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Portfolio separation with α-symmetric and pseudo-isotropic distributions

Nils Chr. Framstad*†, version: February 27, 2011

Abstract. The pseudo-isotropic multivariate distributions are shown to satisfy Ross' stochastic dominance criterion for two-fund monetary separation. The classical case of separation under abence of risk-free investment opportunity, admits a few particular generalizations to k-fund separation for $(1+1/k)$-norm symmetric variables if k is odd.

Key words and phrases: Portfolio separation, mutual fund theorem, stochastic dominance, pseudo-isotropic distributions, K-isotropic distributions.

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0 Introduction

Portfolio separation – i.e. the property of reducing the dimension of a portfolio optimization problem without welfare loss to the agents in question – has been treated extensively since Tobin [16]. There are two main directions: one being the characterization of preferences which admit the property for all suitable probability distributions (the standard work being Cass and Stiglitz [1], but see even the modern probabilistic approach of Schachermayer et al. [15]). The other, which is the focus of this paper, being the characterization of distributions which admit separation for all suitable preferences. The classical reference is Ross [13], who consider preferences compatible with first-order stochastic dominance, and distributions which admit such an ordering. Subsequently, Owen and Rabinovitch [12] and Chamberlain [2] establish that the elliptical (also frequently referred to as «elliptically contoured») distributions satisfy Ross' conditions for two-fund separation. Their setting is a mean–variance trade-off – if necessary, considering merely the underlying uniform distribution on the elliptical contour, without having to make any integrability condition on the returns distribution itself.

The classical results, valid for the elliptical distributions, are two-fund monetary separation both in the presence of a «riskless» numéraire opportunity (in which case it can be taken as one of the funds), and in the absence of such. This paper sets out to generalize to the

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pseudo-isotropic distributions, a class where there is a dispersion measure which is symmetric and positively homogeneous, and which, together with the excess returns entering via a location parameter, characterize the portfolio return distribution completely. While the case without a riskless opportunity does not universally generalize (and if does, might require more funds), the presence of riskless opportunity will yield the two-fund separation result almost trivially from the setup. And certainly, there are such cases long known, at least assuming the riskless opportunity: For vectors of i.i.d. symmetric α-stable random variables (and linear transformations thereof), portfolio separation was established already by Fama [3]. The reader should be warned against the literature's inconsistent treatment of the α-stable laws, dubbed by Hall [6] as a «comedy of errors» – the symmetric α-stable vectors which are *elliptical*, are those for which the underlying spectral measure is *uniform* on the unit sphere ([14, Proposition 2.5.2] – this fact answering [12, footnote 4], who use «symmetric» precisely for those which are rotational-invariant). On the contrary, vectors of i.i.d. symmetric α-stables, have characteristic functions of the form $\exp(-||\boldsymbol{\theta}||_\alpha^\alpha) = \exp(-\sum|\theta_i|^\alpha)$. α-(quasi-) norm dependence is the canonical form of dispersion measure treated in the relevant literature, since [5], and is conjectured to be the only «useful» dispersion measures admitting the constructions needed for the setup (Koldobsky [10] citing Misiewicz [11]).

The exposition will be given in a single-period framework. The dynamic setting with intermediate consumption are covered by consuming excess return, as in the continuous-time case (see this author [4], based on an approach of Khanna and Kulldorff [8]) – provided the distribution is infinitely divisible. Hence discrete time covered herein, is more general in terms of probability distributions. In section 1, the single period model will be established with a fairly general – but ad hoc – two-fund monetary separation theorem. The essential facts required to fit the pseudo-isotropic distributions into the theorem, will then be given section 2. Then section 3 gives a version for risk averse agents, giving some particular generalizations of the classical case without safe investment opportunity.

1 The market and the preferences

Consider a single period investment in a numéraire (enumerated with as the zeroth coordinate) returning X_0 per monetary unit invested, and another n investment opportunities with returns vector assumed to possess the structure $X_0\mathbf{1} + \boldsymbol{b}R_0 + \boldsymbol{X}R$, so that the *portfolio return* from investments $\boldsymbol{\xi}$ – any vector, permitting short sale and borrowing without market frictions – in the n opportunities and $w - \mathbf{1}^\dagger\boldsymbol{\xi}$ (where w is initial wealth) in the numéraire, will be

$$wX_0 + \boldsymbol{\xi}^\dagger\big(\boldsymbol{b}R_0 + \boldsymbol{X}R\big), \tag{1}$$

where the «†» superscript denotes transposition. Here, the probability distribution of $\boldsymbol{b}R_0 + \boldsymbol{X}R$, where R_0 and R will be nonnegative random variables, will be specified conditional on X_0, and $\boldsymbol{X}$ will be assumed independent of (X_0, R_0, R). The location parameter $\boldsymbol{b}$ is assumed constant. Notice that we will not assume $\boldsymbol{X}$ to have finite mean, but we will later assume it symmetric about the origin.

The market will be assumed free of arbitrage opportunities. Redundant investment opportunities (i.e., non-zero portfolios which perfectly replicate the numéraire return) can be assumed removed from the model, and we shall assume so. In view of the above, it represents no loss of generality to interpret – or even formally assume – X_0 as a «risk-free» return. We shall therefore use the term «monetary separation» to refer to separation where the numéraire can be chosen as one of the funds.

In our context, where we consider the separation properties of the distributions (and not of the preferences, at least not until subsection 3), we define portfolio separation as follows:

1.1 Definition (k-fund (monetary) separation). The returns distribution admits *k-fund monetary separation* if there exist vectors («funds») $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{k-1}$ such that for any agent whose preferences are compatible with first-order stochastic dominance of (1), and any portfolio $\boldsymbol{\xi}$, there exist $Q_1, \ldots, Q_{k-1}$ so that the agent (weakly) prefers

$$\boldsymbol{\xi}^* = Q_1\boldsymbol{f}_1 + \cdots + Q_{k-1}\boldsymbol{f}_{k-1} \tag{2}$$

to $\boldsymbol{\xi}$. (The kth fund is then the numéraire.) The distribution admits *$k-1$-fund separation* if in addition the numeraire investment $w - \mathbf{1}^\dagger\boldsymbol{\xi}^*$ vanishes identically. △

For the case of $\boldsymbol{X}|X_0$ being elliptically distributed and located at zero, then the characteristic function $\phi(\boldsymbol{\theta}) = \mathsf{E}[e^{\mathrm{i}\boldsymbol{\theta X}}|X_0]$ takes the form $\psi(\boldsymbol{\theta}^\dagger\boldsymbol{M}\boldsymbol{\theta})$, where the matrix $\boldsymbol{M}$ is positive definite. If $\boldsymbol{f}$ solves the problem

$$\max_{\boldsymbol{\xi}}\ \boldsymbol{b}^\dagger\boldsymbol{\xi} \quad \text{subject to} \quad \boldsymbol{\xi}^\dagger\boldsymbol{M}\boldsymbol{\xi} = 1,$$

then the family $\{Q\boldsymbol{f}\}_{Q\geq 0}$ will yield a portfolio return distribution (of (1)), which first-order stochastically dominate any other possible portfolio returns in the market. This is two-fund monetary separation, reducing the portfolio optimization problem to the one-dimensional allocation between $\boldsymbol{f}$ and the numéraire.

Realizing that the homogeneity and real-valuedness of the quadratic form are the keys, we can immediately formulate a much more general result:

1.2 Theorem. *Assume that the characteristic function of $\boldsymbol{X}|X_0$ admits the representation*

$$\phi(Q\boldsymbol{\theta}) = \psi(Q\,c(\boldsymbol{\theta})) \quad \forall\ Q \geq 0, \qquad \text{where } c \text{ takes values in } [0,\infty). \tag{3}$$

Assume the market is free of arbitrage opportunities and of redundant investment opportunities. Then $c = 1$ is attained. Assume that $\boldsymbol{f}$ solves the problem $\max_{\boldsymbol{\xi}}\ \boldsymbol{b}^\dagger\boldsymbol{\xi}$ subject to $c(\boldsymbol{\xi}) = 1$. Then there is two-fund monetary separation.

Proof. The distribution of $\boldsymbol{X}|X_0$ is uniquely given by $Q\,c(\boldsymbol{\xi})$; note that if $c(\boldsymbol{\xi}) = 0$, then $\boldsymbol{\xi}^\dagger\boldsymbol{X} = 0$, and under the assumption of no arbitrages nor redundant opportunities, this implies $\boldsymbol{\xi} = \mathbf{0}$. Assume $\boldsymbol{f}$ to solve the maximization problem as stated. Now for arbitrary $\boldsymbol{\xi} \neq \mathbf{0}$, we have $\phi(\boldsymbol{\xi}/c(\boldsymbol{\xi})) = \psi(1)$. Let $Q = c(\boldsymbol{\xi})/c(\boldsymbol{f}) = c(\boldsymbol{\xi}) > 0$, so that $\phi(Q\boldsymbol{f}) = \phi(\boldsymbol{\xi})$. In terms of first-order stochastic dominance of the return distribution, we have by assumption that $\boldsymbol{f}$ dominates $\boldsymbol{\xi}/Q$, implying that $Q\boldsymbol{f}$ dominates $\boldsymbol{\xi}$. □

Notice that the result holds also when the portfolios are restricted to a family H of half-lines from the origin (i.e., the property that $\boldsymbol{\xi} \in H$ implies $Q\boldsymbol{\xi} \in H$, all $Q \geq 0$). This covers the generalization to the case where there are more sources of randomness than there are investment opportunities, i.e. $\boldsymbol{X}$ has more coordinates: If $\boldsymbol{\xi} = \boldsymbol{\Sigma\zeta}$ where $\boldsymbol{\zeta}$ is our choice variable, we are restricted to optimizing over $\boldsymbol{\xi}$ being in the image of the $\boldsymbol{\Sigma}$ mapping.

2 Origin-symmetric star bodies and pseudo-isotropic distributions

Following e.g. Jasiulis and Misiewicz [7, Definition 3], a *pseudo-isotropic distribution* is one which satisfies (3) with $\psi(-Q) = \psi(Q)$ (i.e., is *symmetric*).

An *origin-symmetric star body* K, following e.g. Koldobksy [10], is an origin-symmetric closed set with a continuous boundary crossed precisely twice by each line through the origin. He introduces the K-quasinorm notation $||\boldsymbol{\xi}||_K = \min\{a > 0;\ \boldsymbol{\xi}/a \in K\}$. Evidently, the K-quasinorms satisfy the conditions of c in (3). In [9, ch. 6], he refers to the corresponding pseudo-isotropic distributions as «K-isotropic».

As mentioned in the introduction, it is conjectured that the only non-constant positive-definite functions satisfying (3) – are the ones for which c is the quasi-norm of a subspace of L^α for $\alpha \in (0, 2]$, i.e. K is the unit ball of such a space. This would be fairly analogous to the α-stable class, and indeed, the conjecture does hold under the additional assumption that some moment is finite. Koldobsky [10] further restricts the possible counterexamples to functions that, in his words, *«must exhibit rather odd behaviour at both the origin and infinity.»*

For $\alpha \leq 1$, the L^α unit balls are not only non-convex sets – indeed, their complements intersected with any orthant is a convex set (the first-quadrant part of the epigraph defining any component as a convex function of the others). This motivates the formulation of the following theorem:

2.1 Theorem. *Consider the setup of Theorem 1.2, where in addition $\boldsymbol{X}$ is pseudo-isotropic (i.e. ψ real symmetric). Then there is two-fund separation, where the risky fund $\boldsymbol{f}$ can be taken as an extreme point of the convex hull of the unit sphere $c \geq 1$. In particular, if these extremals are on the axis, then one shall only invest in one opportunity, namely the one with highest excess return b_i.*

Proof. Theorem 1.2 immediately yields two-fund separation. Geometrically, the maximization of $\boldsymbol{b}^\dagger\boldsymbol{\xi}$ has to be obtained on an extreme point. □

As a consequence of non-diversification, we immediately have that if components are independent and $c = ||\cdot||_\alpha$ with $\alpha \leq 1$, then $\mathsf{E}|X_i| = \infty$, $i \geq 1$.

Again it is worth addressing the case where $\boldsymbol{\xi}$ is restricted to a family H of half-lines: Notice that this may rule out extremals of the convex hull of the unit ball and invalidate the statement. If H is a cone, then we intersect before taking convex hull. And the «investing in one opportunity» property must be transformed accordingly if there is a volatility matrix $\boldsymbol{\Sigma}$.

3 Risk aversion and the $||\cdot||_\alpha$ case, no borrowing or no risk-free opportunity

There are cases where a risk-averse agent will require fewer funds. For example, if $R\boldsymbol{X}$ is spherical and (for simplicity) bounded, and all b_i identical, then we can take $\boldsymbol{f} = \mathbf{1}$. Now assume that the numéraire cannot be invested in (i.e. the restriction $\mathbf{1}^\dagger\boldsymbol{\xi} = w$ applies); then second-order stochastic dominance will lead to choosing $wn^{-2}\boldsymbol{f}$, which is a single fund – but in order to satisfy risk-seekers, we need another fund $\perp \mathbf{1}$ in order to attain the appropriate variance. Hence there is valuable insight from the choices of risk-averse agents; the fund required by those, will minimize risk for given return, and the additional fund required for the full class of greedy agents, will merely serve the purpose of boosting dispersion.

In the absence of integrability, the usual second-order stochastic dominance concept needs some refinement. A risk-averse agent has a negative attitude towards dispersion, and in the pseudo-isotropic case, $\boldsymbol{X}$ is symmetric around zero. The following defintions are natural for the purposes of this paper:

3.1 Definition (Risk aversion and portfolio separation). Assume $R\boldsymbol{X}$ to be pseudo-isotropic. An agent is called *risk averse* if $\boldsymbol{\xi}^*$ is (weakly) preferred to $\boldsymbol{\xi}$ whenever

$$c(\boldsymbol{\xi}^*) - c(\boldsymbol{\xi}) \leq 0 = (\boldsymbol{\xi}^* - \boldsymbol{\xi})^\dagger \boldsymbol{b}. \tag{4}$$

The returns distribution admits *k-fund monetary separation among risk averse agents* if there exist $k-1$ vectors $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{k-1}$ such that for any risk-averse agent, and any portfolio $\boldsymbol{\xi}$, there exist $Q_1, \ldots, Q_{k-1}$ so that the agent (weakly) prefers

$$\boldsymbol{\xi}^* = Q_1\boldsymbol{f}_1 + \cdots + Q_{k-1}\boldsymbol{f}_{k-1} \tag{5}$$

to $\boldsymbol{\xi}$. The distribution admits $k-1$*-fund separation among risk averse agents* if in addition the numeraire investment $w - \mathbf{1}^\dagger\boldsymbol{\xi}^*$ vanishes identically. △

The purpose of the following result is to show separation results among risk-averse agents if borrowing is not allowed (i.e. if $\mathbf{1}^\dagger\boldsymbol{\xi} \leq w$ is imposed) or if no risk-free investment opportunity exists (i.e. imposing $\mathbf{1}^\dagger\boldsymbol{\xi} = w$). We want to cover cases where the function c is a L^α quasinorm after a linear transformation $\boldsymbol{\Sigma}$. This amounts to optimizing over $\boldsymbol{\xi} = \boldsymbol{\Sigma}\boldsymbol{\zeta}$; assuming $\boldsymbol{\Sigma}$ invertible, we might just as well write transform $\boldsymbol{b}$ and $\mathbf{1}$ accordingly, and assume the risk-averse agent to

$$\min_{\boldsymbol{\xi}} ||\boldsymbol{\xi}||_\alpha^\alpha \qquad \text{subject to} \quad \boldsymbol{a}^\dagger\boldsymbol{\xi} = d, \quad \boldsymbol{r}^\dagger\boldsymbol{\xi} = w \quad \text{resp.} \ \leq w \tag{6}$$

where the first constraint (with $\boldsymbol{a} = \boldsymbol{\Sigma}^{-1}\boldsymbol{b}$) is the yield requirement d – notice that by symmetry, we can assume all $a_i \geq 0$ – and the latter (with $\boldsymbol{r} = \boldsymbol{\Sigma}^{-1}\mathbf{1}$) is the absence of numéraire investment opportunity (resp. of borrowing). We then have the following:

3.2 Theorem. *Consider the $||\cdot||_\alpha$-isotropic market with $\alpha = 1 + 1/k$ where k is an odd natural number. Then the cases of no borrowing resp. no numéraire investment opportunity, admit $k+2$-fund resp. $k+1$-fund separation among risk averse agents.*

Proof. Consider the case where $\alpha > 1$, where the problem is convex and smooth; the Lagrangian stationarity condition is

$$|\xi_i|^{\alpha-1} \operatorname{sign} \xi_i = \gamma a_i + \lambda r_i \tag{7}$$

$$\text{i.e.} \qquad \xi_i = |\gamma a_i + \lambda r_i|^k \operatorname{sign}(\gamma a_i + \lambda r_i) \tag{8}$$

$$\text{which equals} \qquad (\gamma a_i + \lambda r_i)^k \qquad \text{if } k \text{ is odd.} \tag{9}$$

Expanding the power, and collecting $\gamma^j\lambda^\ell$ terms in separate vectors, we have $k+1$ vectors which do not depend on the Lagrange multipliers.

For $\alpha \leq 1$, $n > 2$ we have a minimization problem with a concave Lagrangian, i.e. corner solution, so that some coordinate is zero. Repeat the problem with dimension $n-1$ until the two-dimensional problem, where the constraints (with equality) form a singleton. □

Part (a) is a direct generalization the 2-fund separation result for the elliptical (i.e. $\alpha = 2$) case. Just like for $\alpha = 2$, there will of course be course be special cases where further reductions are possible. For example, if if $\boldsymbol{a}$ is a multiple of $\boldsymbol{r}$, then the location is uniquely given in terms of wealth, and any risk averse agent will simply choose the minimum dispersion portfolio. It should be remarked that if k is even, it is not at all straightforward to determine the sign of ξ_i. To see this, assume $\boldsymbol{\Sigma} = \boldsymbol{I}$ and impose the yield requirement $\boldsymbol{b}^\dagger\boldsymbol{\xi} \geq d$ and the no borrowing (i.e. $\mathbf{1}^\dagger\boldsymbol{\xi} \leq w$) condition: Then $\gamma \geq 0 \geq \lambda$, and the sign is not easy to determine. On the other hand, those agents who would have wanted to leave a position in the numéraire, will face a positive λ if barred from doing so. For these agents, we will automatically obtain positivity and can use the representation (7).

Finally, the case $\alpha \leq 1$ again leads to non-diversification:

3.3 Theorem. *For $\alpha \leq 1$, a solution to problem (6) has at most two non-zero coordinates.*

Proof. For $n > 2$ where we do have a proper optimization problem, the concavity of the Lagrangian leads to corner solution. Now remove one of the coordinates and repeat the argument until only two variables remain. □

4 Discussion

It is easily seen that the portfolio separation results herein reduce to the classical ones by simply putting $\alpha = 2$, both in the case with safe investment and in the $1 + \frac{1}{\text{odd}}$ case without. Certainly, there are ramifications in special cases.

One can notice that if the set K is not convex, then a separation property does not require it to scale homogeneously, as long as the maximum returns vector scales linearly. For example, let K_Q be the Q-ball in $\mathsf{L}^{\alpha(Q)}$ where $\alpha(Q)$ is increasing from 0 to 1, and let $\hat{c}(\boldsymbol{\xi})$ be the unique Q for which $\boldsymbol{\xi}$ hits the boundary of K_Q. Then – if such a probability distribution exists! – the solution for the problem described in Theorem 2.1 is as in the L^1-pseudo-isotropic case: the optimal portfolio is always to hold only one investment opportunity (the one with the highest b_i, assuming the «$\boldsymbol{\Sigma}$» transformation being the identity). However, the $\alpha \leq 1$ case corresponds to nonintegrability, i.e. extremely heavy tails.

Of course, even for convex K_Q which do not scale homogeneously, we might have the maximum returns vector scaling linearly. However, a separation property will then be critically dependent upon the value of the location, and could be destroyed by a small perturbance in $\boldsymbol{b}$.

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