EQUATIONS VS: IDENTITIES IN MACROECONOMICS

by

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(Editor's note: The original version of this paper was written in Norwegian and published in "25 Economic Essays in Honour of Erik Lindahl, 21 November 1956". It was later published as a Memorandum from the Institute of Economics, University of Oslo, May 28 1963. The translation has been done by Olav Bjerkholt and Jon Vislie, Department of Economics, University of Oslo, on request from the international community for having access to Haavelmo's ideas. The translation seems to be somewhat delayed as seen from the following letter to Haavelmo from Ragnar Frisch, dated August 23, 1961, found in the Frisch-archives, Frisch writes:
"Dear Trygve,
This is just to tell you that a few days ago when I was looking for an article in the 1956-Festschrift to Erik Lindahl, I found (I don't know whether I should say fortunately or unfortunately) your article "Equations vs. identities.
I was so "absorbed" that I found myself reading your paper rather than completing some presssing work I was doing which had to be finished before I should go abroad.
Yours Ragnar
P.S. If this paper is not already translated to a "language", you have to see that this is done immediately even though it should only appear as a memorandum.")

1. *Introduction*

In economics we often use expressions like "functional relationship", "relation", "equation", "identity", in a somewhat sloppy and non-discriminatory way. This practice has led economists into various logical tangles. For instance, consider the statement: "We have the same number of relations as unknowns, and therefore the system is determined". From mathematics we know that this is only a qualified truth. Most economists thus know that we should here at least say "independent relations". But what does that mean? One of the rules in this connection is, as well known, that a count of the number of relations one must disregard pure identities. However, some economists denote pure accounting relations as "trivial identities", while at the same time find these relations highly relevant for obtaining determined systems. There is a confusing ambiguity (or equivocation) as to the notion of identity appearing here.

In the following we will try to direct attention to the difference between the concepts of “equation” and the concept of “identity”. This issue has already been subject to some discussion in economics, either more fundamentally or in relation to specific economic theories.

One fundamental discussion can be found in a paper by Professor J. Marschak in Econometrica 1942.[1] Marschak here seizes on the bad habits often displayed by economists by first putting up valid equations from a definitional point of view, and then discuss whether these relations can be “out of equilibrium”. Marschak discusses some instances of such contradictions in terms found in the economic literature and then sorts out the issues in an instructive way. However, his account becomes less valuable than what it could have been, as he makes no attempt at defining precisely what is really meant by an “identity” as opposed to an “equation”.

In a number of related papers in Quarterly Journal of Economics in 1939[2] several American economists attempted to end the prolonged debate over whether “saving is always equal to investment”. The question was formulated as follows: Is $I = S$ an equation or is it an identity? That this attempt at elucidation of the issue did not lead to the desired result, one can convince oneself about by reading the cited articles, and partly by observing that the debate has continued with undiminished strength.

Some examples from recent economic literature show that the topic under consideration has not been exhausted.

[1] J.Marschak: “Identity and Stability in Economics: A Survey”, Econometrica Vol. 10, No.1, 1942.

[2] A.P.Lerner: “Saving and Investment: Definitions, Assumptions, Objectives”, Quarterly Journal of Economics, August 1939, pp. 611-619. O.Lange: ”Saving in Process Analysis”; ibid. pp. 620-622. Myra Curtis: “Saving and Savings”; ibid.pp. 623-626. F.A.Lutz: “Final Comment”; ibid. 627-631.

In Professor Schumpeter's last grand work[3] we find the following passage (p. 970): "Plain common sense can indeed indicate certain conditions that must be fulfilled if such a unique set of values – a 'solution' – is to exist. Thus, the equations must be genuine equations and not mere identities (such as x is x)." (This passage was supplemented by the following footnote: "But identities that express the fact that x and y, which occur in the rest of the system, are really identical ($x \equiv y$) permit suppression of either x or y and thus may contribute toward determinateness just as much as does an equation.")

Another example of reasoning related to the issue under consideration can be found on page 1096 in Schumpeter's volume: "In itself there is nothing new about what has come to be called the Fisher or Newcomb-Fisher equation. It simply links the price level (P) with (1) the quantity of money in circulation (M); (2) its 'efficiency' or velocity (V); and (3) the (physical) volume of trade (T). Let us express this by writing $P = f(M,V,T)$. To this functional relation the Fisher equation imparts the particular form: $P = f(M,V,T) = \frac{MV}{T}$ or $MV = PT$. Again, this equation is <u>not</u> an identity but an equilibrium condition. For Fisher did not say that MV is the same thing as PT or that MV is equal to PT by definition: given values of M,V,T tend to <u>bring about</u> a determined value of P, but they do not simply <u>spell</u> a certain P …."

Another quote from Schumpeter's book might illustrate the point. In a section about Keynesian macroeconomics we read (p. 1180): "By means of those three basic functions or schedules a system of three equilibrium conditions (equations) and one identity can be written that will, with the quantity of money as an externally imposed datum, and under proper assumptions, uniquely determine interest, investment, and either savings or consumption and can be extended to include also other variables such as Keynes's wage rates."

[3] J.A.Schumpeter: History of Economic Analysis. New York. Oxford University Press 1954.

In Professor Alvin Hansen's book A Guide to Keynes,[4] we find the following contribution to the fundamental discussion (p. 105): "Section IV makes it evident that Keynes saw quite clearly the difference between (1) saving and investment being 'equal' (identity equations) and (2) saving and investment being in 'equilibrium' (behavior equations)"…. "Identity equations, being purely tautological, explain nothing. To say that on Nov 1, 1950, the amount of wheat purchased in the Chicago market was equal to the amount of wheat sold does not help to explain wheat prices. Similarly, as noted above identity equations such as $MV = PT$ and $I = S$ explain nothing."

Other examples can surely be found to illustrate the various divergent ideas that prevail as to what are equations and what are identities in economic theory. There might also be people saying that the discussion usually is about real economic differences, and not about logical formalism (e.g. the difference between Keynes' saving = investment on the one hand and the more "classical" idea about this relation). However, I am inclined to believe that a substantial part of the discussion could have been avoided or that we can avoid it in the future by using a little more time to agree on the exact meaning of the concept of "equation" and "identity".

1. *On Equations and Identities as Mathematical Concepts*

Let $x_1, x_2, ..., x_n$ be n real variables in the usual sense that $x_1, x_2, ..., x_n$ denotes a set of n real numbers to be chosen from a more or less delimited set. It is often expedient to think of a set of values for the variables as a "point" in a n-dimensional right-angled coordinate system. When "equations" between such variables are spoken of in mathematics, the meaning is often a rather narrow

[4] New York, Toronto, London, McGraw-Hill Book Company, Inc., 1953.

definition of the notion of equation, namely a polynomial in the n variables set to zero, e.g. for two variables, $ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0$. But the notion of an equation has over time been enhanced to comprise more general expressions in variables put equal to zero. With regard to the further discussion below we find it necessary to be a bit pedantic in setting out exactly what such an "equation" in fact means.

Let it be given "in advance" that the variables $x_1, x_2, ..., x_n$ are allowed to move freely over certain values, or if we want more generally, over a certain subset of the n-dimensional coordinate system. Let this subset determined in advance be denoted D_n. This subset may for instance comprise all possible values for the variables from minus infinity to plus infinity. However, it might also be the case that we know from the outset that we are not interested in negative values, or that it is not conceivable to consider values of the variables above some magnitude, and so on. It might be the case that such an ex ante restriction is due to our knowledge that we in the analysis will say that the variables have no meaning or validity unless we keep within a certain range for the x's. The ex ante restriction to D_n is important. We have to decide what kind of values for the $x's$ are at all to be considered for each problem at hand.

Let $f(x_1, x_2, , ..., x_n) = 0$ be a well-defined real-valued function of the x's, over the admissible domain D_n. The meaning of the expression

(2.1) $\quad f(x_1, x_2, ..., x_n) = 0$ when the x's are in D_n

can be stated as: Find all sets of values in D_n such that f is equal to zero. Let $D_n(f)$ be the set of all such values. Here we have to distinguish between three cases:

1) No such values exist (the set $D_n(f)$ is empty),
2) there exists a unique set of values for the x's which makes f equal to 0, or
3) there exist a multitude of sets of values for the x's, that make f equal to zero.

In geometric terminology this corresponds to $D_n(f)$ containing respectively 1) zero, 2) one, or 3) several points. We can now offer two definitions:

Definition 1: (2.1) is called an <u>equation</u> (or to be more precise a <u>conditional equation</u>) between the variables $x_1, x_2, \ldots, x_n$ when $D_n(f)$ does not contain all the sets of values in D_n.

Definition 2: (2.1) is called an <u>identity</u> in the variables $x_1, x_2, \ldots, x_n$ when $D_n(f)$ contains all sets of values in D_n.

The x's above are usually called "the unknowns". We note that when (2.1) is an <u>equation</u>, and thus $D_n(f)$ "smaller" than D_n, we get, by requiring that (2.1) is satisfied, "more information about the x's than we knew from the outset". If, on the other hand, (2.1) is an identity, it doesn't tell us anything new about the x's.

Then what interest can it have to study identities in the sense defined above? They seem to be rather worthless. However, there are in fact a number of different reasons for us to care about identities.

The most obvious reason is that we are often unable to see whether what appears to be an equation eventually turns out to be an identity or not. We have to find that out by investigating the functional form f. In mathematics there are some useful rules developed for such purpose. For example, if f is a polynomial and x_i can take any value between a_i and $b_i > a_i$, for $i = 1, 2, ..., n$, our polynomial can be identically equal to zero over the domain of the $x's$, only if all the coefficients of the polynomial are equal to zero. (As we shall see, this result can be applied to discuss eco-circ relations as they are simple polynomials.)
But there are other important reasons why we should worry about the notion of identity. One reason is that the terminology is alternating, even in pure mathematics. Therefore it happens that relations which define a new variable y, as a function of the "old" variables $x_1, x_2, ..., x_n$, say as $y = g(x_1, x_2, ..., x_n)$ over a given domain D_n, is being called a "definitional identity". However, in my opinion, it seems easier and safer simply to use the denotation <u>definitional equation</u>. In fact if <u>both</u> y <u>and</u> the x's are considered as variables or "unknown" in the analysis, a definitional equation will in general <u>not</u> be an identity in the variables $y, x_1, x_2, ..., x_n$. A definitional equation is doubtlessly in general to be conceived as a condition on y when the x's are given.
Another reason why the notion of identity must be treated carefully is that the question whether (2.1) is an identity or not will depend upon the definition of the domain D_n. If $D_n(f)$ is smaller than the original set D_n, we could for instance start over again using $D_n(f)$ as the admissible set for the x's, and then (2.1) will of course be an identity defined over this narrower set. Therefore, it is not sufficient only to look at "the shape of f" to determine whether we have an identity or not.
A third, but not less important reason for being on the alert, is the fact that a relation can in one sense be an identity, while in another sense it is not. We can go

wrong by falsely concluding that "an identity does not provide anything new". The crucial question is whether the relation under scrutiny is an identity in <u>those things we want say something about</u>, the "unknowns", or whether it is an identity with regard to variations in other things that are not the unknowns of our problem. A typical example is the following: Let $f(x_1, x_2, ..., x_n)$ be a function of the variables $x_1, x_2, ..., x_n$. Suppose that we assert e.g. that

$$(2.2) \quad f(x_1, x_2, ..., x_n) + a\frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_n} = 0, \quad (a = \textit{constant})$$

is satisfied for all values $x_1, x_2, ..., x_n$ over some set; viz. (2.2) is an identity in the x's for a given domain D_n. For an arbitrarily chosen f and an arbitrarily chosen domain D_n, such an assertion will in general not be true. Suppose however that it is not the x's that are the unknowns of the problem, but instead <u>the shape of the function f</u>.In that case what was stated about (2.2), is a contribution to delimit a class of functions f. By requiring (2.2) to be an identity in the x's, it becomes an equation in the functional form f (often called a functional equation). Typical examples of such functional equations are difference and differential equations, where time is the only independent x-variable. Then it is, of course, not time that is the unknown of the problem, but the shape of certain time functions. That such equations hold as "identities in time", is then far from a "trivial identity".

From these rather brief mathematical remarks, we might tentatively draw the following conclusion:

It is too imprecise to say that a relation is an "equation" or an "identity" without further comments. What is most important, when it comes to an identity, is to add a thorough specification about <u>what the relation is an identity for</u>; i.e. for what kind of variations the identity is satisfied. Otherwise we will get into real trouble when claiming that an identity "does not tell us anything". From this it follows that it is of little help to use the sign $\equiv$ for an identity instead of the sign = for an equation, as we must in any case have to add a specification of in what sense the relation under scrutiny is an equation or an identity.

We shall now check out the conceptual apparatus we have introduced on a couple of controversial relations in economic theory.

2. *What is meant by "Saving equal to Investment"?*

Suppose we study a macroeconomic system with three variables, income = *R*, investment = *I*, and saving = *S*. At the outset we require

$$(3.1) \qquad I = S$$

Is this an equation or is it an identity? Let us think of some possible values of the three variables R, I, S, "before we start formulating a theory about them". Then we can think of all values for the three variables we at all will take into consideration represented by a domain D_3 in a three-dimensional $R - I - S$ space.

As long as *R*, *I*, and *S*, are regarded as free variables, it is immediately obvious that (3.1) not in general an identity according to our definition above. To write down (3.1) is obviously a first step towards narrowing down the variation possibilities for the three variables R, I, S.

Now we could argue, of course, that we take (3.1) as self-evident and hence we could just as well have started with a smaller D_3-set, say $D_3' \subset D_3$, with the property that (3.1) is satisfied for all values in this restricted set. ($I = S$ will then correspond to a plane in the $R - I - S$ space.) It may seem like a quite philosophical question whether we "for other reasons" in advance have been able to narrow down the set D_3' in such a way that (3.1) becomes an identity on this set or whether we shall say that we have "used" (3.1) to find this narrowing and consequently (3.1) is a conditional equation in the variables as defined originally over a larger domain. The question is however more than just a formality. It is a question about the real economic meaning of the theory that (3.1) is a part of. We can explain this more distinctly by the following line of reasoning.

Assume that as a theory for further narrowing down of D_3' we put up

$$(3.2) \qquad S = f(R)$$

$$(3.3) \qquad I = g(R)$$

where f and g are given functions ("propensity to save", and "propensity to invest"). Within the set D_3' for which (3.1) is satisfied, the variables *R, I, S* will be restricted to such variations so that also (3.2) and (3.3) hold. Then it might be the case that the domain for these variables will be narrowed down to a single point in the *(R, I, S)*-space ("unique solution"). On this contracted domain of variation (viz., a single point), it will obviously be the case that all three equations hold "identically". However it seems obvious that in the economic line of reasoning that leads us to such a, possibly, unique solution we have moved around in the *R-I-S*-space outside the solution point itself. Yes, it is really obvious that, when we reason about the hypothetical experimental meaning of (3.2) and (3.3), we have moved outside the set D_3' delimited by (3.1). The meaning of (3.2) is that we have a theory about what savings would be for alternative values of *R*. And the same for *I*. (3.2) and (3.3) are two curves, and we obviously do not mean that these curves coincide. The whole point is exactly that the two functions *f* and *g* are not the same function. (This might of course be the case if e.g. the decisions about savings and investment were always made by the same group of people and thus were just two different names for the same economic decision). In our line of economic reasoning we move obviously in a set D_3 where neither (3.1), (3.2) nor (3.3) hold identically.

There is nevertheless a reason why we might perhaps say that (3.1) is "more of an identity" than (3.2) or (3.3), namely that (we may think of) that the theories (3.2) and (3.3) are not "correct", while (3.1) "must hold". There are thinkable alternatives to (3.2) and (3.3). Perhaps it is now this, now that which is a good theory. But regardless of how the theories (3.2) and (3.3) must be for being considered good theory, we may be sure that we will stick to (3.1). Perhaps we might then say that we regard (3.1) as an "identity with regard to variations in the shape of the theories (3.2) and (3.3)". But here we should be aware that if we are dead certain that a specific function *f* in (3.2) was the only realistic possibility, then we could just as well call (3.2) an identity. In this way we could end up with a completely "diluted" notion of identity, without any claim of interest.

Another matter is whether we find it expedient to use the terms saving and investment in such a way that we should demand (3.1) to be satisfied. To a certain extent this might be a question purely of expediency, a question of which specification of variables we find most useful in each specific case.

Some economists have even stronger spoken against calling (3.1) an identity, than we have done above. They claim that (3.1) is just an "equilibrium condition", i.e. if *I* and *S* are different, then something "will happen" which make them equal. Against this argument others assert that "*I* and *S* can never differ". This disagreement can perhaps be traced back to some confusion of concepts. There are implicitly six variables involved (perhaps more) instead of the three mentioned above; namely our three variables *R, I, S*, and then something like "ex ante" variables; income (R^*), investment (I^*) and savings (S^*). Our system of equations above can then be interpreted as

$$(3.1a) \qquad I = S$$

$$(3.2a) \qquad S^* = f(R^*)$$

$$(3.3a) \qquad I^* = g(R^*)$$

Here there are so far only three equations between six variables. There is no contradiction letting *(3.1)* prevail, along with $I^* \neq S^*$. We have to supplement our story to go into a discussion of the type just outlined. Thus by adding the following equilibrium conditions

$$(3.4) \qquad S^* = I^*$$

$$(3.5) \qquad I^* = I$$

$$(3.6) \qquad R^* = R$$

both parties in the debate can be right, as long as each party is careful in specifying whether they use the variables with or without asterisk.

3. Some General Remarks about Eco-circ Relations

The relation $I = S$ is an example of an eco-circ relation. Other examples are: *National Income* = *consumption* + *real investments* + *exports* – *imports, Net Investment* = *gross Investment* – *depreciation, disposable Income* = *revenue* – *taxes* + *subsidies;* and so on. We can make the same kind of considerations regarding these equations as we did about $I = S$ above: If all the magnitudes entering an eco-circ relation are to be considered as variables, as "unknowns" in our analysis, these eco-circ relations will not be considered as identities for a general variation of the variables. This follows directly from the theorem saying that a polynomial, with coefficients different from zero, cannot be identically zero for a general variation of the

variables involved. However, it might perhaps be the case that we in advance want to consider the possible domain of variations for the variables as a more delimited domain where the eco-circ relations hold as identities. In general it seems necessary to consider them as equations defined over a larger domain, due to the discussion of the remaining behavioural relations of the system. (Cf. the discussion of *(3.1)*.)
It is often reasonable to consider an eco-circ relation as an equation that <u>defines</u> a certain variable. But this is not always the case. If we for instance have a relation like, *income = consumption + investment*, it is far from obvious that we want to consider this as a definition of one of three variables that enter the relation. It might be the case that we want to regard the three variables as of completely equal standing in the eco-circ relation. To what extent one of the three magnitudes, say *the income*, possibly can be said to be determined by the others, is often not revealed until it is apparent from the other relations that are added into a more complete theory about the variables.

The question whether eco-circ relations are relations that "always hold" can be viewed from different aspects. One of these is as discussed above whether such relations are understood as identities or as regular equations. Another aspect of the question is whether we "have to" introduce such relations, whether they are "self-evident", or relations that are above discussion, and so on.

If an eco-circ relation is to be regarded as a definition of a variable, e.g. the definition of income p.a., as the value of *consumption p.a. plus the net value of all investment goods produced p.a.*, then the validity of the relation cannot be contested. But the rationale of determining the three entities such that the relation holds can of course be discussed. If observed income is derived in this way, by adding data for consumption and investment, then the situation is quite clear. But in that case it is not at all obvious that “income”, defined in such a way, becomes an “interesting” variable. For instance, if we, when compiling a complete macroeconomic theory, want to consider “income” as a demand-motivating factor, then it is not at all obvious that the eco-circ income concept as defined above has much explanatory power. It might even be the case that this concept of income concept defined turns out to be a poor starting point for a “revised” demand-motivating income concept.

Eco-circ relations are based on a specification of variables and an algebra taken over from standard book-keeping. We usually take it more or less for granted that eco-circ magnitudes are important economic variables because they can be defined on the basis of realized or hypothetical accounting principles. Economic theory has made progress by adapting concepts from book-keeping to the overall economy. But problems arise when we attempt to use the eco-circ concepts in behavioural or technological relations. Here it often seems to be a kind of substitution relation between the degree of complexity of the shape of the relations themselves and the degree of the observational complexity of the involved variables. The following is a typical example: From an eco-circ point of view it seems reasonable to define “capital” as an integral of “net investment”. Suppose that we consider the capital defined in this way as a factor of production. Then we get entangled into countless difficulties. First we meet with a number of index-problems. But this is not our worst problem. It seems to be almost hopeless to establish an eco-circic concept of investment corresponding to “finished investment goods in use”. Furthermore, it seems almost hopeless to establish an eco-circ concept of depreciation corresponding to the technical reduction in the “productive capacity” of capital. By choosing a measure for capital satisfying the <u>additive</u> properties we find in accounting, we risk placing ourselves in an unfortunate position with regard to the more technological part of the theory. On the other hand, if we should prefer to have a really simple relationship between "capital" and "output", we might perhaps be induced to choose a definition of capital that would be virtually impossible to measure statistically. The practical solution seems to be somewhere in between these extremes. It might for instance be the case that we could have a better definition of capital, from a production theoretical aspect, by defining the rate of growth of capital as a more or less complicated increasing function of total output, without demanding that this rate of growth should correspond to some additive fraction of income, that we call savings.

One of the most important arguments for eco-circ relations necessarily playing a role in economic theory is that we work with transactions between groups of people and individuals. In transactions one party is credited, the other party debited. But even this argument is not entirely convincing, because it is not absolutely given that we acquire the best insight in the economic activities of the society by adopting the axiom of invariance of value in asset transactions (even though it would be cumbersome to choose other axioms in this case).

The conclusion we can draw from the discussion above is hardly that we should shy away from the variables that we naturally are lead towards through eco-circ considerations. But we should perhaps be more aware than what has been the case so far that there are other types of economic variables that can be as important or more meaningful, yes, that there even could be determined economic macro systems with much explanatory power, and, yet, without a single eco-circ relation.