affects our numerical results only slightly without losing the key insights on how to deal with tipping points.

Initial fossil fuel use is not measured in energy units, but in Giga tons of carbon. Technical progress affects both growth in the production of final goods and carbon efficiency. We assume that the economy is on a balanced growth trajectory so that these two rates of technical progress are the same and thus that both the rate of technical progress in final goods production and the rate of decline in emissions per unit of fossil fuel use occur at the rate $\bar{g}$.

With a constant pure rate of time preference of $\bar{\rho} > 0,$ the social welfare function is

(4) $$W = \int_0^\infty e^{-\bar{\rho}t} U\left(C(t)\right) dt,$$

where $U(.)$ is a concave function with a constant coefficient of relative intergenerational inequality aversion (IIA) or coefficient of relative risk aversion (RRA), both denoted by $\eta > 0$, and thus also a constant elasticity of intertemporal substitution (EIS) given by $1/\eta > 0$. Strictly speaking, intergenerational inequality aversion (IIA) makes no sense in a model with infinitely lived households. We assume, however, that households stand in for a sequence of dynastically linked generations, so that IIA can then be thought of as measuring the aversion of the dynasty towards consumption fluctuations across different generations.

Sections 3, 4 and 5 deal with expected utility analysis: maximise $\mathrm{E}[W]$ subject to (1)-(3). Section 6 allows for non-expected utility analysis. Section 7 discusses how to decentralise the optimum in a market economy.

3. Social optimum: post-catastrophe problem

After the catastrophe all uncertainty is resolved. Defining intensive-form variables $c \equiv C / A,$ $q \equiv Q / A,$ $k \equiv K / A,$ $f \equiv F / A$ and $x \equiv X / A,$ the after-catastrophe problem is thus

(5) $$\underset{c,f,x}{\text{Max}} \int_T^\infty \frac{c^{1-\eta}}{1-\eta} e^{-\rho(t-T)} dt$$

subject to the dynamic equations

(6) $$\dot{k} = e^{-\chi P} Bg(k, f, x) - d_F f - d_X x - (\delta + \bar{g})k - c \text{ and}$$

(7) $$\dot{P} = f - \gamma P,$$

where $\rho \equiv \bar{\rho} + (\eta - 1)\bar{g}$, $g(k, f, x) \equiv G(k, f, x, 1)$ and T is the random starting point. This is an optimal control problem that can be solved with Pontragin’s Maximum Principle. For this purpose we need the *social cost of carbon*, abbreviated by SCC and denoted algebraically by s, which is defined as the present discounted value of all future global warming damages resulting from emitting one ton of carbon today.

Proposition 1: The solution of the after-catastrophe problem (5), subject to (6) and (7), is given by the 4-dimensional dynamical system

(8) $\dot{k} = y(s,k,P,B) - sy_s(s,k,P,B) - c,$ given $k(T)$,

(9) $\dot{P} = -y_s(s,k,P,B) - \gamma P,$ given $P(T)$,

(10) $\dot{c} = (r-\rho)c/\eta,$

(11) $\dot{s} = (r+\gamma)s - \chi q,$

where

(12) $y(s,k,P,B) \equiv \max_{f,x} \left[e^{-\chi P} Bg(k,f,x) - (d_F + s)f - d_X x - (\delta + \bar{g})k \right],$

(13) $r \equiv y_k(s,k,P,B)$ and $q \equiv e^{-\chi P} Bg(k, f^*, x^*)$ with $(f^*, x^*) = \arg\max[y(s,k,P,B)].$

Proof: The basic proof is given in appendix A. The maximisation in the Hamiltonian function on the amount of fossil fuel *f* and renewable energy *x* is effectively static. It follows that we can conveniently define *y* in (12) as the maximum output, net of the cost of energy and the depreciation and growth charges of capital, with $y_s = -f$, $y_P = -\chi q$ and $y_B = q/B,$ and implement this in (8) and (9). Note that part of the costs of fossil fuel *f* results from the shadow value or co-state for the stock of atmospheric carbon *P* divided by the marginal utility of consumption, which corresponds to the SCC. The shorthand *r* denotes the social rate of interest, corrected for depreciation and trend growth. The shorthand *q* denotes the maximum output.

The Euler equation (10) gives optimal consumption growth. The dynamics of the social cost of carbon (SCC) are given by (11). Equations (8)-(11) are a 4-dimensional non-homogeneous saddle-path system, where (k,P) are the predetermined and (c,s) the non-predetermined variables. The solution gives a mapping of the non-predetermined on the predetermined variables and time (the stable manifold).

Since the convergence of the Ramsey growth dynamics is much faster than that of the carbon cycle, a good approximation is to suppose that for purposes of calculating the SCC the Ramsey growth dynamics has converged and the economy is on a balanced growth path. If there are no pending catastrophes (as will be the case after the tip has occurred), this leads to an optimal

SCC that is proportional to GDP[12]: $s \cong \Gamma q$ with $\Gamma \equiv \dfrac{\chi}{\rho+\gamma}$ and $\rho = \bar{\rho} + (\eta - 1)\bar{g}.$ The optimal SCC also increases in the damage coefficient χ, and decreases in the decay rate of atmospheric carbon γ and the rate of time preference $\bar{\rho}.$ If intergenerational inequality aversion exceeds unity ($\eta > 1$), the SCC is low if trend growth $\bar{g}$ is large and future generations are rich, as current generations are then less willing to make sacrifices to curb future global warming.

Given the stable after-tip manifolds (denoted by superscript A) for aggregate consumption, $c(t) = c^A\left(k(t), P(t), B(t), t-T\right),$ and the SCC $s(t) = s^A\left(k(t), P(t), B(t), t-T\right), t \geq T,$ the value function $V\left(k(t), P(t), B(t), t-T\right)$ follows from the Hamilton-Jacobi-Bellman (HJB) equation

$$(14) \qquad \rho V = U(c^A) + U'(c^A)\left[y(s^A, k, P, B) - c^A + \gamma s^A P\right], \quad t \geq T,$$

where $V_k = U'(c^A)$ and $V_P = -s^A U'(c^A).$ As a shorthand, we will use $V^0(k,P) \equiv V(k,P,1,0).$

4. Social optimum: before-catastrophe problem

The before-catastrophe problem is

$$(15) \qquad \underset{c,f,x}{\text{Max}}\ \mathrm{E}\left[\int_0^T \frac{c^{1-\eta}}{1-\eta} e^{-\rho t} dt + e^{-\rho T} V\left(k(T), P(T), 1, 0\right)\right] = \int_0^\infty H\left(P(T)\right) e^{-\int_0^T H(P(t))dt} \left[\int_0^T \frac{c^{1-\eta}}{1-\eta} e^{-\rho t} dt + e^{-\rho T} V\left(k(T), P(T), 1, 0\right)\right] dT$$

subject to (6) and (7) with $B(t) = 1$ for $0 < t \leq T$. The second part of (15) has substituted the exponential probability density function for the hazard rate. The cumulative density function is $1 - \exp(-\int_0^T H\left(P(t)\right)dt).$ [13] The probability that the catastrophe has *not* occurred in the period

[12] The mean lag for economic convergence in the neoclassical growth model is about 50 years corresponding to Barro's (2015b) "iron" rule of 2% per year convergence. This is a relatively short lag compared to the long time-delays in the carbon stock and temperature dynamics. A simple rule giving the SCC as a constant fraction of GDP, derived under the assumption that economic convergence has fully taken place, therefore performs well in the market economy and gets very close to the welfare attained in the first-best optimum (Rezai and van der Ploeg, 2016). Nordhaus (1991) already obtained such a simple rule under the assumption that the Ramsey growth block of the model has converged. Golosov et al. (2014) shows that the simple rule holds *exactly* in the expression for the SCC that is proportional to GDP, provided utility is logarithmic, production is Cobb-Douglas, capital depreciates fully each period, and fossil fuel extraction requires no capital.

[13] With a constant carbon stock, the mean arrival time of the tip and its standard deviation equal $1/H(P)$. This distribution has skewness 2; the median arrival time, $\ln(2)/H(P)$, is less than the mean arrival time.

up to time T is thus $\exp(-\int_0^T H(P(t))dt)$. The exponential density function is memoryless, but the mean arrival time depends on the atmospheric carbon stock which changes with time.

Proposition 2: The solution of the before-catastrophe problem (15), subject to (6) and (7) is given by the 4-dimensional dynamical system consisting of (8) and (9), with initial conditions k_0 and P_0 and B = 1, plus

$$(16) \qquad \dot{c} = \frac{1}{\eta}(r+\theta-\rho)c, \quad \theta \equiv H(P)\left[\frac{V_k^0 - U'(c)}{U'(c)}\right] = H(P)\left[\left(\frac{c}{c^A(k,P,1,0)}\right)^{\eta} - 1\right],$$

$$(17) \qquad \dot{s} = \left[r+\gamma+\theta+H(P)\right]s - \left[\chi q + H'(P)\left(\frac{W-V^0}{U'(c)}\right) - H(P)\frac{V_P^0}{U'(c)}\right],$$

where the before-catastrophe value is given by

$$(18) \qquad W = \frac{1}{\rho+H(P)}\left[\frac{\eta}{1-\eta}c^{1-\eta} + c^{-\eta}y(s,k,P,1) + \gamma s c^{-\eta}P + H(P)V^0\right].$$

Proof: Following Polasky et al. (2011) and Lemoine and Traeger (2014), the HJB equation in the before-catastrophe value function $W(k,P)$ has B = 1 and equals

$$(19) \qquad \begin{aligned} \rho W(k,P) = \operatorname*{Max}_{c,f,x} \Bigg\{ & \frac{c^{1-\eta}}{1-\eta} + W_k\left[e^{-\chi P}g(k,f,x) - d_F f - d_X x - (\delta+\bar{g})k - c\right] \\ & + W_P(f-\gamma P) - H(P)\left[W(k,P) - V(k,P,1,0)\right]\Bigg\}. \end{aligned}$$

The last term in the maximand of (16) shows the expected capitalised loss from a catastrophe that occurs at some unknown future date. The optimality conditions give $c^{-\eta} = W_k$, $e^{-\chi P}g_X(k,f,x) = d_X$, and $e^{-\chi P}g_f(k,f,x) = d_F + s$, where the SCC is $s \equiv -W_P / W_k$. Total differentiation of (16) with respect to time and using these optimality conditions, one gets

$$(20) \qquad \begin{aligned} & \left[(\rho+H)W_k - HV_k^0 - \dot{W}_k - rW_k\right]\dot{k} \\ & + \left[(\rho+H+\gamma)W_P - HV_P^0 + H'(W-V^0) - \dot{W}_P + \chi q W_k\right]\dot{P} = 0. \end{aligned}$$

Insisting that (20) holds for all $\dot{k}$ and $\dot{P}$ gives the Pontryagin conditions (16) and (17), and the before-catastrophe value follows from (19).

Equation (16) is the before-catastrophe Euler equation. It shows an important difference from (10): if consumption immediately after the calamity is lower than immediately before, the tilt of the Euler equation is higher to reflect the need for precautionary saving to prepare for the pending catastrophe. The magnitude of the extra saving required depends on the additional required return θ, which increases in the risk of the hazard and thus the degree of global warming (proxied by P) and in the drop in consumption (in case it drops) at the time of the catastrophe. The Euler equation states that optimal consumption growth is proportional to the marginal net product of capital *plus* the extra required return θ *minus* the rate of time preference ρ. However, if consumption just after the tip is higher than just before the tip, $\theta < 0$ and saving is less than in the absence of the tipping point.

Three factors determine whether consumption jumps up or down at the tipping point. First, it is important to note that after tipping, the economy has to adjust to the new conditions and a new optimal consumption path has to be determined. This adjustment may require that consumption jumps up or down relative to long-run consumption before the tip. For low values of IIA η (or high values of EIS, $1/\eta$), the large upward jump in consumption may place consumption immediately after the tip above what it is immediately before. Second, if the tip occurs early enough, before-catastrophe consumption is below the long-run level. This implies that if consumption would jump down at the tipping point in case it had reached a level close to the long-run level, it may now jump up. Third, if the impact of the tipping point is slow, the effect of preparing for possible tipping is mitigated. This implies that if consumption would jump down in case of an immediate effect, it may now jump up, depending on the lag of the impact. In the calibrated simulations in Section 10 below, we will demonstrate some possible outcomes. We discuss these three factors in more detail using a phase diagram in section 5.

Equation (17) gives the before-catastrophe dynamics of the optimal SCC. Compared with the after-catastrophe dynamics (11), we note two differences. First, the rate used to discount marginal global warming damages includes both the additional required return on saving and the hazard rate, $\theta + H(P)$. This rate increases with factor productivity and as the marginal product of capital is higher before the catastrophe than after the catastrophe, this lowers the SCC. Second, the conventional marginal damages resulting from gradual change χq are supplemented with those from abrupt change (the last terms in the second pair of square brackets in (17)). The term $H'(P)(W - V^0)/U'(c) > 0$ corresponds to the expected marginal loss of a catastrophe and pushes up the before-catastrophe SCC. This reflects the desire to avert the risk or, more precisely, to postpone the expected arrival of the catastrophe by curbing global warming. The term $-H(P)V_P^0/U'(c) > 0$ reflects that emitting one ton of carbon pushes up global warming and leads to gradual output losses after the catastrophe and a lower after-tip value. This increases the drop in value at the time of a catastrophe (‘raises the stakes’) and thus pushes up the before-catastrophe SCC. The optimal SCC implied by (17) is the present discounted value of all future expected marginal and non-marginal output damages:[14]

$$(21)\quad s(t) = \int_t^\infty e^{-[r_F(t')+\gamma](t'-t)} m(t')dt' \quad \text{with} \quad m \equiv \chi q + \left[H'(P)(W - V^0) - H(P)V_P^0 \right] / U'(c),$$

where the rate used to discount expected marginal damages, $r_F \equiv r + \theta + \gamma + H(P),$ includes the hazard rate, the additional required return to deal with the pending tip, and the rate of decay of atmospheric carbon. The long-run value, $r_F = \bar{\rho} + (\eta - 1)\bar{g} + \gamma + H(P),$ increases in the rate of pure time preference, $\bar{\rho}$, and, if IIA exceeds unity ($\eta > 1$), in rising affluence.

The SCC is large if the drops in future welfare from climate calamities and the marginal hazard rate are large. The *slope* of the hazard function thus pushes up the SCC. The *level* of the hazard rate depresses the SCC via the higher discount rate but pushes it up via the raising-the-stakes effect.[15] A hazard rate that rises with global warming increases over time, so that the required adjustment to saving to deal with the pending calamity rises over time. Typically, the SCC rises to curb the risk of the calamity by curbing fossil fuel use, carbon emissions and global warming.

Finally, a doomsday scenario occurs if the catastrophic shock is so devastating that it destroys the economy completely ($B(t) = 0, t \geq T$). In that case, the Euler equation (16) becomes $\dot{c} = \left[r - H(P) - \rho \right] c / \eta$ as $\theta = -H(P) < 0,$ $\theta = -H(P) < 0,$ so the hazard rate $H(P)$ is *added* to the discount rate ρ. Before tipping, consumption is then higher and capital accumulation is *lower* than in the naive outcome as one expects the world to come to an end after the catastrophe. With life after the tip, we usually have $B(t) > 0, t \geq T$, $\theta > 0$ and capital accumulation is *higher* (cf. Polasky et al., 2011).

5. Post-tip dynamics and the pre-tip saving response

To gain better understanding on how the post-tip dynamics affects the pre-tip savings response, we use a simplified version of our model by assuming that there are no smooth production damages from global warming and that the hazard rate for the tip does not increase with global

[14] Lemoine and Traeger (2014) give a similar decomposition of the marginal costs in case of a potential tipping point in their formula (3). They distinguish the differential welfare impact, i.e., the hazard rate times the difference in marginal values, and the marginal hazard effect, i.e., the marginal hazard rate times the difference in values. We end up, in (21), with a decomposition of the effects of gradual damages, curbing risk and raising the stakes on the optimal SCC. Via θ we give the determinants of the required amount of precautionary saving.

[15] This term is zero in the absence of conventional gradual damages (i.e., if $\chi = 0$).

warming (i.e., $\chi = H'(P) = 0$). Hence, there is no need to price carbon before or after the tip and we can focus entirely on how tipping affects the savings response.

The post-tip economy corresponds to the usual Ramsey growth model. Consumption follows the stable saddle-path $c^A(k)$ towards the steady state $(k^A, y^A(k^A))$ where y^A denotes the net production function after the tip.[16] As the interest rate drops after the catastrophic shock to productivity,[17] the economy dis-saves in the post-tip period and consumption has to jump immediately after the tip to place the economy on the stable saddle-path $c^A(k)$. Similarly, the naïve pre-tip economy corresponds to the usual Ramsey growth model, but with steady state $(k^N, y^B(k^N))$ where y^B denotes the net production function before the tip. Comparing these outcomes, net production simply moves down at the tip, and the steady state moves down and to the left because marginal net production decreases. The stable saddle-path $c^A(k)$ can cut y^B at different points (see figure 1). When it cuts y^B to the right of $(k^N, y^B(k^N))$, it is clear from equation (16) that the steady state $(k^B, y^B(k^B))$ of the pre-tip economy lies between the point $(k^N, y^B(k^N))$ and the intersection point of $c_2^A(k)$ and y^B, because the precautionary return θ is positive. This implies that if the pre-tip economy is close to the steady state, consumption will jump down at the tip. Depending on the value of IIA, it can happen that the stable saddle-path $c_1^A(k)$ cuts y^B to the left of $(k^N, y^B(k^N))$, which would imply the opposite, but in our calibration this is not the case. However, if the catastrophic shock has an impact delay, the stable saddle-path $c^A(k)$ moves to the left and cuts y^B to the left of $(k^N, y^B(k^N))$ for a sufficiently large impact delay which is relevant in our calibration. This explains why consumption may jump up at the tip, with precautionary dissaving as a result. This happens for a sufficiently large impact delay.

If we are in the situation that consumption jumps down at the tip, if the pre-tip economy is close to the steady state, it may very well happen that consumption jumps up at the tip when it occurs early and the pre-tip economy is still developing. The upward stable saddle-path $c_2^B(k)$ before the tip towards the steady state $(k^B, y^B(k^B))$ is steeper than the upward stable saddle-path $c_2^A(k)$ after the tip (see figure 1 and appendix B). This implies that $c_2^B(k) < c_2^A(k)$ for low levels of k, so that consumption jumps up, and that $c_2^B(k) > c_2^A(k)$ for high levels of k, so that consumption jumps down. It follows that the effect on saving is complicated, because the precautionary return θ moves from negative values to positive values depending on whether the realised date of the tip is early or late. The additional pre-tip saving response needed to deal with the

[16] For our post-tip Ramsey growth model with Cobb-Douglas production function the speed of convergence is given by (B3) in appendix B. Convergence is thus fast is the share of capital in value added α is small and the depreciation rate δ and the discount rate $\rho = \bar{\rho} + (\text{IIA} - 1)g$ are large.

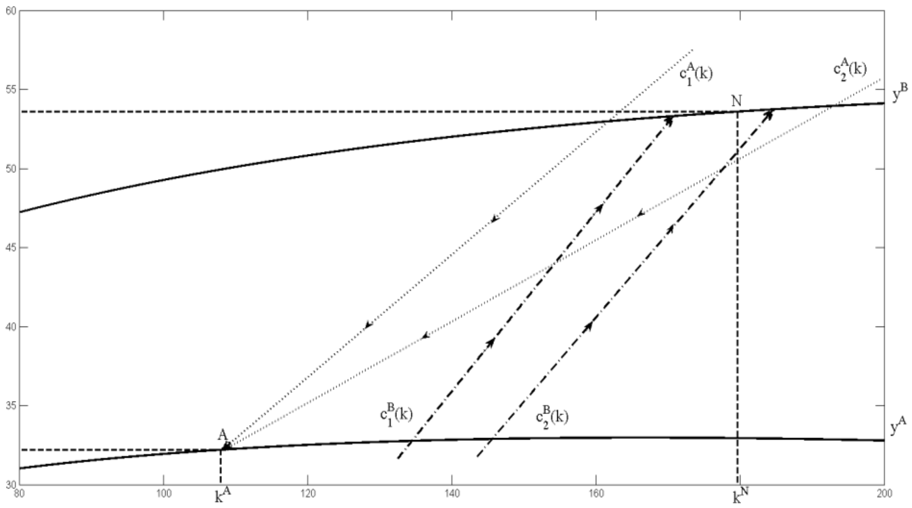
[17] If the tipping shock hits the capital stock instead of total factor productivity, the interest rate would rise and the post-tip economy would save and invest to rebuild the capital stock.

expected tip decreases in the initial value of pre-tip consumption and thus depends on all these values of θ

$$(22) \qquad c(0) = c^{P^*} \exp\left(-\int_0^\infty \left[r\left(k(T)\right) + \theta(T) - \rho \right] dT \right) \leq c^{P^*}.$$

In general, the effect on saving is ambiguous. It is very well possible that the net effect is precautionary saving, which in the steady state corresponds to $\theta > 0$ and a downward jump in consumption, whereas an early realisation of the tip shows an upward jump in consumption. In the simulations below, with the extended model, we will show these possibilities.

Figure 1: Phase diagram for pre-tip and post-tip dynamics



Key: The pre-tip and after-tip net production functions are y^B and y^A, respectively. The naïve and after-tip steady states are N and A, respectively. The stable saddle-paths for pre-tip and after-tip consumption are $c^B(k)$ and the $c^A(k)$ locus, respectively. Case 1 corresponds to a relatively large impact delay where consumption jumps up immediately after the tip. Case 2 corresponds to a small impact delay where consumption jumps up or down immediately after the tip.

Consider first the case for a small impact delay, to the right of the naïve steady state (k^N, $y^B(k^N)$), when the economy has moved a long way along $c_2^B(k)$ before the tip strikes (high T). Consumption jumps down as consumption and capital have grown enough to have risen above the after-tip saddle-path $c_2^A(k)$. If the tip strikes in the early phases of economic convergence (low T), consumption jumps up to the after-tip saddle-path $c_2^A(k)$. This situation arises if consumption and capital have not reached high levels yet in the pre-tip phase, and saving and investment rates are already high. This upward jump in consumption has (together with all the other jumps for all the different realisations of the time of the tipping point) been accounted for by a negative contribution to the additional saving response. Consider now the case for a large impact delay, to the left of the naïve steady state (k^N, $y^B(k^N)$). The after-tip saddle-path $c_1^A(k)$ now lies above the before-tip saddle-path $c_1^B(k)$, so that consumption jumps up at all stages of development.

Summing up, the additional saving response needed to deal with the pending tip is negatively affected by the upward jumps in consumption during the early phases of economic development and positively affected by the downward jumps in consumption during the later phases. The above analysis only applies if the hazard rate is constant and smooth damages from global warming are absent. The simulations and table 3 of section 10 show that with a mean exponential impact lag of 50 years or longer, consumption jumps up at the time of the tip for all possible realisations of the tipping point. This is due to the fact that the stable saddle-path $c_1^A(k)$ moves to the left of the steady state (k^N, $y^B(k^N)$) and thus above the stable saddle-path $c_1^B(k)$.

6. Separating intergenerational inequality aversion and risk aversion

The expected utility framework has RRA = IIA = η. It is important to explain that in the climate change literature households stand in a sequence of infinitely lived households and the IIA can be thought of as measuring the aversion of the dynasty towards consumption fluctuations across different generations. To disentangle RRA and IIA, we use the continuous-time recursive utility framework for Kreps-Porteus (1978) preferences based on temporal resolution of risk developed by Duffie and Epstein (1992, example 3, section 4), which is based on the discrete-time approach of Epstein and Zin (1989). The basic idea is that in the standard case, welfare *W(t)* can be rewritten recursively. The integrand $e^{-\bar{\rho}(s-t)}U(c(s))$ can be replaced by $U(c(s))-\bar{\rho}W(s)$, which is called the aggregator function. This aggregator function shows up in the (stationary) Hamilton-Jacobi-Bellman equation (19) (partly on the left-hand and partly on the right-hand side), which is a recursive formulation of the optimisation problem. Instead of expected utility, with Duffie-Epstein preferences one maximises

$$(23)\qquad \tilde{W}(t)=\mathrm{E}\left[\int_t^\infty \Phi\left(c(t'),\tilde{W}(t')\right)dt'\right] \text{ with } \Phi(c,\tilde{W})\equiv\frac{\rho}{1-\eta_I}\frac{c^{1-\eta_I}-\left[(1-\eta_R)\tilde{W}\right]^{\frac{1-\eta_I}{1-\eta_R}}}{\left[(1-\eta_R)\tilde{W}\right]^{\frac{\eta_R-\eta_I}{1-\eta_R}}},$$

where $\tilde{W}(t)$ or $\tilde{W}(k,P)$ is the value function, $\Phi(c,\tilde{W})$ is the aggregator function for Duffie-Epstein preferences, $\eta_I>0$ is the IIA, $\eta_R>0$ is the RRA, and $\rho\equiv\bar{\rho}+(\eta_I-1)\bar{g}$. We focus on a preference for early resolution of uncertainty, so assume that the dislike of risk exceeds that of intertemporal fluctuations: $\eta_R > \eta_I$.[18] In fact, we assume that $\eta_R > \eta_I > 1$, which is also the empirically relevant case (e.g., Vissing-Jørgensen and Attanasio, 2003).

The following result extends Proposition 2.

Proposition 3: With Duffie-Epstein preferences the Euler equation and the dynamics of the social cost of carbon, defined as $s \equiv -\tilde{W}_P / \tilde{W}_k$, are given by

$$(16') \qquad \frac{\dot{c}}{c} = \frac{1}{\eta_I}\left[r + \theta + \left(\frac{\eta_R - \eta_I}{\eta_R - 1}\right) H(P) \left(\frac{\tilde{W} - \tilde{V}^0}{\tilde{W}}\right) - \rho \right] \text{ with } \theta = H(P)\left(\frac{\tilde{V}_k^0 - \tilde{W}_k}{\tilde{W}_k}\right),$$

$$(17') \qquad \dot{s} = \left[r + \gamma + \theta + H(P) \right] s - m \text{ with } m \equiv \chi q + \left[H'(P)(\tilde{W} - \tilde{V}^0) - H(P)\tilde{V}_P^0 \right] / \tilde{W}_k, \quad 0 \le t < T,$$

where $\tilde{W}_k = \rho c^{-\eta_I} \left[(1-\eta_R)\tilde{W} \right]^{\frac{\eta_R - \eta_I}{\eta_R - 1}}, \quad 0 \le t < T.$

Proof: See appendix C.

Taking a first-order Taylor-series expansion of $\tilde{V}_k^0 / \tilde{W}_k$ around 1 in (16′) gives

$$(16'') \qquad \frac{\dot{c}}{c} = \frac{1}{\eta_I}(r + \theta^* - \rho) \text{ with } \theta^* \cong H(P)\left[\left(\frac{c}{c^A(k,P,1,0)} \right)^{\eta_I} - 1 \right] \left[1 - \left(\frac{\eta_R - \eta_I}{\eta_R - 1} \right) \left(1 - \frac{\tilde{V}^0}{\tilde{W}} \right) \right].$$

The Euler equation (16′) or (16″) indicates that the IIA, not the RRA, determines how slow current generations are willing to sacrifice consumption to limit future global warming. This equation simplifies to the no-tipping version with $\theta^* = 0$ if the catastrophe never occurs or intergenerational inequality aversion is zero. If IIA = RRA, (16′) and (16″) boil down to the expected utility version (16) with $\theta^* = \theta$. If IIA = 1, RRA has no (direct) effect on the uncertainty term and $\theta^* = \theta \tilde{V}_k^0 / \tilde{W}_k < \theta$.[19] If RRA > IIA > 1, the term in the second square brackets is less than one. Further, this term and thus θ^* decrease in RRA and increase in IIA.[20] Lowering IIA or raising RRA thus depress precautionary saving and the optimal carbon tax. Our tipping framework assumes that the tip occurs with certainty at some point of time, albeit that the timing of the catastrophe is uncertain. Risk-neutral policy makers will thus take some action to prevent the catastrophe, which is why the uncertainty term θ^* does not vanish if RRA = 0. The long-run optimal SCC follows from (16′) and (17′):

[18] For further discussion in the context of climate economics, see Traeger (2014).

[19] If RRA $\to \infty$, $\theta^* = \theta \tilde{V}_k^0 / \tilde{W}_k < \theta$ also.

[20] If IIA < 1 and RRA > 1, the term in the second square brackets is smaller than one but increases as RRA goes up. Hence, the effect on saving of higher RRA is zero, positive or negative, depending on whether IIA is equal, smaller or bigger than one.

(24) $$s=\frac{m}{\bar{\rho}+(\eta_I-1)\bar{g}-\left(\frac{\eta_R-\eta_I}{\eta_R-1}\right)H(P)\left(\frac{\tilde{W}-\tilde{V}^0}{\tilde{W}}\right)+\gamma+H(P)}.$$

It is low if the rate of time preference, $\bar{\rho}$, is high and decay of atmospheric carbon is fast. A higher hazard rate depresses the optimal long-run SCC directly but if $\eta_R>\eta_I>1$, boosts it indirectly due to the fall in welfare after the tip. Current generations are less willing to make sacrifices to curb future global warming for the benefit of future richer generations (captured by $\eta_I\bar{g}$), but marginal damages rise in proportion with trend growth in aggregate output (and thus with $\bar{g}$). If the first effect dominates ($\eta_I>1$), higher trend growth depresses the optimal SCC. If the second effect dominates ($\eta_I<1$), higher trend growth pushes up the optimal SCC. If RRA > IIA, the extra term in the denominator pushes up the optimal SCC. The adjustment to saving needed to deal with the tip is driven by the long-run growth-corrected safe interest rate:[21]

(25) $$r=\bar{\rho}+(\eta_I-1)\bar{g}-H(P)\left[\left(\frac{c}{c^A(k,P,1,0)}\right)^{\eta_I}-1\right]\left[1-\left(\frac{\eta_R-\eta_I}{\eta_R-1}\right)\left(1-\frac{\tilde{V}^0}{\tilde{W}}\right)\right].$$

The first term on the right-hand side of (25) is the pure rate of time preference. The second term captures the net effect of rising affluence and rising marginal global warming damages. The third term is the uncertainty term needed to deal with the looming tip. This term is proportional to the hazard of the catastrophe. If consumption falls at the time of the catastrophe, the long-run safe interest rate is below $\rho=\bar{\rho}+(\eta_I-1)\bar{g}$. This boosts the long-run capital stock. If consumption jumps up at the time of the catastrophe, the long-run safe interest rate is pushed above ρ in which case the long-run capital stock is depressed.

7. Market economy: implementation of climate policies and credit constraints

The second fundamental theorem of welfare economics implies that the social optimum can be realised in a decentralised competitive market economy if externalities and information asymmetries are absent. We thus need an augmented version of this theorem that allows for internalisation of the climate externality in aggregate production. There are many different ways of doing this, but one way of doing this is presented in the following proposition.

[21] In principle consumption might jump up at the time of the calamity, if IIA = RRA is low enough. In that case $\theta<0$ and one has negative instead of positive additional saving. However, the last term in (26) is positive if RRA > IAA and demands extra saving (given RRA > 1) which may offset $\theta<0$.

Proposition 4: The social optimum is decentralised in a competitive market economy if a global specific carbon tax, say τ, is levied on emissions and its value is set to the optimal SCC, s, and the revenue is rebated as lump-sum transfers to the private sector.

Proof: See appendix D.

An alternative way of decentralising the social optimum in a competitive market economy is to implement an efficient market for carbon emission permits or a system of quotas. It is no longer necessarily the case that one can implement the social optimum in the market economy if the economy faces stochastic shocks (e.g., Weitzman, 1974) or the government has no access to lump-sum transfers/taxes. In that case, the government has to resort to distorting taxes on labour and capital income and strike a second-best balance between environmental objectives and reducing the burden of the overall tax system (e.g., Bovenberg and de Mooij, 1994; Bovenberg and van der Ploeg, 1994; Goulder, 1995; Jorgenson et al., 2013).

There is no need to subsidise savings to make sure that the necessary saving adjustment (in order to deal with the pending tip) takes place if the private sector internalises the benefits of preparing for a pending climate catastrophe. However, households might fail to take actions to be prepared for a pending climate catastrophe if they are naïve or they find the costs too small to bother about. To attain the social optimum, the government should then offer, for example, a saving subsidy financed by lump-sum taxes which should be set to the socially optimal precautionary return, θ^*. A more realistic market failure occurs if some households are credit constrained and unable to prepare for the risk of catastrophic tip. In that case, the government would find it socially optimal to step in and smooth consumption for these households. The decentralisation of climate policies then requires the government to appropriately price carbon and supplement this with temporary subsidies to credit-constrained households to enable them to smooth consumption by engaging in the correct amount of saving in the face of the pending catastrophe. Of course, the government should undo financial frictions and irrational private behaviour, regardless of whether there is a climate externality or not.

8. Calibration

We calibrate to the world economy and the global climate system. We need to calibrate the parameters in the production function and the welfare function, and in the accumulation of capital and the stock of atmospheric carbon, with initial conditions. The parameter reflecting gradual damages originates from the earlier literature. The specification of the hazard rate and the size and the impact delay of the productivity shock is tentative, but we try to take reasonable numbers. First, we discuss our approach regarding these issues and then we present our full calibration figures in table 1.[22]

Tol (2009) surveys bottom-up approaches to collecting evidence on the monetised effects of global warming on world economic output from a large variety of studies. Nordhaus (2014) and Nordhaus and Sztorc (2013) derive a ballpark figure from these estimates and add a further 25% to allow for the non-monetised impact of global warming to calibrate the damage function of DICE-2013.[23] Their damage function relating output losses to global mean temperature is calibrated in the range 0 to 3 ^{o}C and does not necessarily hold at higher temperatures.

Using a climate sensitivity of 3, there is the standard relationship that global mean temperature relative to its pre-industrial level is given by 3 ln(P/581)/ln(2) where 581 GtC is the pre-industrial stock of atmospheric carbon. This relationship implies that a doubling of carbon stocks leads to 3 ^{o}C higher temperature. This means that production damages can be expressed as a function of the atmospheric carbon stock.[24] These damages are well captured around 2.5 ^{o}C by the functional form used by Golosov et al. (2014) (see (2)) if $\chi = 3.64\times 10^{-5}$.[25] This implies an annual flow damage of 3.64 % of global GDP (roughly 2.6 billion dollars) for each trillion tons of carbon in the atmosphere. These damages imply an output loss of 1.64% of world GDP at 2.5 ^{o}C (i.e., 1035 GtC) and of 2.35% of world GDP at 3 ^{o}C (1162 GtC). This specification of the damage function does not allow for catastrophic events.

Based on survey evidence, Nordhaus (2008) and Nordhaus and Boyer (2000) suggest that with a probability of 6.8% the damages with global warming of 6^o C corresponding to 2324 GtC are catastrophically large, defined as 30% of world GDP. Following Golosov et al. (2014), one could use the unadjusted damage coefficient $\chi_L = 3.64\times 10^{-5}$ at 1035 GtC and the catastrophic damage coefficient $\chi_H = 2.05\times 10^{-4}$ at 2324 GtC (6 ^{o}C).[26] Given a risk of a catastrophe of 6.8%, this yields an expected damage of $0.932\chi_L + 0.068\chi_H = 4.79\times 10^{-5}$. Rather than revising the damage coefficient upwards from 3.64 to 4.79% of global GDP to allow for the risk of catastrophic damages, we explicitly take account of the risk of a catastrophic drop in output.

We thus set $\chi = 3.64\times 10^{-5}$ and allow for a pending catastrophe of a 30% loss in GDP (Δ = 0.3). This figure may be on the low end of the range of possibilities (e.g., Dietz and Stern, 2015).

[22] Data sources are BP Statistical Review 2015, the World Bank Development Indicators 2015 and IPCC 2015. As carbon pricing is currently negligible, we calibrate our model to business-as-usual.
[23] The fraction of output that is lost is 1/(1+0.00267 $Temp^2$), where $Temp$ is global mean temperature relative to pre-industrial temperature in degrees Celsius.
[24] The atmospheric carbon stock is P = 581 exp[$Temp$ ln(2)/3] GtC. For 2.5 ^{o}C this gives 1035 GtC.
[25] χ is chosen such that 1 – exp($-\chi$ (1035-581)) ≅ 1/(1+0.00267 $Temp^2$), where $Temp$ = 2.5 ^{o}C.
[26] The latter figure follows from solving 1 – exp($-\chi_H$ (2324-581)) = 0.3 for χ_H.

We use the linear hazard function $H(P) = 0.012 + 4.3445 \times 10^{-5} \times (P - 1035),\ P \geq 841.$ This implies $H(841) = 0.0036$ at the current stock of carbon, $H(1035) = 0.012$ at 2.5 °C and $H(2324) = 0.068$ at 6 °C. The probabilities that the tip strikes at these carbon stocks in a particular year, given that the tip has not struck before are then 0.36%[27], 1.2% and 6.8%, respectively (i.e., $H(P(t))dt \approx H(P(t)) \times 1$). This relates to the survey evidence reported in Nordhaus and Boyer (2000) and Nordhaus (2008), and is used by Golosov et al. (2014). The expected arrival time of the tip drops from 280 years now to 83 years at 2.5 °C and 15 years at 6 °C. Global warming thus makes the pending catastrophe more imminent. The cumulative risk of the tip (one minus the survival rate) depends on the particular time path of hazard rates. If the hazard rate rises linearly as temperature rises to 6 °C at the start of the next century, the cumulative risk of a tip during this century is 96% (using the formula given in footnote 9). However, if policy makers adhere to the 2015 Paris agreement to limit global warming to 2 °C, the cumulative risk of a tip during this century drops to 37%.

As a sensitivity exercise, we also give results for when the initial hazard rate is either half that value or zero whilst adjusting the slope coefficient in the hazard function so that $H(2324) = 0.068$ still holds. To complete our specification of the pending catastrophe, we compare $\varphi = 0.1$, where the mean impact of the shock is a decade, and $\varphi = 0.02$, where the mean impact of the shock is half a century. A period of half a century is still short for most of the climate tipping points, but we will show that precautionary saving will switch at this point, and this effect will just be stronger when we take longer delays for the impact. Note that fifty years is a mean lag but the actual impact of the catastrophe stretches out over the whole period from now to infinity.

This specification of the hazard, size and speed of the catastrophe is tentative for the simple reason that climate catastrophes with big GDP losses have not occurred yet. However, our objective with this specification of the catastrophe is, on the one hand, methodological to show how this results in an upward bias of the SCC and the need for adjustments in saving, and, on the other hand, quantitative to see how our estimates differ in magnitude from the upward revision of the SCC, if an expected value approach is adopted as in Golosov et al. (2014).

The first two rows of table 1 summarise our specification so far. The third row indicates our assumption of a mean life for atmospheric carbon of 200 years. The fourth row sets our benchmark coefficients of the IIA and RRA to 2 and the pure rate of time preference to 1% per year. We also study in section 10 sensitivity with respect to changing the RRA to 3 and the IIA to 1.33, which are close to the estimates of 3 and 1.5 in Pindyck and Wang (2013, Table 1). To

[27] This is a bit higher than the figure of 0.25% suggested by the expert views solicited in Kriegler et al. (2009) and used in Lontzek et al. (2015).

match the benchmark market interest rate of 5% per year for when policy makers do not take account of the pending tip, we raise for the case where IIA is lowered the pure rate of time preference from 1% to 2.34% per year. This keeps ρ constant and allows us to focus at the uncertainty effect (on θ^*).

Table 1: Calibration and functional forms

Gradual global warming damages	$\chi = 3.64 \times 10^{-5}$, so flow damage of 3.64% of GDP/TtC
Catastrophic output losses	$\Delta = 0.3$, $\varphi = 0.1$ and $\varphi = 0.02$ $H(P) = 0.012 + 4.3445 \times 10^{-5} \times (P - 1035),\ P \geq 841.$
Carbon dynamics	$\gamma = 1/200$, $P_0 = 841$ GtC
Social preferences	$\eta_I = \eta_R = 2$, $\bar{\rho} = 0.01$ (or $\eta_I = 1.33$, $\eta_R = 3$, $\bar{\rho} = 0.0234$), $\rho = 0.03$
Production function	$g(k, f, x) = \Xi k^{\alpha} \left[\omega f^{1-1/\varepsilon} + (1-\omega) x^{1-1/\varepsilon} \right]^{\frac{\beta}{1-1/\varepsilon}}$, $\alpha = 0.3$, $\varepsilon = 3.5$, $d_F = 0.504$, $d_X = 17.8$, $\beta = 0.0688$, $\omega = 0.9352$, $\bar{g} = 0.02$ (also 0.01), $\delta = 0.065$, $\Xi = 14.51$, $K_0 = 160$ trillion \$.

The aggregate production function is Cobb-Douglas in capital and the energy aggregate, and the production function for the energy aggregate is CES in fossil fuel and renewable energy (see appendix E). We suppose a 30% share of capital ($\alpha = 0.3$), so the golden rule, $\alpha q / k = \rho + \delta + \bar{g} = 0.115$, gives a long-run capital stock in efficiency units of 197 trillion US dollars. The initial capital stock is set to 160 trillion US dollars, which is 80% of that value to reflect catching up in part of the world. We set the elasticity of substitution between fossil fuel and renewable energy, ε, to 3.5. Papageorgiou et al. (2015) estimate this elasticity to be 2 for the electricity generating sector and almost 3 for the non-energy sectors. We set it a bit higher in view of the long time-scales that are involved when analysing climate policy and the resulting increased possibilities for substitution. We suppose production cost for fossil fuel of 8.9 \$/BTU[28] or 504 \$/tC and for renewable energy of $d_X = 17.8$ \$/BTU. Global fossil fuel use f in 2013 is 9.9 GtC or 560 million Giga BTU and renewable use in 2013 is approximately 12 million Giga BTU. The fossil fuel and renewable energy shares in aggregate output are 6.59% and 0.29%, respectively. The total energy share β in aggregate output is thus 6.88%, so $\beta = 0.0688$. We use this to calibrate around 2013 with a negligible carbon price to get the

[28] We take an average for the costs of oil, natural gas and oil. These costs are roughly 14, 6.5 and 4 US \$ per million BTU, which yields a weighted average of 8.9 US \$ per million BTU.

share parameter for fossil fuel in the CES sub-production function: $\omega = 0.9352.$ Finally, the constant in the production function $\Xi = 14.51$ is set to match 2013 world GDP.

In the benchmark we have a trend rate of growth of $\bar{g} = 2\%$ per year, which is the average historical growth rate of the world economy. Our benchmark rate to discount damages is thus $\rho = \bar{\rho} + (\eta_I - 1)\bar{g} = 0.03$ or 3% per year. We also consider a variant with $\bar{g} = 1\%$ per year.

9. Naïve and expected value approaches to the optimal carbon tax and post-tip outcomes

Before we present our core estimates of the optimal carbon tax and required adjustments to saving needed to deal with the tip, we discuss before-catastrophe and after-catastrophe outcomes that do not take account of possible tipping. Table 2 gives the steady states of various business-as-usual (BAU) and optimal outcomes, both before and after the tip. The first column gives the before-tip BAU scenario for when no action is undertaken to curb gradual climate damages or the risk of a calamity. The carbon stock rises to 1600 GtC. The next column shows what we call the naïve optimisation outcome when gradual damages are internalised with a carbon tax of 85 $/tC that grows at a rate of 2% per year, but no action is undertaken to lower the risk or adjust saving to prepare for the possibility of a catastrophe. The carbon stock drops to 1226 GtC. The third column does not take account of the possibility of tipping either but adjusts the damage coefficient used to calculate the optimal SCC upwards to take account of possible higher future damages, i.e. $\mathrm{E}[\chi] = 4.79 \times 10^{-5}$. Actual climate damages to production are unaffected as the tip has not happened yet. This leads to a higher optimal SCC of 111 $/tC, which curbs the carbon stock from 1600 to 1128 GtC. The optimal carbon tax rises roughly at the rate of 2% per year. As a result of the more aggressive carbon tax, long-run aggregate output, capital and consumption are somewhat higher than in the naïve outcome and higher still than under BAU.

Table 2 also gives the BAU and optimal scenarios for after the tip. The catastrophe results in big drops in output and thus in aggregate capital stock and in fossil fuel and renewable use. As a result of the fall in economic activity, the long-run carbon stock under BAU falls to 945 GtC. The optimal after-catastrophe carbon tax is driven by lower marginal damages and is also lower as a result of the reduced economic activity, i.e., 50 $/tC. This induces a drop in the carbon stock from 945 to 803 GtC whilst economic activity remains low.

The final column of table 2 shows the results for the expected value procedure used by Nordhaus (2008), Nordhaus and Boyer (2000), and Golosov et al. (2014). The expected flow damage coefficient, $\mathrm{E}[\chi] = 4.79 \times 10^{-5}$, allows for catastrophic risk and bumps up our ballpark estimate of the optimal long-run SCC to 109 $/tC. The difference with the third column is that now the current production damages from global warming are driven by the expected value coefficient, E[χ], rather than by the lower actual value coefficient, χ_L.

Table 2: BAU and optimal before- and after-catastrophe steady states

	Before catastrophe			After catastrophe		Expected value
	BAU	Naïve	Adjusted	BAU	Optimal	
Capital k (T $)	209.9	212.0	212.5	123.9	124.0	208.5
Carbon stock P (GtC)	1600	1226	1128	945	803	1114
Consumption c (T $)	57.1	58.2	58.4	33.7	33.9	57.3
Carbon tax s ($/tC)	0	84.6	111.4	0	49.5	109.3
World GDP q (T $)	80.5	81.3	81.5	47.5	47.5	79.9
Temperature (ºC)	4.4	3.2	2.9	2.1	1.4	2.8
χ in damages	χ_L	χ_L	χ_L	χ_L	χ_L	E[χ]
χ in rule for SCC	0	χ_L	E[χ]	0	χ_L	E[χ]
Δ	0	0	0	0.3	0.3	0

Key: The BAU outcomes have no policy response whatsoever. The naïve outcome ignores tipping and internalises smooth damages with low damages. The adjusted outcome adjusts the damage coefficient upwards to allow for the small risk of catastrophic damages. The optimal outcome after the catastrophe internalises the non-catastrophic damages and is optimal because the tip is no longer a threat. The final expected value outcome uses the expected damage coefficient in both the damages and in the calculation of the SCC. Capital, consumption, the carbon tax and world GDP are measured in 2013 US dollars and efficiency units, so are in 2013 equivalents corrected for productivity growth. The parameter χ is the flow output damage of global warming and **Δ** the size of the catastrophic shock. Sensitivity of the optimal SCC with respect to RRA, IIA, $\bar{\rho}$, and $\bar{g}$ is discussed in appendix F.

The adjustment paths for the after-tip consumption and the optimal carbon tax follow from the numerical after-tip stable manifolds (see appendix G). For our benchmark calibration these manifolds immediately after the catastrophe has struck follow from (G6) and (G5) and are

$$(26)\qquad c^A(T) = c^{A*}\left(\frac{k(T)}{k^{A*}}\right)^{0.438}\left(\frac{P(T)}{P^{A*}}\right)^{-0.0251} e^{\frac{0.0458}{0.119+\varphi}} e^{\frac{-0.00006}{0.035+\varphi}} \quad \text{and}$$

$$(27)\qquad s^A(T) = s^{A*}\left(\frac{k(T)}{k^{A*}}\right)^{0.719}\left(\frac{P(T)}{P^{A*}}\right)^{-0.0166} e^{\frac{0.1066}{0.119+\varphi}} e^{\frac{-0.0247}{0.035+\varphi}},$$

where 0.119 and 0.035 are the two eigenvalues with positive real part of the Jacobian of the post-tip system and the penultimate column of table 2 gives the after-catastrophe steady states (denoted by an asterisk). With a mean lag of a decade, the last two terms in (26) and (27) amount to an upward adjustment of 23% and 35%, respectively.[29] With a mean impact lag of half a century, the adjustments necessary to put the after-tip economy on its stable manifold are 39% and 37%, respectively. A slower speed of impact, φ, from 0.1 to 0.02 thus curbs the need for extra saving (as $c^A(T)/c(T)$ increases) and reduces the required carbon tax immediately after the tip. Using (26) and (27), we can evaluate the after-tip value function from (14) which is needed to solve the optimal before-tip outcomes. Note that for $\varphi = 0.0254$ consumption c^A will be equal to the naïve steady-state consumption level 58.2, so that speeds of impact higher than 0.0254 corresponding to a mean impact lag of 39 years imply that consumption jumps up, if the pre-tip economy is close to the steady state (see section 5). If the economy is in the early stages of economic development, consumption may also jump up if the cut-off is lower than 39 years.

10. Quantitative assessment of before-tip carbon tax and saving adjustments

10.1. Transient responses

The optimal policy simulations for different dates at which the catastrophe starts striking the economy (10, 25 and 90 years) are shown in figure 2.[30] The transient optimal pre-tip paths corresponding to a catastrophe with a mean lag of a decade and a mean lag of half a century are denoted by dashed-dotted and dashed lines, respectively. The post-tip outcomes, starting when the tip starts striking in 2023, 2038 or 2303, are given by dotted lines. The solid lines present the transient paths for the naïve optimal outcomes. If it takes longer on average for the tip to have its full impact (50 instead of 10 years) there is no need for capital accumulation (the dashed line is below the sold line) and thus consumption becomes higher in the pre-tip phase. Consequently, fossil fuel demand, emissions and global warming are less and thus emissions are eventually priced lower in the pre-tip phase, but are priced slightly higher initially. In comparison, the naïve optimal outcome which does not anticipate the tip, prices carbon even lower (witness the solid line being below the dashed and thus also below the dashed-dotted line) as the need to curb the risk of a catastrophe is not taken into account. As a result, the naïve optimal outcome ends up with more global warming, even more than the optimal outcome when the tip is expected to last only a decade and capital accumulation increases.

Figure 2 also indicates that at the time the tip starts striking, i.e., 2023, 2038 and 2303, the carbon tax jumps upwards to make possible the lower carbon taxes in the long run that are

[29] With an abrupt catastrophe ($\varphi \rightarrow \propto$), the adjustments are zero. If the mean lag is a century, the adjustments are 42% and 32%, respectively. Due to the negative sign on the smaller eigenvalue, the carbon tax adjustment is non-monotonic.
[30] The numerical algorithm we use for the policy simulations in this section is based on log-linear approximations around the steady state and is outlined in appendix G.

Figure 2: Optimal responses to pending catastrophe

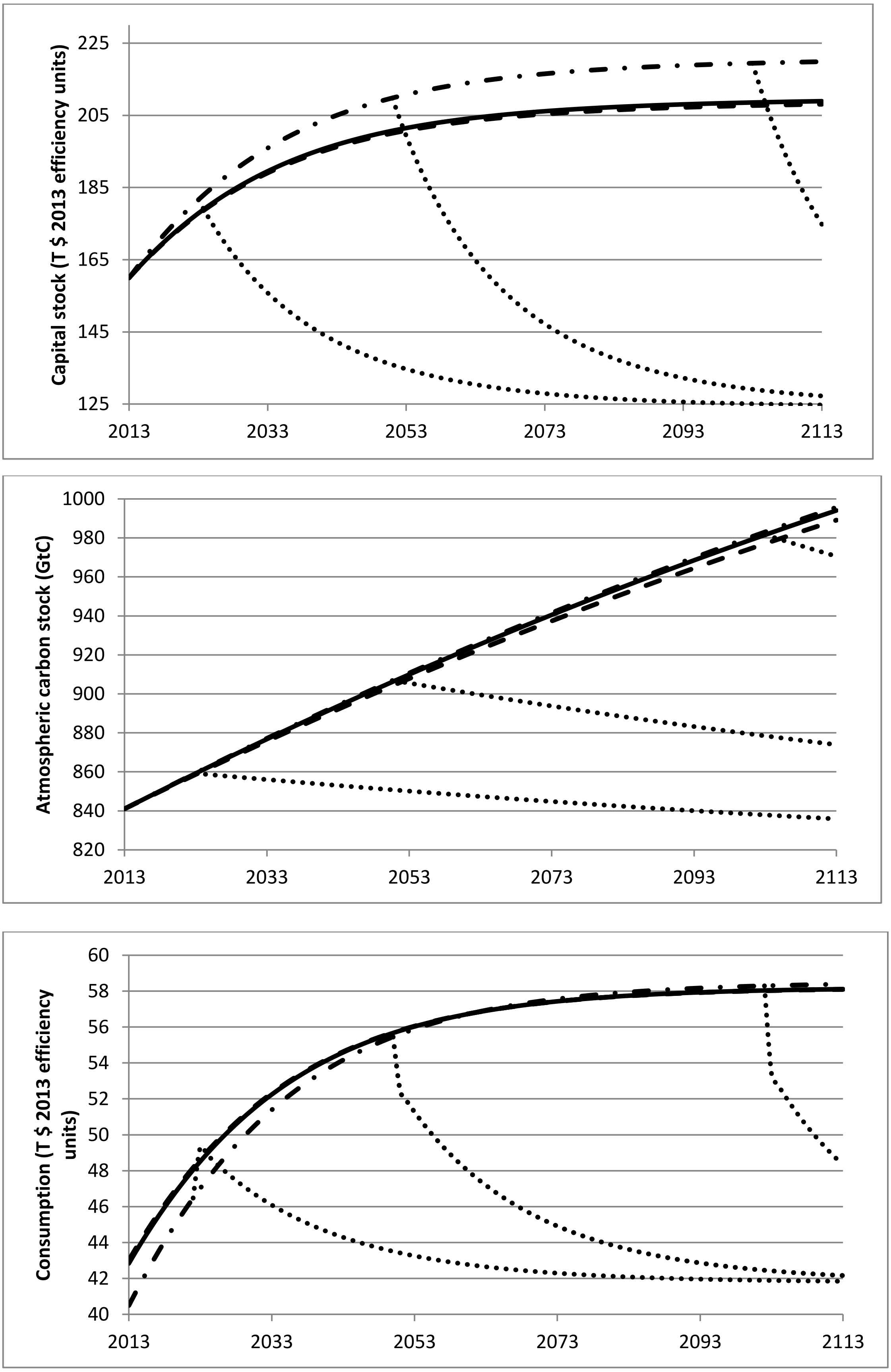
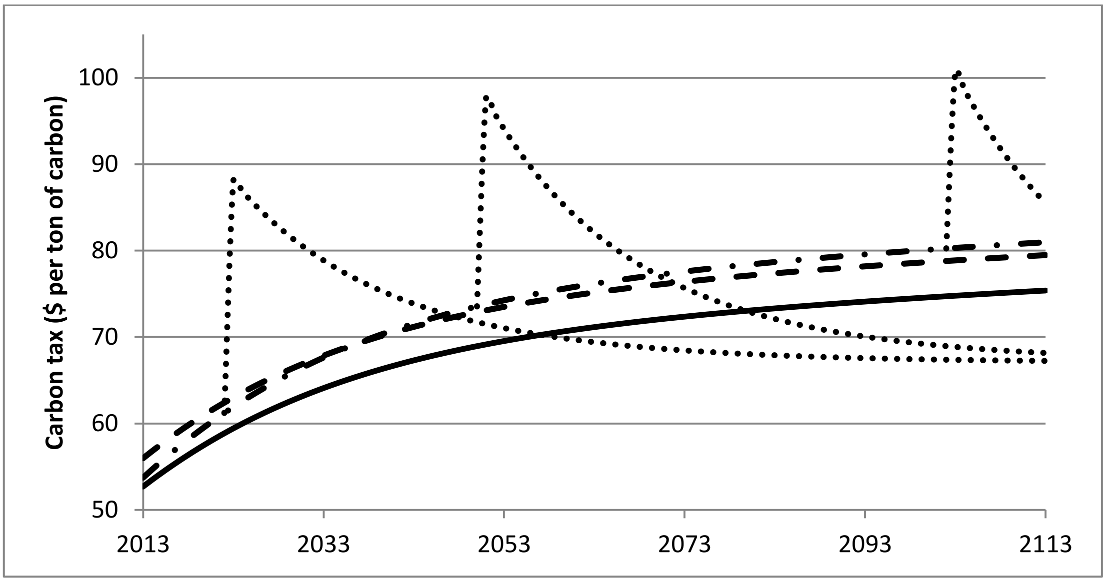


Figure 2 cont.: Optimal responses to pending catastrophe



Key: Dashed-dotted and dashed lines give the before-tip optimal outcomes for a mean impact lag of 10 and 50 years. Dotted lines indicate after-tip outcomes for when the tip starts in 2023, 2038 or 2103 and the mean impact lag is 10 years. Solid lines give the naïve optimal outcomes.

required when the economy runs at a much lower level of economic activity including fossil fuel use and emissions.[31] Table 3 indicates that the upward jump in the carbon tax is bigger for a catastrophe whose full impact is felt more quickly and also that the jumps are smaller the longer it takes for the catastrophe to materialise.

Table 3: Jumps in consumption and carbon taxes for different realisations of tip date (%)

	Mean impact lag of tip is decade				Mean impact lag of tip is 50 years			
Date tip	T = 10	T = 25	T =90	T= ∞	T = 10	T = 25	T = 90	T = ∞
$\Delta c(T)$	4.8	-0.9	-8.5	-8.6	13.1	6.3	1.2	1.0
$\Delta \tau(T)$	42.4	39.9	25.4	11.8	39.5	33.9	24.3	11.4

Consumption jumps up on impact of the tip by 4.8% if it starts striking early as in 2013, but has to jump down by 0.8% or 8.5% if the tip starts striking not until 2038 and 2103, respectively. So if the tip occurs quickly, consumption must make a discrete jump upwards as the catastrophic shock in total factor productivity necessitates decumulation of capital as the economy moves to a much lower level of activity in the post-tip phase. If the tip occurs later, consumption jumps

[31] Lemoine and Traeger (2014) also have an upward jump in the carbon tax at the time of the tip, but their tip corresponds to a sudden increase in climate sensitivity which then works slowly through the climate system over a longer time scale whilst our tip corresponds to a drop in total factor productivity whose full impact takes time to materialise. Another difference is that our interest is in highlighting the consumption and capital responses to a pending tip in total factor productivity.

downwards. However, with a mean impact lag of half a century, consumption never jumps downwards when the tip occurs. In fact, for longer mean impact lags the upward jumps become even bigger. We conclude that jumps in consumption are thus smaller and become negative for a later realisation date of the tip and a quicker impact of the tip on damages.

10.2. Sensitivity of before-catastrophe steady states

Table 4 gives the steady states corresponding to the before-tip paths of figure 1 (column 2). Comparing the base line with a fast impact to the naïve optimisation outcome (column 1), we see that the rate of return on capital drops from 3 to 2.6% per year so it is optimal to accumulate 5% more capital to be better prepared for the catastrophe when it comes. Furthermore, it is optimal to price carbon in the long run at 91 \$/tC rather than at \$85/tC. The atmospheric carbon stock nevertheless only drops a bit from 1226 to 1222 GtC, since the additional capital accumulation engenders more fossil fuel demand and thus more carbon emissions.

We have already noted in our discussion of (28) and (29) that a slower impact curbs the need for additional saving to deal with the tip. In fact, a slower impact of the tip corresponding to a mean impact lag of 50 instead of 10 years eliminates the need for precautionary capital accumulation in the long run (column 3). Capital falls a little with respect to naïve optimisation as can be seen from the slight increase in the long-run interest rate from 3 to 3.04% per year. Hence, fossil fuel demand does not rise as much and thus the long-run carbon tax only has to be 88.8 \$/tC.

Table 4: Sensitivity of before-catastrophe steady states

	Naïve	Base line: fast impact (φ=0.1)	Slower impact (φ=0.02)	Zero initial hazard	IIA = 1.33	IIA = 1.33 $\bar{\rho}$=2.34%	RRA = 3
Capital k (T \$)	212.0	223.14	211.0	222.0	258.0	219.5	223.1
Carbon stock P	1226	1222	1208	1230	1124	1219	1222
Consumption c (T \$)	58.2	58.5	58.2	58.4	59.5	58.4	58.5
Carbon tax s (\$/tC)	84.6	90.8	88.8	88.1	132.1	89.9	90.7
Interest rate (%/year)	3	2.60	3.04	2.64	1.57	2.73	2.60

A more pessimistic growth scenario of 1% per year (not reported in table 4) depresses the benchmark growth-corrected discount rate from 3% to 2% per year. The uncertainty effect ensures that the long-run interest rate is even lower: 1.67% per year. The optimal carbon tax is pushed up from 91 to 138 \$/tC and capital increases in the long run substantially from 223 to 296 T\$. This still corresponds to an increase of 5% relative to BAU, since lower growth also leads to higher capital accumulation (281 T$). As a result of the high carbon tax, global warming is much less, with a long-run stock of atmospheric carbon of only 1150 GtC.

Reducing the initial hazard rate to zero, whilst raising the coefficient on P in the hazard function to 4.5853E-5 so that $H(2324) = 0.068$ still holds, needs less precautionary capital accumulation as the interest rate rises from 2.60 to 2.64% per year (column 4 of table 4). Consequently, it yields less fossil fuel use and emissions, and thus the carbon tax can be set a bit lower. Halving the initial hazard has effects which are very similar. In contrast, doubling the hazard rates at each level of global warming (not reported in table 4) leads to a quicker onset of the catastrophe and therefore policy makers respond with more precautionary capital accumulation (231 T$ instead of 223 T$) resulting from a lower long-run interest rate (2.35% instead of 2.6% per year) and a higher carbon tax (102 $/tC instead of 91 $/tC). As a result, global warming is less with a lower long-run atmospheric carbon stock (1194 GtC instead of 1222 GtC).

The last three columns of table 4 give the steady-state before-tip outcomes for Duffie-Epstein preferences. Lowering the IIA from 2 to 1.33 so that $\rho = \bar{\rho} + (\text{IIA} - 1)g$ decreases from 3% to 1.66% per year leads to a much more ambitious response to the pending climate catastrophe (column 5).[32] First, precautionary capital accumulation compared with the outcome under naïve optimisation is 22% instead of 5.3% as can be seen from the much lower rate of interest (1.57 instead of 2.6% per year). Second, as current generations sacrifice less to curb future global warming, the optimal carbon is tax much higher (132 instead of 91 $/tC). Hence, the long-run atmospheric carbon stock is substantially lower.

Of course, this large boost to capital formation is mostly due to the lower discount rate. The next column therefore shows the pure uncertainty effects of lowering intergenerational inequality aversion from 2 to 1.33, whilst adjusting the pure rate of time preference to keep ρ at 3% per year, and thus shows the pure uncertainty effects of this change in social preferences on optimal climate policies (column 6). In line with our discussion of the Duffie-Epstein Euler equation (16"), we now have much lower capital accumulation (3.5% higher than the naïve outcome) to deal with the tip and also a lower optimal carbon tax. Raising relative risk aversion from 2 to 3, keeping IIA = 2, also has minimal effects with slightly less precautionary capital accumulation and a somewhat lower carbon tax (column 7), which are also in the direction explained in our discussion of (16").

[32] Traeger (2014) applies Epstein-Zin preferences to a non-tipping problem with RRA = 9.5 and IIA = 0.67 from Vissing-Jørgensen and Attanasio (2003). These estimates are based on financial decisions of private investors. In our tipping problem there is no solution if IIA is too low.

11. Concluding remarks

The optimal response to a pending catastrophe consists of a two-pronged strategy. The first part is to price carbon more vigorously than is justified by marginal global damages alone in order to curb the risk and prolong the arrival of a catastrophe. This depends crucially on the sensitivity of the hazard rate with respect to global warming. Hence, if climate change is still far away but approaches more rapidly when the globe warms up, optimal climate policy requires a high price of carbon. The second part of the strategy is to adjust saving and capital accumulation in order to dampen the discrete change in consumption at the time of the calamity and be better prepared for when disaster starts to strike. The magnitude of this prudence effect depends on the hazard rate itself as well as size of the catastrophic drop and the time it takes for this to materialise. The required saving adjustment is positive if full impact of the tipping point is felt immediately or the mean impact lag is not too large but is negative if the mean impact lag is larger.

Consumption and capital increase in the pre-tip phase as the economy develops. Consumption jumps up immediately after the tip if it strikes in the early phases of economic development when the economy is already saving and investing a lot and jumps down if it strikes later and economic growth is lower. After the tip the interest rate drops instantaneously and from then on the economy dis-saves and consumption and capital fall towards the after-tip steady state. The additional saving response needed to deal with the tip takes full account of all possible future expected upward and downward jumps in consumption corresponding to early and later dates of the realisation of the tip and allows for, respectively, their negative and positive contribution to the additional saving response.

We have shown that this two-pronged strategy is more ambitious for a more patient society with little intergenerational inequality aversion. If this aversion exceeds one, more pessimistic growth prospects for the global economy makes current generations more willing to price carbon vigorously and to engage in more saving to deal with the pending tip. With Duffie-Epstein preferences we show that precautionary saving is increased if aversion to risk exceeds that to intertemporal fluctuations (and both exceed one). Furthermore, the magnitude of this effect is larger if intergenerational inequality aversion is big and risk aversion is low. Higher hazard rates also make climate policy more ambitious. Our framework makes adjustments to the optimal carbon tax and required saving to deal with a looming tipping point explicit.

With our benchmark values for the pure rate of time preference (1% per year), intergenerational inequality and risk aversion (2), and trend growth of the world economy (2% per year), adjusting the damage coefficient upwards to deal with a pending drop in aggregate output of 30% boosts the long-run optimal carbon tax from $85 under naïve optimisation (i.e., ignoring the tip altogether) to $109 $/tC in efficiency units. It does not allow for precautionary saving. Our tipping point approach boosts the long-run optimal carbon tax from $85 to only $91 $/tC in efficiency units and allows for 5% additional precautionary capital accumulation to be better prepared for the looming climate catastrophe if the mean impact lag is a decade.

Catastrophes that have a mean impact lag of half century instead of a decade require negative instead of positive additional saving to deal with the tip and thus less fossil fuel use and carbon emissions. As a result, the carbon tax is set lower. As capital (except perhaps for buildings and some infrastructure) is much shorter lived than the time it takes for climate damages to build up, capital accumulation seems a poor hedge against climate change.[33]

Lowering IIA (to 1.33) boosts precautionary capital accumulation from 5% to 22% and the required long-run optimal carbon tax increases from $91 to 132 $/tC. However, if one focuses on the pure uncertainty effect by adjusting the pure rate of time preference to fix the implied long-run interest rate, one has in line with the comparative statics of our Duffie-Epstein Euler equation only a slightly lower capital stock and carbon tax. Raising relative risk aversion has negligible negative effects on precautionary saving and the carbon tax.

Our calibration and policy simulations are merely illustrative and highlight the drivers of the upwards adjustment of the global carbon tax and the adjustments in saving needed to deal with pending catastrophes. It is important to use more detailed numerical integrated assessment models to investigate these issues, which are typically formulated in discrete rather than continuous time. Cai et al. (2015), Lontzek et al. (2015) and Lemoine and Traeger (2014) have already made important steps in this direction, and Lemoine and Traeger (2016) and Cai et al. (2016b) extend the methodology to allow for multiple tipping points.

Apart from precautionary capital accumulation to better smooth consumption when a catastrophe hits, the world might invest in institutions to curb the risk of climate change (e.g., norms and networks that facilitate the implementation of climate policy or CCS). Similarly, spending money now to slow down global warming acts as an insurance to offset the small risk of a ruinous catastrophe (Weitzman, 2007). Next to these mitigation efforts, specific adaptation capital may be needed to soften the impact of a catastrophe (e.g., seawalls, storm surge barriers, dune reinforcement and creation of marshlands as protection to sea level rises, crop relocation, diversifying tourist attractions, adjusting rail and roads to cope with warming and drainage).[34]

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[33] We are grateful to a referee for pointing this out to us.

[34] The optimal amount of specific adaptation capital increases in the degree of global warming and in the stock of physical capital (van der Ploeg and de Zeeuw, 2014).

A narrative based on costs of climate catastrophes at higher temperatures rather than on costs of gradual damages at moderate temperatures may offer a more convincing narrative to policy makers. More climate science research is thus needed to improve information on the type of catastrophes that can occur, the time it takes to have their full impact and the different hazard and marginal hazard rates of the various disasters that can occur (cf. Lenton and Ciscar, 2013).

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Appendix A: Proof of Proposition 1 for after-catastrophe system

The Hamiltonian function for the problem (5) subject to (6) and (7) is defined as

(A1) $$H \equiv \frac{c^{1-\eta}}{1-\eta} + \lambda\left[e^{-\chi P}Bg(k, f, x) - d_F f - d_X x - (\delta+\bar{g})k - c\right] + \mu(f - \gamma P),$$

where λ and μ are the undiscounted co-states corresponding to (6) and (7). The first-order conditions follow from Pontryagin's Maximum Principle:

(A2) $$\partial H / \partial f = e^{-\chi P}Bg_f(k, f, x) - d_F\lambda + \mu = 0, \quad \partial H / \partial x = e^{-\chi P}Bg_x(k, f, x) - d_X\lambda = 0,$$

(A3) $$\partial H / \partial c = c^{-\eta} - \lambda = 0,$$

(A4) $$\rho\lambda - \dot{\lambda} = \partial H / \partial k = e^{-\chi P}Bg_k(k, f, x)\lambda - (\delta+\bar{g})\lambda,$$

(A5) $$\rho\mu - \dot{\mu} = \partial H / \partial P = -\chi e^{-\chi P}Bg(k, f, x)\lambda - \gamma\mu,$$

Combining (A3) and (A4) one gets the Euler equation (10). Combining (A5) with (A4) and defining $s \equiv -\mu / \lambda$ gives (11). Given that the steady state is a constant, it is straightforward to verify that the transversality conditions for this problem are satisfied.

Appendix B: Post-tip and pre-tip dynamics

Here we analyse the post-tip and pre-tip dynamics of a special case of our model with $H'(P) = \chi = 0,$ no energy demands, and Cobb-Douglas production function $g(k) = \Xi k^{\alpha}$.

Post-tip dynamics

Proposition 1 for the post-tip economy simplifies to $\dot{k} = \Xi k^{\alpha} - (\delta+\bar{g})k - c,$ given $k(T),$ and $\dot{c} = (r-\rho)c/\eta,$ where $r(k) = \alpha\Xi k^{\alpha-1} - (\delta+\bar{g}).$ Using hats to indicate deviations from the post-tip steady state (e.g., $\hat{k} \equiv k - k^{A^*}$), linearisation around the post-tip steady state yields

(B1) $$\dot{\hat{k}} = \rho\hat{k} - \hat{c}, \text{ given } k(T), \quad \text{and} \quad \dot{\hat{c}} = -(1-\alpha)(\rho+\delta+\bar{g})\hat{k}c^{A^*} / \eta k^{A^*}.$$

Conjecturing that $\lambda^A > 0$ is the slope of the post-tip saddle-path and substituting this into (B1), we get $\dot{\hat{c}} = -(1-\alpha)(\rho+\delta+\bar{g})\hat{k}c^{A^*} / \eta k^{A^*} = \lambda^A\dot{\hat{k}} = \lambda^A(\rho-\lambda^A)\hat{k}.$ Picking the positive solution of the quadratic $\lambda^{A^2} - \rho\lambda^A - (1-\alpha)(\rho+\delta+\bar{g})c^{A^*} / \eta k^{A^*} = 0$ yields the slope of the saddle-path:

(B2) $$\lambda^A = \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta+\bar{g})c^{A^*} / \eta k^{A^*}} > \rho.$$

Hence, the saddle-path is steeper if the pure rate of time preference ρ, the depreciation rate δ and the trend growth are higher and the capital share in value added α and the IIA = η are smaller. It is also steeper than the locus $\dot{\hat{k}} = 0.$ The speed of convergence along the post-tip saddle-path corresponds to the absolute value of the negative eigenvalue:

(B3) $$\frac{1}{2}\sqrt{\rho^2 + 4(1-\alpha)(\rho+\delta+\bar{g})c^{A^*} / \eta k^{A^*}} - \frac{1}{2}\rho.$$

Before-tip dynamics

Equation (16) of Proposition 2 boils down to

$\dot{c} = \frac{c}{\eta}\left(r(k) + \theta - \rho\right)$ with $\theta = H\left[\left(\frac{c}{c^A(k,.,1,0)}\right)^{\eta} - 1\right]$, Denoting the before-tip prudent steady state by an asterisk, the post-tip dynamics can be linearised as:

$$\dot{\hat{k}} = (\rho - \theta^*)\hat{k} - \hat{c}, \text{ given } k(T), \text{ and}$$

(B4) $$\dot{\hat{c}} = -\left[(1-\alpha)(\rho - \theta^* + \delta + \bar{g})\frac{c^*}{\eta k^*} + (\theta^* + H)\frac{c^*}{c^{A^*}}\lambda^A\right]\hat{k} + (\theta^* + H)\hat{c}.$$

Conjecturing that $\lambda^B > 0$ is the slope of the pre-tip saddle-path and substituting in (B4), we get the quadratic $\lambda^{B^2} - (\rho + H)\lambda^B - (1-\alpha)(\rho - \theta^* + \delta + \bar{g})\frac{c^*}{\eta k^*} - (\theta^* + H)\lambda^A = 0.$ Picking the positive solution gives the slope of the pre-tip saddle-path:

(B5) $$\lambda^B = \frac{1}{2}(\rho + H) + \frac{1}{2}\sqrt{(\rho + H)^2 + 4(1-\alpha)(\rho - \theta^* + \delta + \bar{g})c^* / \eta k^* + 4(\theta^* + H)\lambda^A} > \rho.$$

If $H = 0$, (B5) and (B2) are the same. A positive hazard H pushes up the pre-tip slope and this typically ensures that $\lambda^B > \lambda^A.$ Hence, the $c^B(k)$ locus is steeper than the $c^A(k)$ locus in figure 1.

Appendix C: Derivation optimal climate policy with Duffie-Epstein preferences

After-tip problem

The after-tip objective, $\tilde{V}(t) = \mathrm{E}\left[\int_t^\infty \Phi\left(c(t'), \tilde{V}(t')\right)dt'\right], \ t \geq T,$ gives the HJB equation

(C1) $$\rho\left(\frac{1-\eta_R}{1-\eta_I}\right)\tilde{V} = \underset{c,f,x}{\mathrm{Max}}\left\{\frac{\rho}{1-\eta_I}c^{1-\eta_I}\left[(1-\eta_R)\tilde{V}\right]^{\frac{\eta_I - \eta_R}{1-\eta_R}} + \tilde{V}_P\dot{P} + \tilde{V}_k\dot{k}\right\}, \quad t \geq T.$$

This yields the optimality conditions $\rho c^{-\eta_I}\left[(1-\eta_R)\tilde{V}\right]^{\frac{\eta_I-\eta_R}{1-\eta_R}}=\tilde{V}_k$ and $s=-\tilde{V}_P/\tilde{V}_k,\ \ t\geq T.$ Together with equations (6)-(8) this allows us to solve for $\tilde{V}$ from the after-tip HJB equation:

(C2) $$\tilde{V}=\frac{1}{1-\eta_R}\left[(1-\eta_I)\rho V\right]^{\frac{1-\eta_R}{1-\eta_I}},\quad t\geq T.$$

The optimality condition for consumption thus boils down to the usual $c^{-\eta_I}=V_k,\ \ t\geq T,$ which is independent of relative risk aversion as uncertainty has by that time been resolved.

The before-tip problem

The before-catastrophe HJB equation is

(C3) $$0=\underset{c,f,x}{\text{Max}}\left\{\Phi(c,\tilde{W})+\tilde{W}_P\dot{P}+\tilde{W}_k\dot{k}-H(P)(\tilde{W}-\tilde{V}^0)\right\},\quad 0\leq t<T.$$

The optimality condition for pre-tip aggregate consumption in (B3) is

(C4) $$c^{-\eta_I}=\left[(1-\eta_R)\tilde{W}\right]^{\frac{\eta_R-\eta_I}{1-\eta_R}}(\tilde{W}_k/\rho),\quad 0\leq t<T.$$

Using (23) and equations (6) and (7), the HJB equation (C3) can be rewritten as

(C5) $$\rho\left(\frac{1-\eta_R}{1-\eta_I}\right)\tilde{W}=\underset{c,f,x}{\text{Max}}\left\{\frac{\rho}{1-\eta_I}c^{1-\eta_I}\left[(1-\eta_R)\tilde{W}\right]^{\frac{\eta_I-\eta_R}{1-\eta_R}}+\tilde{W}_P(f-\gamma P)\right.$$
$$\left.+\tilde{W}_k\left[e^{-\chi P}g(k,f,x)-d_F f-d_X x-(\delta+\overline{g})k-c\right]-H(P)(\tilde{W}-\tilde{V}^0)\right\},\quad 0\leq t<T.$$

Totally differentiating (C5) with respect to time and using the optimality condition (C4), we get

(C6) $$\left[\left\{\left(\frac{1-\eta_R}{1-\eta_I}\right)\rho+H-\Gamma\right\}\tilde{W}_k-H\tilde{V}_k^0-\dot{\tilde{W}}_k-r\tilde{W}_k\right]\dot{k}$$
$$+\left[\left\{\left(\frac{1-\eta_R}{1-\eta_I}\right)\rho+H+\gamma-\Gamma\right\}\tilde{W}_P-H\tilde{V}_P^0+H'(\tilde{W}-\tilde{V}^0)-\dot{\tilde{W}}_P+\chi q\tilde{W}_k\right]\dot{P}=0,$$

where $\Gamma\equiv\frac{\rho c^{1-\eta_I}}{1-\eta_I}(\eta_I-\eta_R)\left[(1-\eta_R)\tilde{W}\right]^{\frac{\eta_I-1}{1-\eta_R}}$. Due to the gradual impact of the tip, $\tilde{V}_P$ also depends on time. Since we are interested in $\tilde{V}_P^0$ only, we can abstract from this term in (C6). Insisting that (C6) holds for every $\dot{k}$ and $\dot{P}$ leads to the Pontryagin conditions which using (C4) give the dynamics of consumption, *c*,

(C7) $$\frac{\dot{c}}{c}=\frac{1}{\eta_I}\left[r+\theta+\left(\frac{\eta_R-1}{1-\eta_I}\right)\rho+\Gamma-\left(\frac{\eta_R-\eta_I}{1-\eta_R}\right)\frac{\dot{\tilde{W}}}{\tilde{W}}\right]\ \text{ with }\ \theta\equiv H(P)\left(\frac{\rho\tilde{V}_k^0-\tilde{W}_k}{\tilde{W}_k}\right),$$

and equation (17′) for the SCC, $s \equiv -\tilde{W}_P / \tilde{W}_k$. Using the definition of Γ and

(C8) $$\dot{\tilde{W}} = \rho\left(\frac{1-\eta_R}{1-\eta_I}\right)\tilde{W} - \frac{\rho}{1-\eta_I}c^{1-\eta_I}\left[(1-\eta_R)\tilde{W}\right]^{\frac{\eta_I-\eta_R}{1-\eta_R}} + H(P)(\tilde{W}-\tilde{V}^0),$$

we thus establish that the Euler equation (C7) boils down to equation (16′).

Appendix D: Implementing the command optimum in the market economy

Consider the following competitive market economy. Firms choose factors of production *K*, *L*, *F* and *X* to maximise profits, $e^{-\chi P}BG(K,F,X,AL)-(d_F+\tau)F-d_X X-wL-(r_M+\delta)K,$ where *w* is the wage and r_M the (growth-corrected) market interest rate. They operate under perfect competition and take global warming and the carbon tax as given. Firms thus set the marginal product of capital to its user cost, $e^{-\chi P}BG_K(K,F,X,AL)=r_M+\delta,$ the marginal product of fossil fuel to its user cost, $e^{-\chi P}BG_F(K,F,X,AL)=d_F+\tau,$ the marginal product of renewable energy to its cost, $e^{-\chi P}BG_X(K,F,X,AL)=d_X,$ and the marginal product of labour to the wage, $e^{-\chi P}ABG_{AL}(K,F,X,AL)=w.$ After the tip, households maximise lifetime utility (4) subject to their budget constraint $\dot{\Upsilon}=r_M\Upsilon+wL+Z-C,$ where Υ are the financial assets held by them and *Z* the lump-sum payments received from the government. This yields the after-tip Euler equation, $\dot{C}=(r_M-\bar{\rho})C/\eta.$ Before the tip, households maximise expected lifetime utility taking account of the hazard $H(P).$ This gives the before-tip Euler equation $\dot{C}=(r_M+\theta_M-\bar{\rho})C/\eta,$ where $\theta_M=H(P)\left[\frac{V^0_{M,K}-U'(C)}{U'(C)}\right]$ is the precautionary return and $V_M(K,P,B,t-T)$ the after-tip value function for households. Ignoring government debt (w.l.o.g. due to Ricardian equivalence), the government budget constraint is $\tau F=Z.$ Capital market equilibrium demands that assets held by households equal the equity supplied by firms, $\Upsilon=K.$ Labour market equilibrium requires $L = 1$. Goods market equilibrium requires that supply equals total consumer and investment demand, $Q-d_F F=C+\dot{K}+\delta K.$ We will now establish that if one sets $\tau=s,$ the market economy exactly replicates the social optimum of sections 3, 4 and 5.

The after- and before-catastrophe carbon taxes follow from (11) and (17) or (17′), respectively. The carbon dynamics (3) or (7) hold in both the command optimum and market economy. The household budget constraint, the capital market equilibrium condition and the government budget constraint imply that the goods market is in equilibrium (Walras' law):

(D1) $$\begin{aligned}\dot{\Upsilon}=\dot{K}=r_M K+wL+Z-C &= Q-(d_F+\tau)F-d_X X-\delta K+Z-C \\ &= Q-d_F F-d_X X-\delta K.\end{aligned}$$

This is the material balance equation (2) or (6) of the command optimum. The growth-corrected market interest rate corresponds to the social interest rate, $r_M - \bar{g} = r,$ so the after-catastrophe Euler equation for the market economy in intensive form, i.e., $\dot{c}/c = (r_M - \bar{\rho})/\eta - \bar{g} =$ $(r + g - \bar{\rho})/\eta - \bar{g} = r - \rho,$ corresponds to the after-catastrophe Euler equation for the social optimum (10). The corresponding before-catastrophe Euler equation for the market economy can be written in intensive form as $\dot{c}/c = (r_M + \theta_M - \bar{\rho})/\eta - \bar{g}.$ Since $\theta_M = \theta$ and $V_M = V,$ this corresponds to the before-catastrophe Euler equation for the social optimum (16). One can easily verify that with Duffie-Epstein preferences the Euler equation for private agents with the appropriate price of carbon imposed also corresponds to the socially optimal Euler equation (16′). This establishes that the command optimum is replicated in the market economy if there is a specific carbon tax that is set to the optimal SCC and revenues are rebated as lump sums.

Appendix E: Energy demands and aggregate production

We use the following production function:

(E1) $$g(k, f, x) = \Xi k^{\alpha} \left[\omega f^{1-1/\varepsilon} + (1-\omega) x^{1-1/\varepsilon} \right]^{\frac{\beta}{1-1/\varepsilon}}.$$

The optimality conditions for energy are

(E2) $$\frac{\omega \beta q f^{-1/\varepsilon}}{\omega f^{1-1/\varepsilon} + (1-\omega) x^{1-1/\varepsilon}} = d_F + s \quad \text{and} \quad \frac{(1-\omega) \beta q x^{-1/\varepsilon}}{\omega f^{1-1/\varepsilon} + (1-\omega) x^{1-1/\varepsilon}} = d_X.$$

These give $x = \left(\frac{1-\omega}{\omega} \right)^{\varepsilon} \left(\frac{d_F + s}{d_X} \right)^{\varepsilon} f,$ so conditional energy demands are

(E3) $$f = \frac{\beta q}{(d_F + s) D} \quad \text{and} \quad x = \frac{\beta q (D-1)}{d_X D} \quad \text{with} \quad D \equiv 1 + \left(\frac{1-\omega}{\omega} \right)^{\varepsilon} \left(\frac{d_F + s}{d_X} \right)^{\varepsilon - 1}.$$

Using this to calibrate for 2013 with no carbon price gives $\omega = 1 / \left[1 + \frac{d_X}{d_F} \left(\frac{x}{f} \right)^{\frac{1}{\varepsilon}} \right] = 0.9352.$

Putting (E3) into the production function gives $q = \left\{ e^{-\chi P} B \Xi k^{\alpha} \left(\frac{\beta}{d_F + s} \right)^{\beta} \left[\omega^{\varepsilon} D \right]^{\frac{\beta}{\varepsilon - 1}} \right\}^{\frac{1}{1-\beta}}.$ The growth-corrected return on capital is $r = (\alpha q / k) - \delta - \bar{g}.$

Appendix F: Sensitivity of naïve and adjusted optimal before- and after-tip carbon tax

Table F1 shows the sensitivity of the optimal carbon tax reported in table 2 to intergenerational inequality aversion, more patience and lower growth prospects. Lowering (increasing) the IIA implies that society is more (less) willing to sacrifice consumption to curb future global warming and so the optimal carbon tax is pushed up (down). For example, if the IIA is 1 (4) instead of 2, the optimal carbon tax in the naïve outcome rises (falls) from 85 to 198 (40) $/tC.

Lowering the rate of pure time preference from 1% to 0.1% per year as in Stern (2007), the naïve optimal carbon tax rises to 114 $/tC. A drop in the trend rate of growth from 2% to 1% per year, boosts the optimal carbon tax from 85 to 118 $/tC. As future generations will be richer, current generations are prepared to make more sacrifices to curb global warming. Adjustments to the optimal carbon tax are similar for the other non-BAU columns.

Table F1: Sensitivity of the naïve and adjusted optimal long-run carbon taxes

	Before catastrophe			After catastrophe		Expected value
	BAU	Naïve	Adjusted	BAU	Optimal	
Carbon tax *s* ($/tC)	0	84.6	111.4	0	49.5	109.3
IIA = 1	0	198	260	0	116	255
IIA = 4	0	40	52	0	23	51
$\bar{\rho}$ = 0.1% per year	0	114	150	0	67	147
$\bar{g}$ = 1% per year	0	118	156	0	69	153

Appendix G: Numerical algorithm

We solve the post-catastrophe 4-dimensional saddlepath system (8)-(11) by loglinearising it around the steady state (indicated by asterisks). We thus get

$$\text{(G1)} \qquad \dot{x} = Ax + a = \begin{pmatrix} A_{pp} & A_{pn} \\ A_{np} & A_{nn} \end{pmatrix}\begin{pmatrix} x_p \\ x_n \end{pmatrix} + a \text{ with } x_p \equiv \begin{pmatrix} \ln(k/k^*) \\ \ln(P/P^*) \end{pmatrix} \text{ and } x_n \equiv \begin{pmatrix} \ln(c/c^*) \\ \ln(s/s^*) \end{pmatrix},$$

where x_p denotes the vector of predetermined variables and x_n of non-predetermined variables, and the matrix A follows from the state transition matrix of the linearised system, i.e.,

$$\hat{A}_{pp} = \begin{pmatrix} r + s\xi_1 & -\chi q - s\xi_2 \\ \xi_1 & -\xi_2 - \gamma \end{pmatrix}, \quad \hat{A}_{pn} = \begin{pmatrix} -1 & -s\xi_3 \\ 0 & -\xi_3 \end{pmatrix},$$

$$\text{(G2)} \qquad \hat{A}_{np} = \begin{pmatrix} \frac{c}{\eta_I}\frac{\alpha}{k}\left(\xi_4 - \frac{q}{k}\right) & -\frac{c}{\eta_I}\frac{\alpha}{k}\xi_5 \\ \frac{\alpha s}{k}\left(\xi_4 - \frac{q}{k}\right) - \chi\xi_4 & \left(\chi - \frac{\alpha s}{k}\right)\xi_5 \end{pmatrix} \text{ and } \hat{A}_{nn} = \begin{pmatrix} 0 & -\frac{c}{\eta_I}\frac{\alpha}{k}\xi_6 \\ 0 & r + \gamma + \left(\chi - \frac{\alpha s}{k}\right)\xi_6 \end{pmatrix},$$

where $\xi_1 \equiv \frac{\alpha}{1-\beta}\frac{f}{k}$, $\xi_2 \equiv \frac{\chi f}{1-\beta}$, $\xi_3 \equiv \frac{f}{d_F + s}\left[1 + \frac{\beta}{(1-\beta)D} + \left(\frac{D-1}{D}\right)(\varepsilon - 1)\right]$ and

$\xi_4 \equiv \frac{\alpha}{1-\beta}\frac{q}{k}$, $\xi_5 \equiv \frac{\chi q}{1-\beta}$, $\xi_6 \equiv \frac{\beta q}{(1-\beta)(d_F + s)D}$, and using $A_{ij} = \hat{A}_{ij} x_j^* / x_i^*$ to convert to the loglinearised system. The time-varying forcing vector for the post-catastrophe system is

(G3) $\quad a(t) = e^{-\varphi(t-T)}\hat{a}, \quad t \geq T \quad \text{with} \quad \hat{a} \equiv \Delta\left(\frac{q}{Bk} + \frac{sf\xi_5}{k}, \frac{f\xi_5}{P}, \frac{\alpha q}{\eta_I k}\xi_5, \left(\frac{\alpha s}{k} - \chi\right)\frac{q}{s}\xi_5\right)'$,

where $\xi_5 \equiv \frac{1}{(1-\beta)B}$. Spectral decomposition gives $A = M\Lambda M^{-1} = N^{-1}\Lambda N$, where the diagonal matrix $\Lambda = \begin{pmatrix} \Lambda_p & 0 \\ 0 & \Lambda_n \end{pmatrix}$ has the eigenvalues of A on its diagonal. The eigenvalues associated with the predetermined variables are collected in the diagonal sub-matrix Λ_p and have negative real parts. The others have positive real parts and are collected in the diagonal sub-matrix Λ_n. Diagonalisation of (G1) gives $\dot{y} = \Lambda y + e^{-\varphi(t-T)}n$ for $y = Nx$ with $n = N\hat{a}$, which has solution

(G4) $\quad y_{p,i}(t) = e^{\lambda_{p,i}(t-T)} y_{p,i}(T) + \frac{n_{p,i}}{\lambda_{p,i} + \varphi}\left[e^{\lambda_{p,i}(t-T)} - e^{-\varphi(t-T)}\right], i = 1, 2,$ and

(G5) $\quad y_{n,i}(t) = -\frac{n_{n,i}}{\lambda_{n,i} + \varphi}, i = 1, 2, \quad t \geq T,$

where $y_p(T) = M_{pp}^{-1}\left[x_p(T) - M_{pn} y_n(T)\right]$. The solution to (G1) is then $x(t) = My(t), t \geq T$. The stable manifold converges asymptotically to $x_n(t) = M_{np} M_{pp}^{-1} x_p(t), t \geq T$, and at $t = T$ it equals

(G6) $\quad x_n(T) = M_{np} M_{pp}^{-1} x_p(T) + (M_{nn} - M_{np} M_{pp}^{-1} M_{pn}) y_n(T).$

Equation (G6) gives $c(t) = c^A\left(k(t), P(t), B(t), t - T\right), s(t) = s^A\left(k(t), P(t), B(t), t - T\right), t \geq T$, and post-catastrophe welfare from the HJB equation (14). Note that these depend via (G5) on the speed of adjustment of catastrophe, φ. If the catastrophe were abrupt ($\varphi \to \infty$), the second term on the right-hand side of equation (G6) vanishes.

Before-catastrophe system

We simulate the general system with Duffie-Epstein preferences discussed in section 6. The solution of the before-catastrophe system (8), (9), (16′) and (17′) proceeds along similar lines with two differences. First, the solution simplifies to $y_{p,i}(t) = e^{\lambda_{p,i}t} y_{p,i}(0), i = 1, 2,$ with $y_p(0) = M_{pp}^{-1} x_p(0)$, and $x_n(t) = M_{np} M_{pp}^{-1} x_p(t), 0 \leq t < T$. Second, one has to take care of the additional terms in (16′)-(17′) to allow for the pending catastrophe. The expressions for A_{pp} and A_{pn} must be re-evaluated for the different values of the before- and after-catastrophe steady states. This also holds for A_{np} and A_{nn}, but in addition one has to allow for the extra terms in (16′)-(17′) by adding the following terms to $\hat{A}_{np}$ and $\hat{A}_{nn}$:

$$\text{(G7)} \quad \begin{pmatrix} \xi_6\left(\dfrac{\tilde{V}^0\tilde{W}_k}{\tilde{W}^2}-\dfrac{\tilde{V}_k^0}{\tilde{W}}\right)+\dfrac{c\theta_k}{\eta_I} & \xi_6\left(\dfrac{H'(\tilde{W}-\tilde{V}^0)}{H\tilde{W}}+\dfrac{\tilde{V}^0\tilde{W}_P}{\tilde{W}^2}-\dfrac{\tilde{V}_P^0}{\tilde{W}}\right)+\dfrac{c\theta_P}{\eta_I} \\ \theta_k s+\xi_7 & (\theta_P+H')s+\xi_8 \end{pmatrix} \text{ and } \begin{pmatrix} \dfrac{c\theta_c}{\eta_I} & 0 \\ \theta_c s & \theta+H \end{pmatrix},$$

where $\xi_6 \equiv \dfrac{c}{\eta_I}\left(\dfrac{\eta_R-\eta_I}{\eta_R-1}\right)H$, $\xi_7 \equiv \dfrac{H'(\tilde{W}-\tilde{V}^0)-H\tilde{V}_P^0}{\tilde{W}_k^{\,2}}\tilde{W}_{kk}-\dfrac{H'(\tilde{W}_k-\tilde{V}_k^0)-H\tilde{V}_{Pk}^0}{\tilde{W}_k}$ and

$\xi_8 \equiv \dfrac{H'(\tilde{W}-\tilde{V}^0)-H\tilde{V}_P^0}{\tilde{W}_k^{\,2}}\tilde{W}_{kP}-\dfrac{H'(\tilde{W}_P-\tilde{V}_P^0)-H\tilde{V}_{PP}^0}{\tilde{W}_k}-\dfrac{H''(\tilde{W}-\tilde{V}^0)-H'\tilde{V}_P^0}{\tilde{W}_k}$ with

$$\text{(G8)} \quad \begin{aligned} &\theta=\left(\frac{\rho c^{-\eta_I}}{\tilde{W}_k}-1\right)H \text{ with } \theta_c=-H\frac{\eta_I\rho c^{-\eta_I-1}}{\tilde{W}_k}, \\ &\theta_k=-\frac{\rho H c^{-\eta_I}\tilde{W}_{kk}}{\tilde{W}_k^{\,2}} \text{ and } \theta_P=\frac{\theta H'}{H}-\frac{\rho H c^{-\eta_I}\tilde{W}_{kP}}{\tilde{W}_k^{\,2}}, \end{aligned}$$

where $\tilde{W}_k=\rho c^{-\eta_I}\left[(1-\eta_R)\tilde{W}\right]^{\frac{\eta_I-\eta_R}{1-\eta_R}}$, $\tilde{W}_P=-s\tilde{W}_k$, $\tilde{W}_{kk}=-\dfrac{\eta_I\tilde{W}_k}{c}c_k+\left(\dfrac{\eta_I-\eta_R}{1-\eta_R}\right)\dfrac{\tilde{W}_k^{\,2}}{\tilde{W}}$ and

$\tilde{W}_{kP}=s\dfrac{\eta_I\tilde{W}_k}{c}c_k-s\left(\dfrac{\eta_I-\eta_R}{1-\eta_R}\right)\dfrac{\tilde{W}_k^{\,2}}{\tilde{W}}$. The after-catastrophe expressions in (G8) are calculated from the stable manifold, which we denote with superscript *A*. We use $V_k^0=c^A(k,P,1,0)^{-\eta_I}$, $V_P^0=-s_P^A(k,P,1,0)c^A(k,P,1,0)^{-\eta_I}$ and $V_{kP}^0=-\eta_I c^A(k,P,1,0)^{-\eta_I-1}c_P^A(k,P,1,0)$ in (G8).

Our spectral decomposition algorithm is perhaps not at the frontier of numerical methods, but it gives answers that make sense for more theoretically oriented continuous-time problems and it has the merit of transparency. Other accurate numerical infinite-horizon optimisation methods for discrete-time problems approximate with a terminal value, use efficient discretisation, and maximise welfare directly rather than solving the first-order conditions (e.g., Cai et al., 2016a).