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Pareto Improvements of Nash Equilibria in Differential Games

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Last 10 Memoranda

No 18/11	Erik Biørn, Knut R. Wangen *Models of Truncation, Sample Selection, and Limited Dependent Variables: Suggestions for a Common Language*
No 17/11	Steinar Holden , Victoria Sparrman *Do Government Purchases Affect Unemployment?*
No 16/11	Ola Lotherington Vestad *Who pays for occupational pensions?*
No 15/11	Mads Greaker, Michael Hoel *Incentives for environmental R&D*
No 14/11	Halvor Mehlum , Karl Moene , Ragnar Torvik *Mineral Rents and Social Development in Norway*
No 13/11	André K. Anundsen, Eilev S. Jansen *Self-reinforcing effects between housing prices and credit: Evidence from Norway*
No 12/11	Nils Chr. Framstad *Portfolio Separation with α-symmetric and Psuedo-isotropic Distributions*
No 11/11	Jørgen Modalsli *Inequality and growth in the very long run: inferring inequality from data on social groups*
No 10/11	Eilev S. Jansen, Tord S. H. Krogh *Credit conditions indices: Controlling for regime shifts in the Norwegian credit market*
No 09/11	Karin Jacobsen , Kari H. Eika, Leif Helland, Jo Thori Lind, Karine Nyborg *Are nurses more altruistic than real estate brokers?*

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Pareto improvements of Nash equilibria in differential games.

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Abstract This paper yields sufficient conditions for Pareto inoptimality of controls forming Nash equilibria in differential games. In Appendix a result yielding existence of open loop Nash equilibria is added.

Keywords: Differential games, Nash equilibria, Pareto improvements.

JEL Classification C73

Introduction. In static one-shot games, Nash equilibria are frequently not Pareto optimal. Thus, not seldom cooperation can improve payoffs to all players. In particular, in two person games, if a Nash equilibrium consist of strategies in the interior of the strategy sets, Pareto improvements are usually possible. Similar results hold for dynamic games where either open loop or closed loop controls are allowed. Sufficient conditions for Pareto improvements to be possible are stated and proved below. In Appendix a result on existence of open loop Nash equilibria is added, useful for an example on Pareto inoptimality presented below. (The assumptions differ from those appearing in existence results found in the references.)

Let us first consider the simple case of a static one shot game with m players. Let

r_i be the strategy of player i, R_i his strategy set (a given interval),
$F^i(r_1, ..., r_m)$ the payoff to player i.

Let first $m = 2$ and let (r_1^*, r_2^*) be a Nash equilibrium.

Assume that

(i) r_i^* are interior points in R_i
and that the following condition on partial derivatives holds:

(ii) $\partial F^1(r_1^*, r_2^*)/\partial r_2 \neq 0$, $\partial F^2(r_1^*, r_2^*)/r_1 \neq 0$.

Then (r_1^*, r_2^*) is not Pareto optimal.

To see this, note that by moving the strategy slightly each player can increase the payoff of the opponent in the first order, with only a second order effect on one's own payoff. (The condition (ii) can most often be expected to hold:

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[2] Comments from K.Sydsæter have been very useful in order to improve the exposition.

For example, F^1 is not optimized with respect to r_2, so chances are high that $F_2^1 \neq 0$.)

For more than two players, sufficient conditions for Pareto inoptimality are more demanding: If (i) holds, and, for all i, all $F_{r_i}^j(r_1^*, ..., r_m^*)$, $j \neq i$, differ from zero and have the same sign for all j, then a strict Pareto improvement is possible (all players strictly better off).

I. The dynamic game

Let

$x^{(i)} \in \mathbb{R}^{n_i}$ be the state of player i,
$u^{(i)} \in \mathbb{R}^{k_i}$ be the control of player i, taking values in a given set U_i,
$x = (x^{(1)}, ..., x^{(m)}), u = (u^{(1)}, ..., u^{(m)})$,
$f^{(i)}(t, x, u)$ be the instantaneous reward to player i,
$\dot{x} = g^{(i)}(t, x, u)$ be the state equation of player i.

We assume that $f^{(i)}$ and $g^{(i)}$ are C^1 (they take values in $\mathbb{R}$ and $\mathbb{R}^{n_i}$, respectively). There are given natural numbers k_i^* and k_i^{**}, $k_i^* \le k_i^{**}$ and real numbers $\bar{x}_k^i$, $k = 1, ..., k_i^{**} \le n_i$. Define

$$A_i = \{x \in \mathbb{R}^{n_i} : x_i = \bar{x}_k^i, \text{ for } k = 1, ..., k_i^*, x_i \ge \bar{x}_k^i \text{ for } k = k_i^* + 1, ..., k_i^{**}\}.$$

Let $a^i \in \mathbb{R}^{n_i}$ be a fixed vector for which $a_k^i = 0$, $k \le k_i^{**}$, and denote by Problem i the problem

$$\max_{u^{(i)}(.)} W_i^{u^{(1)}(.),...,u^{(m)}(.)}, \text{ where} \tag{1}$$

$$W_i^{u^{(1)}(.),...,u^{(m)}(.)} = a^i x^{u^{(1)}(.),...,u^{(m)}(.)}(T) \; + \int_0^T f^{(i)}(t, x^{u^{(1)}(.),...,u^{(m)}(.)}(t), u^{(1)}(t), ..., u^{(m)}(t))dt, \tag{2}$$

subject to

$$\dot{x}^{(j)} = g^{(j)}(t, x, u^{(1)}(t), ..., u^{(m)}(t)), \; x(0) = x_0^j, \; x_0^i \text{ given,} \tag{3}$$

$$x^{u^{(1)}(.),...,u^{(m)}(.)(j)}(T) \text{ free for } j \ne i, \; x^{u^{(1)}(.),...,u^{(m)}(.)(i)}(T) \in A_i \tag{4}$$

where $x^{u^{(1)}(.),...,u^{(m)}(.)}(.)$ is the solution of the m vector equations (3) and $x^{u^{(1)}(.),...,u^{(m)}(.)(j)}(.)$ is the solution of the j'th vector equation. So player i seeks a control $u^{(i)}(.) \in U_i$ such that, given $u^{(j)}(.)$, $j \ne i$, the criterion in (1) is maximized, subject to the condition $x^{(i)}(T) \in A_i$. He/she is forced to have $x^{(i)}(T) \in A_i$ satisfied, but he/she disregard all conditions $x^{(j)}(T) \in A_j$, $j \ne i$, (to have the latter conditions satisfied is not his/her problem but the problem of the other players!) Player i takes into consideration the influence of $u^{(i)}$ on all $x^{(j)}$. We shall also consider problems where there are no end conditions, (all A_i are replaced by $\mathbb{R}^{n_i}$).

In a special case considered below, (**C**), $f^{(i)}$ and $g^{(i)}$ will not depend on $u^{(j)}$, $j \ne i$. Still player i takes into account the influence of $u^{(i)}$ on all $x^{(j)}$, $j \ne i$, via the influence of $x^{(i)}$ on these entities, (or more precisely, he takes into consideration the interplay between all the $x^{(j)}$'s when choosing $u^{(i)}$).

To condition (4), there corresponds a transversality condition (explained in more detail below):

$$
\begin{aligned}
k &\leq k_i^* : \text{no information on } p_k(T) \\
k &= k_i^*+1, ..., k_i^{**} : \ p_k(T) \geq 0, p_k(T) = 0 \text{ if } x_k(T) > \bar{x}_k^i. \\
k &> k_i^{**} : p_k(T) = a_k^i.
\end{aligned} \tag{5}
$$

II. Open loop controls Assume that all controls are simply measurable time functions (open loop controls). Let $\{\hat{u}^{(i)}(.)\}_i$ be an open loop Nash equilibrium consisting of piecewise continuous functions, with corresponding solutions (vector functions) $\hat{x}^{(i)}(.)$, i.e. for each i, given $\hat{u}^j(.)$, $j \neq i$, $\hat{u}^i(.)$ yields maximum in Problem i (see (1)). Let $\hat{x}(t) = (\hat{x}^{(1)}(t), ..., \hat{x}^{(m)}(t))$, $\hat{u}(t) = (\hat{u}^{(1)}(t), ..., \hat{u}^{(m)}(t))$, $p = (p_{(1)}, ..., p_{(m)})$. For $p_0^{(i)} \in \{0, 1\}$, define

$$H^i(t, x, u, p) = p_0^{(i)} f^{(i)}(t, x, u) + \sum_j p_{(j)} g^{(j)}, \tag{6}$$

and define $\mathrm{p}^{(i)}(t) = (p_{(1)}^{(i)}(t), .., p_{(m)}^{(i)}(t))$, where the vector function $p_{(j)}^{(i)}(t)$ is defined by

$$\dot{p}_{(j)}^{(i)} = -H_{x^{(j)}}^i(t, \hat{x}(t), \hat{u}(t), p^{(i)}) \text{ for } j = 1, ..., m, p_{(j)}^{(i)}(T) = 0, j \neq i, \tag{7}$$

and where $p_{(i)}^{(i)}(T)$ satisfies (5) for $x_k(T) = x_k^{(i)}(T)$, $p(T) = p_{(i)}^{(i)}(T)$ (i.e. the components $p_{(i)k}^{(i)}(T)$ of $p_{(i)}^{(i)}(T)$ equals the $p_k(T)$'s appearing in (5)).

Now, by the maximum principle, there exist a $p_0^{(i)} \in \{0, 1\}$ and a function $p^{(i)}(.)$ with the above properties such that $\hat{u}^{(i)}(t)$ maximizes

$$u^{(i)} \to H^{(i)}(t, \hat{x}(t), \hat{u}^{(1)}(t), ..., \hat{u}^{(i-1)}(t), u^{(i)}, \hat{u}^{(i+1)}(t), ..., \hat{u}^{(m)}(t), p^{(i)}(t)). \tag{8}$$

In what follows, some different cases are discussed.

A In the present case, all $A_i = \mathbb{R}^n$, so there are no end constraints and $p_0^{(i)} = 1$. Then, of course, $p_{(j)}^{(i)}(T) = 0$ for all i and j.

Consider first the subcase

$\mathbf{A}_1$ $\ m = 2$.

Let us write down the adjoint equations in this case.
For $i = 1$ (player 1),

$$dp_{(1)}^{(1)}(t)/dt = -f_{x^{(1)}}^{(1)} - p_{(1)}^{(1)} g_{x^{(1)}}^{(1)} - p_{(2)}^{(1)} g_{x^{(1)}}^{(2)}$$

$$dp^{(1)}_{(2)}(t)/dt = -f^{(1)}_{x^{(2)}} - p^{(1)}_{(1)}g^{(1)}_{x^{(2)}} - p^{(1)}_{(2)}g^{(2)}_{x^{(2)}}$$

For $i = 2$ (player 2),

$$dp^{(2)}_{(1)}(t)/dt = -f^{(2)}_{x^{(1)}} - p^{(2)}_{(1)}g^{(1)}_{x^{(1)}} - p^{(2)}_{(2)}g^{(2)}_{x^{(1)}}$$

$$dp^{(2)}_{(2)}(t)/dt = -f^{(2)}_{x^{(2)}} - p^{(2)}_{(1)}g^{(1)}_{x^{(2)}} - p^{(2)}_{(2)}g^{(2)}_{x^{(2)}}$$

All derivatives on the right hand sides are evaluated at $(t, \hat{x}^{(1)}, \hat{x}^{(2)}, \hat{u}^{(1)}, \hat{u}^{(2)})$.

$$\text{Assume all } \hat{u}^{(i)}(t) \text{ interior in } U_i \text{ for all } t. \tag{9}$$

For $j \neq i$, $\hat{u}^{(j)}(t)$ does not usually maximize

$$u^{(j)} \to H^{(i)}(t, \hat{x}(t), \hat{u}^{(1)}(t), ..., \hat{u}^{(j-1)}(t), u^{(j)}, \hat{u}^{(j+1)}(t), ..., \hat{u}^{(m)}(t), p^j(t))$$

and, in fact, chances are high that $H^{(i)}_{u^{(j)}}(t, \hat{x}(t), \hat{u}(t), p^{(j)}(t)) \neq 0$.

A sufficient condition for Pareto inoptimality is as follows.

Theorem 1. Assume $m = 2$, (9) and no end conditions. If for $i = 1, 2$, the derivative $H^{(i)}_{u^{(j)}}(t^j, \hat{x}(t^j), \hat{u}(t^j), p^{(i)}(t^j))$ is nonzero for some continuity point $t = t^j$ of $\hat{u}(.)$ for $j \neq i$, then a strict Pareto improvement is possible.

$\mathbf{A}_2$ $m > 2$.

Then a sufficient condition for Pareto inoptimality is as follows.

Theorem 2. Assume $m > 2$, (9), no end conditions. If, for each j, for some continuity point $t = t^j$ of $\hat{u}(.)$, for some $u^j \in \mathbb{R}^{k_j}$,

$$H^{(i)}_{u^{(j)}}(t^j, \hat{x}(t^j), \hat{u}(t^j), p^i(t^j))[u^j - \hat{u}(t^j)] \tag{10}$$

is nonzero and has the same sign for all $i \neq j$, then a strict Pareto improvement is possible.

Here, instead of the condition related to (10), we can assume that for some point $t^j \in (0, T)$, some $u^j \in \mathbb{R}^{k_j}$ either

$$H^{(i)}_{u^{(j)}}(t^j, \hat{x}(t^j), \hat{u}(t^j+), p^i(t^j))[u^j - \hat{u}(t^j)]$$

or

$$H^{(i)}_{u^{(j)}}(t^j, \hat{x}(t^j), \hat{u}(t^j-), p^i(t^j))[u^j - \hat{u}(t^j)]$$

is nonzero and has the same sign for all $i \neq j$ ($t-$ and $t+$ means left and right limits). This condition implies the condition related to (10). (A similar comment of course pertains also to Theorem 1.)

Theorem 3 Assume $m \geq 2$ and no end conditions. Assume that there exist pairs $(t^j, u^j), t^j \in (0, T)$, $u^j \in U_j$, $j = 1, ..., m$, such that, for any i, either

$$H^{(i)}(t^j, \hat{x}(t^j), \hat{u}^{(1)}(t^j+), ..., \hat{u}^{(j-1)}(t^j+), u^j, \hat{u}^{(j+1)}(t^j+), ..., \hat{u}^{(m)}(t^j+), p^{(i)}(t^j)) - H^{(i)}(t^j, \hat{x}(t^j), \hat{u}(t^j+), p^{(i)}(t^j)) \geq 0,$$

with strict inequality holding for $i \neq j$, or

$$H^{(i)}(t^j, \hat{x}(t^j), \hat{u}^{(1)}(t^j-), ..., \hat{u}^{(j-1)}(t^j-), u^j, \hat{u}^{(j+1)}(t^j-), ..., \hat{u}^{(m)}(t^j-), p^{(i)}(t^j)) - H^{(i)}(t^j, \hat{x}(t^j), \hat{u}(t^j-), p^{(i)}(t^j)) \geq 0,$$

with strict inequality holding for $i \neq j$.

Then a strict Pareto improvement is possible[3].

A weaker condition is sufficient for obtaining this conclusion, see the next theorem.

Theorem 4 Assume $m \geq 2$, no end conditions. Assume that for all j, there exist triples (s^j, t^j, u^j), $s^j > 0, t_j \in (0, T)$, $u^j \in U_j$ such that for each i,

$$\sum_j s^j H^{(i)}(t^j, \hat{x}(t^j), \hat{u}^{(1)}(t^j\pm), ..., \hat{u}^{(i-1)}(t^j\pm), u^j, \hat{u}^{(i+1)}(t^j\pm), ..., \hat{u}^{(m)}(t^j\pm), p^{(i)}(t^j))$$

$$- \sum_j s^j H^{(i)}(t^j, \hat{x}(t^j), \hat{u}(t^j\pm), p^{(i)}(t^j)) > 0, \tag{11}$$

($\pm$ meaning either t^j- or t^j+, which one can depend on j). Then a strict Pareto improvement is possible.

B. Assume no end conditions, that $T = \infty$, and that for some positive constants $\hat{k}, B$ and b, for all x, $|\partial f^{(i)}(t, x, \hat{u}(t))/\partial x_k^{(j)}| \leq B \exp(-bt)$ for all i, j, k and $|\partial g_{k'}^{(i)}(t, x, \hat{u}(t))/\partial x_k^{(j)}| \leq \hat{k}$ for all i, j, k, k'. Here $x_k^{(j)}$ ($g_{k'}^{(i)}$) is the component number k (k') of the vector $x^{(j)}$ ($g^{(i)}$). It is assumed that $\hat{k} \sum_i n_i < b$ holds. In this case, the adjoint functions satisfies a new set of conditions, namely $p^{(i)}(t) = \lim_{T \to \infty} p^{(i)}(t, T)$, where $p^{(i)}(t, T)$ satisfies $\dot{p}^{(i)} = -H_x^i(t, \hat{x}(t), \hat{u}(t), p^{(i)})$, $p^{(i)}(T, T) = 0$. (The maximum conditions for each players are still satisfied by $p^{(i)}(t)$, $p_0^{(i)} = 1$).

[3] This theorem may be of interest in some bang-bang situations. Consider a simple case where $U = [0, 1]$ and $m = 2$. Assume that $\hat{u}_1(t)$ switches from $u = 0$, to $u = 1$ at some $t_1 \in (0, T)$, Then $u_1 = 0$ satisfies the first inequality in Theorem 3 with equality for $i = 1$, and one has simply to test whether this inequality holds for $i = 2$ for $u_1 = 0$. (For $u_1 = 1$, use the second inequality.)

For these $p^{(i)}-$ functions, Theorems 1 - 4 hold (in case of Theorem 1 and 2 (9) is assumed).

C. When the horizon is finite, Theorems 1 and 2 also hold for end constrained problems, provided, for each i, f^i and g^i do not depend on $u^{(j)}$, $j \neq i$, and provided,

(a) the rank of $\pi_i g^{(i)}_{u_i}(T, \hat{x}(T), \hat{u}_i(T-))$ equals k_i^{**}, the number of end constrained states of $x^{(i)}$, $\pi_i = x \to (x_1, ...x_{k_i^{**}})$, $x \in \mathbb{R}^{n_i}$.

The condition (a) can be replaced by the weaker condition (b): For some $K > 0$, 0 is an interior point in $\{\pi_i g^{(i)}_{u_i}(T)[u - \hat{u}_i(T))] : |u| \leq K\} - B$, $B := \{(0, ..., 0, y_{k_i^*+1}, ..., y_{k_i^{**}}) \in \mathbb{R}^{k_i^{**}} : y_{k_1^*+k} \in [0, K]\}$. (Note that (a) and (b) imply $p_0^{(i)} = 1$.)

Example

Let k_i, $i = 1, 2$ be states (real capital belonging to player i). The capital k_i

develops according to

$$\dot{k}_i = \phi_i(k_1, k_2) - C_i,\ 0 \leq C_i,\ k_i(0) = k_i^0 > 0,\ k_i^0 \text{ fixed}, \tag{12}$$

where C_i is the consumption of player i (the control of player i). The constraint on his capital is $k_i(T) \geq k_i^T > 0$, where k_i^T are given numbers $< k_i^0$. Now, ϕ_i is defined and continuous for $k_1, k_2 \geq 0$, continuously differentiable for $k_1, k_2 > 0$, with $\partial\phi_i/\partial k_i > 0$ for $k_1, k_2 > 0$, and either (α) $\partial\phi_i/\partial k_j > 0$ for $k_1, k_2 > 0$, $j \neq i$, or (β) $\partial\phi_i/\partial k_j < 0$ for $k_1, k_2 > 0$, $j \neq i$, and with $\phi_i \geq 0$, $\phi_1(0, k_2) = \phi_2(k_1, 0) = 0$.

The problem of player i is

$$\max \int_0^T v_i(C_i)dt \text{ subject to (12)},\ k_i(T) \geq k_i^T,\ k_j(T) \text{ free},\ j \neq i.$$

where $v_i(C)$, defined on $[0, \infty)$, is continuous, increasing and concave, and is C^1 on $(0, \infty)$, with $\lim_{C\to 0} v_i'(C) = \infty$ and $\lim_{C\to\infty} v_i'(C) = 0$. Assume that an open loop Nash equilibrium $(\hat{C}_i(t), \hat{k}_i(t))$, $i = 1, 2$, exists, with $\hat{k}_i(t) > 0$ for all t. We have

$$H^i = p_0^{(i)} v_i(C_i) + p_i[\phi_i(k_i, k_2) - C_i] + p_j[\phi_j(k_1, k_2) - C_j].$$

Now,

$$\dot{p}^i_j = -p^i_i \phi^{(i)}_{k_j} - p^i_j \phi^{(j)}_{k_j}. \tag{13}$$

Let us first discuss the solutions of the equations (13) in an informal way. Consider first the case $\phi^{(i)}_{k_j} > 0$, ($j \neq i$, here and below) With $p^{(i)}_{(i)}(T) > 0$,

$p^{(i)}_{(j)}(T) = 0$, then for t close to T, $\dot{p}^{(i)}_{(i)}, \dot{p}^{(i)}_{(j)} < 0$, so both $p^{(i)}_{(i)}(t)$ and $p^{(i)}_{(j)}(t) > 0$ for t close to T, and from the equations it is apparent that this will continue backwards to $t = 0$. Consider next the case $\phi^{(i)}_{k_j} < 0$. In that case, still $\dot{p}^{(i)}_{(i)} < 0$ close to T (the dominant term $-p^i_i \phi^{(i)}_{k_{ij}}$ is negative), while $\dot{p}^{(i)}_{(j)} > 0$, (the dominant term $-p^i_i \phi^{(i)}_{k_j}$ is positive), so for t close to T, $p^{(i)}_{(i)}(t) > 0$ and $p^{(i)}_{(j)}(t) < 0$. This will apparently be the case all the way back to 0, because the two terms in the expression for $\dot{p}^{(i)}_{(i)}$ (respectively $\dot{p}^{(i))}_{(j)}$) are both < 0 (respectively > 0).

A formal proof is obtained by a "backwards version" (presented as Lemma 4 in Appendix) of Theorem A.7 (see also Note A.4) in Seierstad and Sydsæter (1987). Consider the case $\phi^{(i)}_{k_j} > 0$: Because $\dot{p}^i_j \leq 0$ for $p^i_i = 0$, $p^i_j \geq 0$, and $p^{(i)}_i(T) \geq 0$, $p^{(i)}_j(T) = 0$, by Lemma 4 in Appendix, we get, for all t, $p^i_j(t) \geq 0$, $i = 1, 2$, $j = 1, 2$. Consider next the case $\phi^{(i)}_{k_j} < 0$: Applying Lemma 4 to the differential equations for $p^{(i)}_{(i)}$ and $-p^{(i)}_{(j)}$ $(j \neq i)$, by the same type of arguments we get, for all t, $p^{(i)}_{(i)}(t) \geq 0$ and $-p^{(i)}_{(j)}(t) \geq 0$ $(j \neq i)$.

By the maximum condition, $p^{(i)}_{(i)}(t)$ cannot be zero, so $p^{(i)}_{(i)}(t) > 0$ for all t. Moreover, $\hat{C}_i$ cannot be $\equiv 0$, because then $d\hat{k}_i/dt = \phi_i \geq 0$, so $\hat{k}(T) \geq k^0_i > k^T_i$, implying $p^{(i)}_{(i)}(T) = 0$, a contradiction. So for some t, $\hat{C}(t) > 0$, and at t $H^i_{C_i} \geq 0$. Combined with $p^{(i)}_{(i)}(t) > 0$ this gives $p^{(i)}_0 = 1$. But then $\hat{C}(s) > 0$ for all s, by the maximum condition. Because $p^{(i)}_{(i)}(t) > 0$ for all t, by the adjoint equations, $p^{(i)}_{(j)}(t) \neq 0$, $j \neq i$, for all $t < T$. Evidently then, $H^i_{C_j} = -p^{(i)}_{(j)}(t) \neq 0$, $j \neq i \in \{1, 2\}$, $t < T$, so a strict Pareto improvement is possible, according to **C** and Theorem 1. (Trivially, the rank condition holds, $\partial\phi_i/\partial C_i = -1$.)

If there are several players and, for each j, $\partial\phi_i/\partial k_j > 0$ for all $i \neq j$, or $\partial\phi_i/\partial k_j < 0$ for all $j \neq i$, then by the same arguments a strict Pareto improvement is possible.

Closed loop controls Assume that all controls appearing in the problems are closed loop control, i.e. functions of t and x. In particular, assume that $\breve{u}^{(j)}(t, x)$, $j = 1, ..., m$, form a closed loop Nash equilibrium, with corresponding solutions $\hat{x}^{(j)}(t)$. Suppose that $\breve{u}^{(j)}(t, x)$, $j = 1, ..., m$, are C^1 for all j. Then $p^{(i)}(.)$ is defined by $\dot{p}^{(i)} = -[(\partial/\partial x)H^{(i)}(t, x, \breve{u}(t, x)), p^{(i)})]_{x=\hat{x}(t)}$[4], where $\breve{u}(t, x) = (\breve{u}^{(1)}(t, x), ..., \breve{u}^{(m)}(t, x))$, $(\hat{x}(t) = (\hat{x}^{(1)}(t), ..., \hat{x}^{(m)}(t)))$. With this change, for $\hat{u}(t) = \breve{u}(t, \hat{x}(t))$, the theorems above still hold.

(So, for $j = 1, ..., m$, adding small constants to $\breve{u}^{(j)}(t, x)$ on small time intervals improves payoffs to all players, when the conditions in the theorems are satisfied.)

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[4] When calculating this derivative, the term containing $\partial\breve{u}^{(j)}(t, x)/\partial x$ drops out due to $H^{(j)}_{u^{(j)}} = 0$.

If there are surfaces in (t,x) -space on which $u^{(j)}(t,x)$ is discontinuous, then allowing jumps in $p^{(i)}(.)$ may sometimes work, see Seierstad and Stabrun (2010), further comments on this case are omitted.

Proofs The proof of Theorem 4 in the no end constraint case follows directly from standard result in control theory, see Lemma 1 in Appendix. To apply the lemma, in (11) we need to (and can) move the points t^j slightly apart and such that they are continuity points of $\hat{u}(.)$ and with the following inequality holding:

$$\sum_j s^j H^{(i)}(t^j, \hat{x}(t^j), \hat{u}^{(1)}(t^j), ..., \hat{u}^{(j-1)}(t^j), u^j, \hat{u}^{(j+1)}(t^j), ..., \hat{u}^{(m)}(t^j), p^{(i)}(t^j)) -$$

$$\sum_j s^j H^{(i)}(t^j, \hat{x}(t^j), \hat{u}(t^j), p^{(i)}(t^j)) =: \gamma^i > 0. \tag{14}$$

Theorems 1 and 2 follow from the same argument, because, for any j, there is a u^j close to $\hat{u}^{(j)}(t^j)$ such that, because t^j is a continuity point, for $i \neq j$ the inequalities in Theorem 3 hold even for 0 replaced by $\alpha_j|u^j - \hat{u}^j(t^j)|$ for some $\alpha_j > 0$, while for $i = j$, the inequalities hold if 0 is replaced by $\geq \varepsilon_j(|u^j - \hat{u}^j(t)|)$, where $\varepsilon_j(.)$ is negative and of the second order in $|u^j - \hat{u}(t)|$. So, choosing $|u^j - \hat{u}^j(t^j)|$ small enough,

$$\alpha_j|u^j - \hat{u}^j(t)| + (1/(m-1))\varepsilon_j(|u^i - \hat{u}^j(t^i)|) \geq (\alpha_j/2)(|u^j - \hat{u}^j(t^j)|),$$

and hence, in a shorthand notation, for any $i \neq j$,

$$\begin{aligned}
&H^{(i)}(t^j, \hat{x}(t^j), ...u^j..., p^{(i)}(t^j)) - H^{(i)}(t^j, \hat{x}(t^j), \hat{u}(t^j), p^{(i)}(t^j)) + \\
&(1/(m-1))[H^{(i)}(t^i, \hat{x}(t^i), ...u^i..., p^{(i)}(t^j)) - H(t^i, \hat{x}(t^i), \hat{u}(t^i), p^{(i)}(t^i))] > \\
&\alpha_j(|u^j - \hat{u}(t)|) + (1/(m-1))\varepsilon_j(|u^i - \hat{u}^j(t^i)|) \geq (\alpha_j/2)(|u^j - \hat{u}^j(t^j)|).
\end{aligned}$$

Summing over $j \neq i$, we get (14), for $s^j = 1$.

Proof in case of C (end restrictions).

Proposition. In case of **C** a strict Pareto improvement $\tilde{u}_i(t)$ of $\hat{u}_i(t), i = 1, ..., m$, exists when it is required that $\tilde{x}^i(T) \in A_i$ for all i, $\tilde{x}^i(t)$, $i = 1, ..., m$, the solution corresponding to $\tilde{u}^j(t)$, $j = 1, ..., m$.

Proof: Let $\{.\}^{(i)}$ be the map $x \to x^{(i)} : \mathbb{R}^{\sum_i n_i} \to \mathbb{R}^{n_i}$. We give only a proof for $k_i^* = k_i^{**}$. An easy modification, using auxiliary controls, gives a proof in the case $k_i^* < k_i^{**}$. Let $C^i(s,t)$ be the resolvent of the equation $\dot{q}^{(i)} = g_x^{(i)}(t, \hat{x}(t), \hat{u}^i(t))q^{(i)}$, ($C^i(t,t) = I$, the identity matrix, $\partial C^i/\partial s = g_x^{(i)}(s)C^i(s,t)$).

Define $g = (g^{(1)}, ..., g^{(m)})$,

$$u_{(j)} = (\hat{u}^1(t^j), ..., \hat{u}^{j-1}(t^j), u^j, \hat{u}^{j+1}(t^j), ..., \hat{u}^m(t^j)).$$

and $h^i := \pi_i\{\sum_j s^j C^i(T,t^j)[g(t^j,\hat{x}(t^j),u_{(j)}) - g(t^j,\hat{x}(t^j),\hat{u}(t^j))]\}^{(i)}$.

I. Assume first that for some $a > 0$ for each i there exist a $\bar{u}_i \in U_i$ such that

$$h^i + a\pi_i[g^i(T,\hat{x}(T),\bar{u}_i) - g^i(T,\hat{x}(T),\hat{u}(T))] = 0,$$

with $\bar{u}_i$ satisfying

$$H^{(i)}(T,\hat{x}(T),\bar{u}^i,p^{(i)}(T)) = H^{(i)}(T,\hat{x}(T),\hat{u}(T),p^{(i)}(T)),$$

where

$$\bar{u}^i = (\hat{u}^{(1)}(T), ..., \hat{u}^{(i-1)}(T), \bar{u}_i, \hat{u}^{(i+1)}(T), ..., \hat{u}^{(m)}(T)).$$

For $\delta \geq 0$, define $\check{u}_\delta(t) =$

$$\sum_j u_{(j)} 1_{[t^j,t^j+\delta s^j]} + (\bar{u}_1, ..., \bar{u}_m)1_{[T-\delta a,T]} + \hat{u}(t)(1 - \sum_j 1_{[t^j,t^j+\delta s^j]} - 1_{[T-\delta a,T]}),$$

Let

$$W_i^u = \int_0^T f^i(t,x^u(t),u^i(t))dt, \ (u = u(t) = (u^1(t), ..., u^m(t))),$$

let $p^{i,0} = (p_1^{i,0}, ..., p_m^{i,0})$ be the solution of

$$dp^{i,0}/dt = -f_x^i(t,\hat{x}(t),\hat{u}^i(t)) - p^{i,0}g_x^i(t,\hat{x}(t),\hat{u}^i(t)), \ p^{i,0}(T) = 0,$$

let $\tilde{p}^i := (0, ..., 0, p_i^i(T), 0, ..., 0)$ and let $q^i(t) = \tilde{p}^i C^i(T,t)$. Using that $dC^i(T,t)/dt = -C^i(T,t)g_x^i$, see p. 272 in Sydsæter et al. (2005), and multiplying by $\tilde{p}^i$ in this equality, we get that $\dot{q}^i = -q^i g_x^i$. As $p^{i,0}(T) + q^i(T) = p^{(i)}(T)$, then $p^{i,0}(t) + q^i(t) = p^{(i)}(t)$, $(\dot{p}^{i,0} + \dot{q}^i = \dot{p}^{(i)})$.

As explained below (dropping writing $\hat{x}$ in $f^{(i)}, g^{(i)}$ and $H^{(j)}$), we have

$$\begin{aligned}
&[dW_i^{\check{u}_\delta}/d\delta]_{\delta=0} = \\
&s^i(f^i(t^i,u^i) - f^i(t^i,\hat{u}^i(t^i))) + \textstyle\sum_j s^j p_j^{i,0}(t^j)(g^j(t^j,u^j) - g^j(t^j,\hat{u}(t^j))) + \\
&a(f^i(T,\bar{u}_i) - f^i(T,\hat{u}(T))) = \\
&s^i(f^i(t^i,u^i) - f^i(t^i,\hat{u}^i(t^i))) + \textstyle\sum_j s^j p_j^{i,0}(t^j)(g^j(t^j,u^j) - g^j(t^j,\hat{u}(t^j))) - \\
&p_{(i)}^{(i)}(T)a(g^i(T,\bar{u}_i) - g^i(T,\hat{u}(T))) = \\
&s^i(f^i(t^i,u^i) - f^i(t^i,\hat{u}^i(t^i))) + \textstyle\sum_j s^j p_j^{i,0}(t^j)(g^j(t^j,u^j) - g^j(t^j,\hat{u}(t^j))) + \\
&p_{(i)}^{(i)}(T)\{\textstyle\sum_j s^j C^i(T,t^j)[g(t^j,u_{(j)}) - g(t^j,\hat{u}(t^j))]\}^{(i)} = \\
&s^i(f^i(t^i,u^i) - f^i(t^i,\hat{u}^i(t^i))) + \textstyle\sum_j s^j\{p^{i,0}(t^j)(g(t^j,u_{(j)}) - g^j(t^j,\hat{u}(t^j))) \\
&+[\tilde{p}^i C^i(T,t^j)]\}(g(t^j,u_{(j)}) - g(t^j,\hat{u}(t^j))) = \\
&s^i(f^i(t^i,u^i) - f^i(t^i,\hat{u}(t^i))) + \textstyle\sum_j s^j p^{(i)}(t^j)(g(t^j,u_{(j)}) - g^j(t^j,\hat{u}(t^j))) = \gamma^i.
\end{aligned}$$

Here we have used, successively, (1): Lemma 1 in Appendix, (2): $f^i(T,\bar{u}_i) - f^i(T,\hat{u}(T)) = -p_{(i)}^{(i)}(T)[g^i(t^i,\bar{u}_i) - g^i(T,\hat{u}^i(T))]$, a consequence of $H^{(j)}(T,\bar{u}_j) = H^{(j)}(T,\hat{u}(T))$, (3): $p_{(i)}^{(i)}(T)(I - \pi_i) = 0$ and

$$\bar{q}^i := h^i + a\pi_i[g^i(T,\hat{x}(T),\bar{u}_i) - g^i(T,\hat{x}(T),\hat{u}(T))] = 0, \tag{15}$$

(the $\bar{u}_i$'s where so chosen above), for any x $p^{(i)}_{(i)}(T)\{x\}^{(i)} = \tilde{p}^i x$, (4): $p^i(t) = p^{i,0}(t) + \tilde{p}^i C^i(T,t)$ and (5): inequality (14).

Let U'' be the set of measurable functions $u(.) = (u_1(.),...,u_m(.))$, $u_j(t) \in U_j$, and let $q^{u,\check{u}}$, $u, \check{u} \in U''$, be defined by

$$\dot{q}^{u,\check{u}}(t) = g_x(t, x^{\check{u}}(t), \check{u}(t))q^{u,\check{u}}(t) + g(t, x^{\check{u}}(t), u(t)) - g(t, x^{\check{u}}(t), \check{u}(t)),$$
$$q^{u,\check{u}}(0) = 0,$$

and let $\bar{q} = (\bar{q}^1,...,\bar{q}^m)$, $\pi x = (\pi_1 x_1,...,\pi_m x_m)$. In fact, $|\pi q^{\check{u}_\delta,\hat{u}}(T) - \delta\bar{q}|$ is of the second order in δ (apply Lemma 1 in Appendix to $q^{\check{u}_\delta,\hat{u}}$ rather than to x^{u_δ}), and $\bar{q} = 0$ by (15). Now, (9) and the rank condition in **C** is easily seen to imply that for some $\varepsilon' > 0$, $B(0,3\varepsilon') \subset \text{clco}\{\pi q^{u,\hat{u}}(T) : u \in U'', \text{essup}|u| \le K\}$ say for $K = 1+\text{essup}|\hat{u}(t)|$. Because $q^{u,\check{u}_\delta}(T)$ is close to $q^{u,\hat{u}}(T)$ uniformly in $u \in U_K := \{u \in U'' : \text{essup}|u| \le K\}$ when δ is small, then for some $\delta'' > 0$, $\text{cl}B(0,2\varepsilon') \subset \text{clco}\pi q^{U_K,\check{u}_\delta}(T)$, $\delta \le \delta''$ (see Lemma 3 in Appendix). For some $\delta^* \in (0,\delta'']$ and some $d' > 0$, for any $\delta \le \delta^*$, any $z \in B(0,\varepsilon')$, and any $d \in (0,d']$, there exists a control $\tilde{u}^d_\delta$, such that

$$\pi\tilde{x}^{\tilde{u}^d_\delta}(T) - \pi x^{\check{u}_\delta}(T) = dz, \ \sigma(\tilde{u}_d,\check{u}_\delta) \le dT, \tag{16}$$

see Lemma 3 in Appendix. Because $|\pi q^{\check{u}_\delta,\hat{u}}(T) - \delta\bar{q}|$ and $|x^{\check{u}_\delta}(T) - q^{\check{u}_\delta,\hat{u}}(T)|$ are of the second order, then, for δ small, $|\pi x^{\check{u}_\delta}(T) - \delta\bar{q}| \le \check{\varepsilon}(\delta)$, for some second order term $\check{\varepsilon}(\delta)$. Fix a $\delta' \in (0,\delta^*]$ such that $2\check{\varepsilon}(\delta)/\varepsilon' \le d'$ when $\delta \le \delta'$. Define a_δ by $2\check{\varepsilon}(\delta)a_\delta/\varepsilon' = -\pi x^{\check{u}_\delta}(T) + \delta\bar{q}$. Then, for $\delta \le \delta'$, $|a_\delta/\varepsilon'| \le 1/2$, or $|a_\delta| \le \varepsilon'/2$ and $da_\delta = -\pi x^{\check{u}_\delta}(T) + \delta\bar{q}$ for $d = 2\check{\varepsilon}(\delta)/\varepsilon'$. By (16), for $\delta \le \delta'$, for some $\tilde{u}_\delta = \tilde{u}^d_\delta$, $\pi x^{\tilde{u}_\delta}(T) - \pi x^{\check{u}_\delta}(T) = da_\delta = -\pi x^{\check{u}_\delta}(T) + \delta\bar{q}$, i.e. $\pi x^{\tilde{u}_\delta}(T) = 0$, $\sigma(\tilde{u}_\delta,\check{u}_\delta) \le dT = 2\check{\varepsilon}(\delta)T/\varepsilon'$.

II. We cannot always find $\bar{u}_j$ with the properties above. By (9) and the rank condition in **C** and the inverse function theorem, there exists a constant $\tilde{K}$ such that we can find a pair $(a,\bar{u})$, $a > 0$, $\bar{u} = (\bar{u}_1,....,\bar{u}_m)$, $\bar{u}$ arbitrarily close to $\hat{u}(T)$, such that for all j, $h^j = -a\pi_j(g^j(T,\bar{u}_j) - g^j(T,\hat{u}^j(T)))$, $|a(\bar{u}_j - \hat{u}^j(T))| \le \tilde{K}$. Now, at $\hat{u}(T)$, $H^j_{u_j} = 0$, so $|H^{(j)}(T,\bar{u}_j) - H^{(j)}(T,\hat{u}(T))| \le \varepsilon(|\bar{u}_j - \hat{u}^j(T)|)$, $\varepsilon(s)$ a second order term in s. Hence $\tilde{K}|H^{(j)}(T,\bar{u}_j) - H^{(j)}(T,\hat{u}(T))|/|\bar{u}_j - \hat{u}^j(T)| \le \gamma^j/2$ for $|\bar{u}_j - \hat{u}^j(T)|$ small enough. So $a|H^{(j)}(T,\bar{u}_j) - H^{(j)}(T,\hat{u}(T))| \le \gamma^j/2$ for $|\bar{u}_j - \hat{u}^j(T)|$ small enough, thus $a(H^{(j)}(T,\bar{u}_j) - H^{(j)}(T,\hat{u}(T))) \ge -\gamma^j/2$. Hence, at $\delta = 0$, instead of having $dW_i^{\check{u}_\delta}/d\delta = \gamma^i$ we get $dW_i^{\check{u}_\delta}/d\delta \ge \gamma^i/2$.

III. By Lemma 2 at $\delta = 0$, $dW_i^{\tilde{u}_\delta}/d\delta = dW_i^{\check{u}_\delta}/d\delta \ge \gamma^i/2$, and the proof is finished.

Appendix. In Appendix, we consider a standard control problem

$$\max_{u(.)} W^{u(.)}(T) \text{ where } W^u(t) = \int_0^t f(t, x^u(t), u(t))dt \tag{17}$$

subject to

$$\dot{x}(t) = g(t, x(t), u(t)), x(0) = x_0, x_0 \in \mathbb{R}^n \text{ fixed, } u(t) \in U \text{ for all } t, \tag{18}$$

(x^u the solution corresponding to $u = u(.)$), U a fixed bounded set, f, g, f_x, g_x continuous.

Let U' be the set of measurable functions with values in U, let $\hat{u}(.) \in U'$ be a given control with corresponding solution (of 18) denoted $\hat{x}(t)$ (assumed to exist), and let $q^{u,\hat{u}}(.)$, $u = u(.) \in U'$, be the solution of

$$\dot{q}^{u,\hat{u}}(t) = g_x(t, \hat{x}(t), \hat{u}(t))q^{u,\hat{u}} + g(t, \hat{x}(t), \hat{u}(t)) - g(t, \hat{x}(t), \hat{u}(t)),\ q^{u,\hat{u}}(0) = 0.$$

Let $C(t, s)$ be the resolvent of the equation $\dot{q}(t) = g_x(t, \hat{x}(t), \hat{u}(t))q$, and let $p^0(.)$ be the solution of

$$\dot{p}^0 = -f_x(t, \hat{x}(t), \hat{u}(t)) - p^0(t)g_x(t, \hat{x}(t), \hat{u}(t)),\ p^0(T) = 0.$$

Let $\sigma(u, u') := \text{meas}\{t : u(t) \neq u'(t)\}$. Let $\{t^j\}$ be a finite set of distinct continuity points of $\hat{u}(.)$, let $u^j \in U$, and let $u_\delta(t) := \sum_j u^j 1_{[t^j, t^j+\delta s^j]}(t) + \hat{u}(t)(1 - \sum_j 1_{[t^j, t^j+\delta s^j]}(t))$, s^j given positive numbers.

Lemma 1

$$[dx^{u_\delta}(T)/d\delta]_{\delta=0} = \sum_j C(T, t^j)s^j[g(t^j, \hat{x}(t^j), u^j) - g(t^j, \hat{x}(t^j), \hat{u}(t^j))]$$

and

$$[dW^{u_\delta}/d\delta]_{\delta=0} = \sum_j s^j\{H(t^j, \hat{x}(t^j), u^j, p^0(t^j)) - H(t^j, \hat{x}(t^j), \hat{u}(t^j), p^0(t^j))\},$$

where

$$H(t, x, u, p) = f(t, x, u) + pg(t, x, u).$$

(The derivatives do exist, as $\delta \geq 0$ they are actually right derivatives.)

Proof For a single point t^j the second equality is proved in Seierstad and Sydsæter (1987), see p. 221. The first equality is proved in many proofs of the maximum principle, see e.g. Lemma 14.1, p. 50 in Fleming and Rishel (1975). Treating W as a state in an augmented system (with states $(W, x_1, ..., x_n)$), yields that $dW^{u_\delta}(T)/d\delta$ equals $\sum_j (1, 0, ..., 0) \cdot s^j \tilde{C}(T, t^j)[g(t, \hat{x}(t), u^j) - g(t, \hat{x}(t), \hat{u}(t))]$, where $\tilde{C}(t, s)$ is the resolvent of the augmented system. The last sum equals the sum in the second equality in the Lemma.

Lemma 2 Assume that $u^\delta \in U'$, $\delta > 0$, are given controls such that $\sigma(u^\delta, u_\delta) \leq \varepsilon(\delta)$, $\varepsilon(\delta)$ a second order term. Then $dx^{u^\delta}(T)/d\delta = dx^{u_\delta}(T)/d\delta$ and $dW^{u^\delta}/d\delta = dW^{u_\delta}/d\delta$. (The derivatives do exist.)

Proof It follows from (8) p. 485 in Seierstad (1970) that $x^{u^\delta}(T) - x^{u_\delta}(T)$ is of the second order in δ. Treating W as a state in an augmented system, yields also for the state W that $W^{u^\delta}(T) - W^{u_\delta}(T)$ is of the second order in δ.

Let $\pi := x \to (x_1, ..., x_{n^*})$, $n^* \leq n$ and as before, let $\hat{u} \in U'$ be a given control for which a solution $x^{\hat{u}}(.)$ exists. Then, by general theory, for any $u \in U'$ $\sigma-$close to $\hat{u}$, $x^u(.)$ exists.

Moreover, $x^{\hat{u}}([0,T])$ is compact, so there exists an open set $B \subset \mathbb{R}^n$ containing $x^{\hat{u}}([0,T])$ such that g, and g_x are bounded on $[0,T] \times B \times U$ by some constant M. For u $\sigma-$ close to $\hat{u}$, $x^u(t) \in B$ for all t.

Lemma 3 If $B(\bar{c}, 3\varepsilon) \subset \text{clco}\pi q^{U',\hat{u}}(T)$ (cl = closure, clco = closed convex hull), then for some $\alpha > 0$, some $d' > 0$, for all $\hat{u}' \in U_\alpha := \{u \in U' : \sigma(u, \hat{u}) \leq \alpha\}$, $B(\bar{c}, 2\varepsilon) \subset \text{clco}\pi q^{U',\hat{u}'}(T)$, and for each $d \in (0, d']$, for each $\hat{u}' \in U_\alpha$ and for each $c \in B(\bar{c}, \varepsilon)$, there exists a $u' \in U'$ such that $dc = \pi x^{u'}(T) - \pi x^{\hat{u}'}(T)$, $\sigma(u', \hat{u}') \leq dT$.

Proof Lemma 3 follows from modifications of almost any proof of the maximum principle. If Theorems 1 and 2 in Seierstad (1970) are used (with $A = U'$, $\partial = \sigma, T = 1$), note that, if $A_d = B(\bar{a}, \check{M}d/2)$ (as here assumed), then (D) in Theorem 1 holds for all $a, a' \in A^d = B(\bar{a}', \check{M}d/4)$ (A^d replacing A_d) for any $\hat{u}' = \bar{a}' \in B(\bar{a}, \check{M}d/4)$, ($\check{e}(.)$ independent of $\bar{a}'$), we then have $a = u, y^+(a) = \pi q^{u,\hat{u}'}(T)$, $y(a) = \pi x^u(T)$, $\check{M} = 2$, $\bar{a} = \hat{u}'$). This follows from the continuous differentiability established on top of p. 485 in Seierstad (1970). (Note that for $u, u' \in U'$, $|g(t, x(.), u(.)) - g(t, x(.), u'(.))|_1$, $|g_x(., x(.), u(.)) - |g_x(., x(.), u'(.))|_1$ $\leq M\sigma(u(.), u'(.))$, for all continuous $x(.)$ with values in B, compare $||.||^*$ in (7) in Seierstad (1970).) From this continuous differentiability, it also follows that, uniformly in u, $q^{u,\hat{u}'}(T)$ is close to $q^{u,\hat{u}}(T)$ when $\hat{u}'$ is $\sigma-$ close to $\hat{u}$. So, using Lemma 11.1 in Seierstad (1975), if $B(\bar{p}, 3\bar{e}) \subset \text{clco}\pi q^{U',\hat{u}}(T)$, then for $\delta' > 0$ small enough $B(\bar{p}, 2\bar{e}) \subset \text{clco}\pi q^{U,\hat{u}'}(T)$ for any $\hat{u}'$ such that $\sigma(\hat{u}', \hat{u}) \leq \delta'$. Choose $d' > 0$ such that $\sup_{0<d\leq d'} \check{e}(d)\check{M} < \bar{e}/16$, see proof of Theorem 1 in Seierstad (1970). Then for $\alpha = \min\{\delta', \check{M}d'/4\}$, for $\bar{a}$ replaced by $\bar{a}' = \hat{u}' \in U_\alpha$ and A_d replaced by A^d, the conclusion in Theorem 1 (in Seierstad (1970)) holds, in fact, for all $d \in (0, d']$, all $\hat{u}' \in U_\alpha$, $B(d\bar{p}, d\bar{e}) + \pi x^{\hat{u}'}(T) = B(d\bar{p}, d\bar{e}) + y(\bar{a}') \subset y(\text{cl}A^d) = \{\pi x^u(T) : u \in U', \sigma(u, \bar{u}') \leq \check{M}d/4 = d/2\}$.

Lemma 4. Consider the vector differential equation $\dot{x} = h(t, x)$, $x(0) = x_0$, x_0 given (h continuous, locally Lipschitz in x) on $[0, T]$ and suppose there exists a vector $\bar{x}$ such that, for all i, $h_i(t, x) \leq 0$ for all $x = (x_1, ..., x_n)$ that satisfy $x_j \geq \bar{x}_j$, $j \neq i$, $x_i = \bar{x}_i$. Then any solution $x(.)$, with $x_j(T) \geq \bar{x}_j$ for all j, satisfies $x_j(t) \geq \bar{x}_j$ for all j, $t \in [0, T]$.

Proof When $x(.)$ exists, by general theory also $\dot{x}^\alpha = h(t, x^\alpha) - \alpha$, $x^\alpha(T) = x(T)$, $\alpha > 0$, α small, has a solution $x^\alpha(t)$ that converges to $x(.)$ as $\alpha \downarrow 0$. Now, for $t < T$, it is easily seen that $x_i^\alpha(t) > \bar{x}_i$ for all i. First, note that for a small interval (r, T), $x_i(t) > \bar{x}_i$ for all i all $t \in (r, T)$: For each i either $x_i^\alpha(T) > \bar{x}_i$, or

$x_i^\alpha(T) = \bar{x}_i$ and in the latter case $\dot{x}_i^\alpha(T) = h_i(T, x^\alpha(T)) - \alpha < 0$, and, in both cases, for some r close to T, $x_i^\alpha(t) > \bar{x}_i$ in (r, T). Evidently, r can be chosen independent of i. Let (s, T) be the largest interval for which $x_i^\alpha(t) > \bar{x}_i$ for all i, all $t \in (s, T)$, and assume by contradiction that $s > 0$. Evidently $x_i^\alpha(s) \geq \bar{x}_i$ for all i. For each i, either (a) $x_i^\alpha(s) > \bar{x}_i$ or (b) $x_i^\alpha(s) = \bar{x}_i$ and in the latter case $\dot{x}_i^\alpha(s) = h_i(s, x^\alpha(s)) - \alpha < 0$. In both cases (a) and (b) $x_i^\alpha(t) > \bar{x}_i$ for t in a small interval (r', r'') around s, in case (b) because $\dot{x}_i^\alpha(t) = h_i(t, x^\alpha(t)) - \alpha < 0$ not only for $t = s$, but for t in a small interval (r', r'') around s. With $x_i^\alpha(r'') \geq \bar{x}_i$ (in fact, $x_i^\alpha(r'') > \bar{x}_i$), this gives $x_i^\alpha(t) > \bar{x}_i$ for all $t \in (r', r'')$. The interval (r', r'') can be taken to be independent of i. Hence, $x_i^\alpha(t) > \bar{x}_i$ for $t \in (r', T)$ for all i, a contradiction. So $x_i^\alpha(t) > \bar{x}_i$ for $t \in (0, T)$ for all i. As $x^\alpha(t) \to x(t)$, $x_i(t) \geq \bar{x}_i$ for all $t \in (0, T)$, and then in $[0, T]$.

Let $|y(.)|_\infty$ be the supnorm $\sup_t |y(t)|$.

Lemma 5 Let g in (18) also depend on a vector $y \subset \mathbb{R}^{k^*}$, so $g = g(t, x, y, u)$. Let g, g_x and g_y be continuous in $[0, T] \times X' \times Y' \times U$, X', Y' open sets. When $\hat{u} = \hat{u}(.)$ is a given control function and y in g is replaced by the given function $\hat{y} = \hat{y}(.) \in Y'' = \{y(.);, y(.) \text{ continuous}, y(t) \in Y'\}$, write $x^{\hat{u},\hat{y}}(.)$ and $q^{u.\hat{u},\hat{y}}(.)$ $(u \in U')$, for the solutions corresponding to $\hat{y}(t), \hat{u}(t)$, ($q^{u.\hat{u},\hat{y}}$) does exist, $x^{\hat{u},\hat{y}}(.)$ assumed to exist, with $x^{\hat{u},\hat{y}}(t) \in Y'$ for all t). If $B(\bar{c}, 3\varepsilon) \subset \text{clco}\pi q^{U',\hat{u},\hat{y}}(T)$, then for some $d' > 0$, for all $y \in B(\hat{y}, d')$, we have that $B(\bar{c}, 2\varepsilon) \subset \text{clco}\pi q^{U',\hat{u}',y}(T)$, and for each $d \in (0, d']$, for each $\hat{y}' \in B(\hat{y}, d'\}$ and for each $c \in B(\bar{c}, \varepsilon)$, there exists a $u' \in U'$ such that $dc = \pi x^{u',\hat{y}'}(T) - \pi x^{\hat{u}',\hat{y}'}(T)$, $\sigma(u', \hat{u}') \leq dT$.

Proof By general theorems on existence, for y $|.|_\infty-$ close to $\hat{y} \in Y''$, and u $\sigma-$close to $\hat{u}$, $x^{u,y}(.)$ exists. Now, $x^{\hat{u},\hat{y}}([0, T])$ is compact, so there exists an open set $B \subset \mathbb{R}^{n+k^*}$ containing $(x^{\hat{u},\hat{y}}([0, T]), \hat{y}([0, T]))$ such that g, g_x and g_y are bounded on $[0, T] \times B \times U$ by some constant M, and such that $(x^{u,y}(t), y(t)) \in B$ for all t, if (u, y) is $\sigma \times |\ |_\infty$ - close to $(\hat{u}, \hat{y})$.

Theorems 1 and 2 in Seierstad (1970) shall be used (with $A = U'$, $\partial = \sigma, T = 1$). Note that, if $A_d = B(\bar{a}, \check{M}d/2)$ (as here assumed), then (D) in Theorem 1 holds for all $a, a' \in A_d$, for any $\hat{y}' \in B(\hat{y}, d) \cap Y''$ ($\check{e}(.)$ independent of $\hat{y}'$ in this ball), where we now have $a = u, y^+(a) = \pi q^{u,\hat{u},\hat{y}'}(T)$, $y(a) = \pi x^{u,\hat{y}'}(T)$, $\check{M} = 2$, $\bar{a} = \hat{u}$). This follows from the continuous differentiability of $\check{E}$ established on p. 484 in Seierstad (1970)[5]. From this continuous differentiability, it also follows that, uniformly in $u, q^{u,\hat{u},\hat{y}'}(T)$ is close to $q^{u,\hat{u},\hat{y}}(T)$ when $\hat{y}'$ is $|.|_\infty-$ close to $\hat{y}$. So, using Lemma 11.1 in Seierstad (1975), if $B(\bar{p}, 3\bar{e}) \subset \text{clco}\pi q^{U',\hat{u},\hat{y}}(T)$, then for $\delta' > 0$ small enough $B(\bar{p}, 2\bar{e}) \subset \text{clco}\pi q^{U,\hat{u},\hat{y}'}(T)$ for any $\hat{y}' \in B(\hat{y}, \delta') \subset Y''$. Choose $d' \in (0, \delta']$ so small that $\sup_{0<d\leq d'} \check{e}(d)\check{M} < \bar{e}/16$, see Proof of Theorem 1 in Seierstad (1970). Then the conclusion in Theorem 1 (in Seierstad (1970))

[5]To see this, one might imagine that the state x is agumented by including even y, with $\dot{y} = v$, $y(0) = y_0$, $v \in \text{cl}B(0, K)$ as an additional trivial state equation, v an auxiliary control. Note that for $u, u' \in U'$, $|g(t, x(.), y(.)u(.)) - g(t, x(.), y(.), u'(.))|_1$, $|g_x(., x(.), y(.), u(.)) - |g_x(., x(.), y(.), u'(.))|_1 \leq M\sigma(u(.), u'(.))$, for all continuous $(x(.)\ , y(.))$ with values in B, compare $||.||^*$ in (7) in Seierstad (1970).)

holds, in fact, for all $d \in (0, d']$, all $\hat{y}' \in B(\hat{y}, d')$, we have $B(d\bar{p}, d\bar{e}) + \pi x^{\hat{u},\hat{y}'}(T) = B(d\bar{p}, d\bar{e}) + y(\bar{a}') \subset y(\text{cl}A_d) = \{\pi x^{u,\hat{y}'}(T) : u \in U',\ \sigma(u, \bar{u}') \leq \check{M}d/2 = d\}$.
Let b' be a positive number and let Y be a set of absolutely continuous functions from $[0, T]$ into $\mathbb{R}^{k^*}$ such that $|\dot{y}(t)| \leq b'$ a.e. and $y(0) = y_0$, y_0 fixed.

Lemma 6. Assume in the situation of Lemma 5 that $Y \subset Y''$ and that Y is closed in sup-norm. For any given $y = y(.) \in Y$, let $\tilde{X}(y)$ be the set of solutions $x^u(.)$ of (18) obtained when $u(.)$ varies through U', and for a given vector z, let $X(y) := \{x(.) \in \tilde{X}(y) : \pi x(T) = z\}$, assumed to be nonempty for each $y \in Y$. For all $y \in Y$, for some $b_y > 0$, all $x(.) \in X(y)$ is assumed to satisfy $|x(.)| \leq b_y$. For a fixed vector a, with $a_k = 0$, $k \leq n_*$, let $X_*(y)$ be the set of $x(.)$ maximizing $ax(T)$ in $X(y)$, i.e. maziming $ax(T)$ subject to $\pi x(T) = z$. For any y, if $x^u(.), u(.)$ is any pair such that $x^u(.) \in X_*(y)$, assume that the necessary conditions (maximum principle) are satisfied for $p_0 = 1$, not $p_0 = 0$, $p_0 \in \{0, 1\}$ the multiplier in the transversality condition $p_k(T) = p_0 a_k$, $k > n_*$ (no information on $p_k(T)$, $k \leq n_*$). Assume that U is compact and, for all (t, x, y), $y \in Y$, that $g(t, x, y, U)$ is convex.

Then $X_*(y)$ is nonempty and has a closed graph as a function of y.

Proof. Recall the measurable selection lemma that $\dot{x}(t) \in g(t, x(t), y(t), U) \Longleftrightarrow \dot{x} = g(t, x(t), y(t), u(t))$ for some measurable $u(.) : J \to U$. See Section 8.3 in Cesari (1983). Moreover, for any y, standard existence theorems (Cesari (1983) Theorem 9.2.i, p. 311) gives that $X_*(y)$ is nonempty. To prove the closed graph property, assume that $x_n = x_n(.) \in X_*(y_n)$, $x_n \to x = x(.)$ in sup-norm, $y_n \to y$ in sup-norm, $y = y(.)$ and $y_n = y_n(.)$ belonging to Y, and let us show that $x \in X_*(y)$.

Consider the "orientor equations" $\dot{x}(t) \in g(t, x(t), y(t), U)$, $\dot{y}(t) \in \text{cl}B(0, b')$, $x(0) = x_0$, $y(0) = y_0$. The result 8.6. in Cesari (1983) p. 299 immediately yields that $x(.) \in X(y)$ as $g(t, x, y, U)$ has Cesari's property (Q). Let us prove that $x(.) \in X_*(y)$. Take any x_* in $X_*(y)$, with corresponding control u_*. The fact that the necessary conditions are not satisfied for $p_0 = 0$ means that for some[6] $\varepsilon > 0$, $B(0, 3\varepsilon) \subset \text{clco}\pi q^{U,u_*,y}(T)$. Then $B(0, 2\varepsilon) \subset \text{clco}\pi q^{U,u_*,y_n}(T)$ for all y_n close to y, see Lemma 5. By Lemma 5, for some $d' > 0$, for all $\hat{y}' \in Y_{d'} := \{y' \in Y : |y' - y| \leq d'\}$, all $d \in (0, d']$,

$$B(0, d\varepsilon) \subset \{x^{u,\hat{y}'}(T) - x^{u_*,\hat{y}'}(T) : u \in U', \sigma(u, u_*) \leq dT\}. \tag{19}$$

Now, for any natural number m such that $1/m \leq d'$, a number $n_m \geq m$ exists, such that $|y_{n_m} - y| \leq 1/m$ ($\Rightarrow y_{n_m} \in Y_{d'}$) and such that $\alpha_{n_m} \leq 1/m$, where $\alpha_n = 2|-x^{u_*,y_n}(T) + x^*(T)|$. Then, $-x^{u_* y_{n_m}}(T) + x^*(T) \in B(0, \alpha_{n_m}) \subset B(0, \varepsilon/m) \subset B(0, d'\varepsilon)$. Thus, by (19), for some $u_{n_m} \in U'$, $\sigma(u_{n_m}, u_*) \leq T/m$, $\pi x^{u_{n_m}, y_{nm}}(T) -$

[6] The origin 0 belongs to $\text{clco}\pi q^{U,u_*,y}(T)$. If 0 is a boundary point, then for some nonzero p^* , $p^*\text{clco}\pi q^{U,u_*,y}(T) \leq p^*0 = 0$, which implies that the maximum principle is satisfied for $p_0 = 0$. So 0 is an interior point.

$x^{u_*,y_{n_m}}(T) = x^*(T) - x^{u_*,y_{n_m}}(T)$, hence $\pi x^{u_{n_m}}(T) = \pi x^*(T) = z$. Now, by optimality, $ax_{n_m}(T) \geq ax^{u_{n_m},y_{n_m}}(T)$ and $ax(T) = \lim_{m\to\infty} ax_{n_m}(T) \geq \lim_m ax^{u_{nm},y_{n_m}}(T) = ax_*(T)$. Hence $x(.) \in X_*(y)$.

Denote by x^{-i} the collection $(x^{(1)}, ..., x^{i-1}, x^{i+1}, ..., x^{(m)})$

Lemma 7 Consider the "Nash problems" (1)-(9) with f^i and g^i containing only $u^{(i)}$.For some numbers K^i, let Y be a set of absolutely continuous functions $y(.) = (y^{(1)}, ..., y^{(m)})$ satisfying $y^{(i)}(0) = x_0^i$ and $|\dot{y}^{(i)}| \leq K^i$, Y closed in sup-norm $\sup_t |y(t)|$. Let $\check{Y} \subset \mathbb{R}^{\sum_i n_i}$ be an open set such that $y(t) \in \check{Y}$ for all t, all $y(.) \in Y$, and assume now that f^i, g^i, f_x^i and g_x^i (exist) and are continuous in $[0,T] \times \check{Y} \times U_i$. Assume that, for each i, U_i is compact and that for all (t, x), $x \in \check{Y}$,

$$N_i(t,x) = \{(f^{(i)}(t,x,u^{(i)}) + \gamma_i, g^{(i)}(t,x,u^i)) : u^{(i)} \in U_i,\ \gamma_i \leq 0\} \text{ is convex.} \tag{20}$$

Assume that for all $y(.) \in Y$, an admissible solution $x^{(i)} = x^{(i)}(.)$ exists when x^{-i} in g is replaced by y^{-i}, (i.e. $x^{(i)}$, satisfies (5) and (9) when $x^{(j)}$ is replaced by $y^{(j)}(.)$, $j \neq i$). Denote the set of such solutions by $\check{X}^i(y)$ and assume, for any $y \in Y$, that some $b_y^i > 0$ exists such that $|x^{(i)}(.)|_\infty \leq b_y^i$ for all $x^{(i)} \in \check{X}^i(y)$. For x^{-i} replaced by y^{-i} in Problem i, denote the set of optimal solutions in $\check{X}^i(y)$ by $X_*^i(y)$. Assume that for any $y = y(.) \in Y$, $X_*(y) := X_*^1(y) \times ... \times X_*^m(y) \subset Y$. If $A_i \neq \mathbb{R}^{n_i}$, assume that, for each $y \in Y$, any pair $(x^{(i),u^{(i)}(.)}(.), u_i(.))$ such that $x^{(i),u^{(i)}(.)}(.) \in X_*^i(y)$ satisfies the necessary conditions only for $p_0 = 1$, not for $p_0 = 0$. Assume that[7]

$$\text{for any } y \in Y,\ X_*^i(y) \text{ is convex.} \tag{21}$$

Then a fixed point $x_* \in X_*(x_*)$, (a sort of equilibrium in open loop controls) exists.

Proof We give a proof only for $f^{(i)} \equiv 0$, $k_i^* = k_i^{**}$. Using auxiliary states, the general case can be derived from this special case. By Lemma 6, $X_*^i(y)$ is nonempty and has a closed graph in sup-norm. Now, $X_*(y) := X_*^i(y) \times .... \times X_*^m(y) \subset Y$, and Y is compact in supnorm. By assumption $X_*^i(y)$ is convex, so $X_*(y)$ is convex. By Kakutani's theorem (see Ch.6, Sec 4 in Aubin and Ekeland (1984)), a fixed point x_* exists in Y such that $x_* \in X_*(x_*)$.

Lemma 8. Assume that no equality terminal restrictions are present. Then the two conditions (20) and (21) can be replaced by the conditions that $f^{(i)}$ and

[7] If $X_*^i(y)$ contains a single point, convexity is trivial. "Almost always", when an optimal control problem is solved, a unique optimal control is found (here for "almost all" y, a unique optimal control migh be expected). But perhaps not for all y : Consider the problem of maximizing $x \to x^3 - yx^2$ in $[-1, 1]$, $y > 0$. Except for $y = 1$, a unique optimal point exists, either $x = 0$, or $x = 1$, for $y = 1$, both are optimal.

$g^{(i)}$ are nondecreasing in[8] $x^{(i)}$ for each (x^{-i}, u, t), that $f^{(i)}$ and g^{i} are concave in $(x^{i}, u^{(i)})$ for each (x^{-1}, t) and that U is convex and compact. Finally, Lemma 8 (as well as Lemma 7) holds even we add the requirements $\psi_k^i(t, x^{(i)(}(t)) \geq 0$ (ψ_k^i given continuous functions), $k = 1, ..., k_i$ on $x^{(i)}(.)$ for $x^{(i)}(.)$ to be admissible, provided the set of admissible solutions, still denoted $\check{X}^i(y)$, is nonempty for all $y \in Y$.

Using an existence result of the type of Theorem 9, p. 135 in Seierstad and Sydsæter (1987), the proof is essentially the same ((21) holds automatically).

Existence of a Nash equilibrium in the example. We now assume that $\dot{k}_1.\dot{k}_2$ are the controls, denote u_1 and u_2. We assume that the $\phi_i's$ have extensions to an open set containing $\{(k_1, k_2) : k_1 \geq 0, k_2 \geq 0\}$. Assume also in the example that C_i is required to belong to a given interval $U_i := [0, M_i]$, $M_i > 0$. and that $\phi_i(k_1, k_2) \leq c + d(k_1^{\gamma} + k_2^{\gamma}) + d' k_1^{\alpha} k_2^{\beta}$, for some positive constants c, d, d', α, β, $\gamma \in (0, 1), \alpha + \beta < 1$. We shall prove existence of a Nash equilibrium in open-loop controls by means of Lemma 8. The monotonicity of $f^{(i)} = v_i(C)$ and $g^{(i)} = \phi_i$ and the concavity is satisfied, as well as compactness of U_i.

Define K by the equality $k_1(0) + k_2(0) + cT + 2dK^{\gamma}T + d'(K^{\alpha+\beta})T = K$, let $\nu_i := \max_{0 \leq k_1 \leq K, 0 \leq k_2 \leq K} \phi_i(k_1, k_2)\}$ and let $Y = \{y = (y_1(.), y_2(.)) : y_i(t) \in [0, K]$, $\dot{y}_i \in [-M_i, \nu_i]$. Given any $y = (y_1, y_2)$, let $k_i(.)$ be any y - admissible solution (meaning, say for $i = 1$, that $\dot{k}_1 = \phi_1(k_1, y_2) - C_i$ for some $C_i(.)$). Let $[0, s]$ be the largest interval on which $k_i(t) < K$ for $t < s$, $i = 1, 2$. Then we have $\dot{k}_i(t) \leq c + d(k_1(t)^{\gamma} + k_2(t)^{\gamma}) + d'(k_1(t)^{\alpha} k_2(t)^{\beta}) < c + 2dK^{\gamma} + d'K^{\alpha+\beta}$ for $t < s$, so $k_i(s) < k_1(0) + k_2(0) + cT + 2dK^{\gamma}T + d'(K^{\alpha+\beta})T = K$ and $[0, s]$ is largest only if $[0, s] = [0, T]$.

From now on consider $i = 1$, the case $i = 2$ has a completely symmetric treatment. Let $\dot{k}_1^* = \max\{\phi_1(k_1^*, 0), \phi_1(k_1^*, K)\}$, $k_1^*(T) = k_1^T/2$. Let $(s, T]$ be the largest interval on which $k_1^*(t) > 0$. Then $k_1^*(s) \geq 0$, but $k_1^*(s) = 0$, gives $k_1^*(.) \equiv 0$ on $[s, T]$ (just insert in the equation and check!), so by uniqueness of solutions[9] even $k_1^*(s) > 0$ and $[s, T]$ is largest only if $s = 0$, (and $k_1^*(.) > 0$ in $[0, T]$). Define $\mu = \min_1 k_1^*(t) > 0$. Then, for any y, for any y - admissible $k_1(.)$ we have automatically $k_1(.) \geq \mu/2 > 0$: If for some t', $k_1(t') \leq \mu/2 < k_1^*(t')$, then on a maximal interval $(t', s), k_1(t) < k_1^*(t)$, but $\dot{k}_1 \leq \dot{k}_1^*$ as long as $k_1 \leq k_1^*$ so even $k_1(s) < k_1^*(s)$, so $s = T$ and $k_1(T) \leq k_1^*(T) = k_1^T/2$, which is impossible. So $k_1(t) \geq \mu/2$ for all t. Hence,, for any y, if $k_i(.)$ is $y -$ admissible, then $(k_1(.), k_2(.)) \in Y$. Moreover, Y is compact in sup-norm. Let $K_*^1(y)$ be the set of optimal solutions in Problem 1, given $y(.)$ (i.e. $y_2(.)$), and let $K_*^2(y)$ be correspondingly defined.[10].

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[8] Provided, for each component number k, $g_k^{(i)} = h_k^i + a_k^i(t) x_k^{(i)}$, $a_k^i(t)$ continuous, ≤ 0, we have that the monotonicity of $g^{(i)}$ in $x^{(i)}$ can be replaced by the monotonicity of $h^i(t, x, u)$ in $x^{(i)}$.

[9] Here the extension property of the ϕ_i's is needed.

[10] If the extension property does not hold, imagine that $k_k(t) \geq k_i^T/2$ for all t, is added as a requirement for $k_i(.)$ to be admissible. (Still, for all $y \in Y$, y - admissible $k_i(.)$ exist.) Using necessary conditions for problems with state restrictions, it is easily seen that $p_i(.)$

The bounds on C_i introduced bounds on u_i. When discussing the necessary conditions, it is easiest to treat the C_i's as the controls. For any $y \in Y$, if any pair $(k_1(.), C_1(.))$, $k_1(.) \in K^1_*(y)$, satisfies $C_1 \equiv 0$, then $k_1(T) > k_1^T$ ($\dot{k}_1 \geq 0$, $k_1^T < k_1^0$). Hence, (for p_1 being the adjoint variable corresponding to k_1), $p_1(T) = 0$, $p_0 = 1$, but then $C_1 \equiv 0$ cannot satisfy the maximum condition. Thus, for any pair $(k_1(.), C_1(.))$, $k_1(.) \in K^1_*(y)$, we have that $C_1(t) > 0$ for some t. We can show that any pair $k_1(.), C_1(.)$, $k_1(.) \in K^1_*(y)$, satisfies the maximum condition only for $p_0^{(1)} = 1$: As $C_1 \in (0, M_1]$ for some t, at this point $H_C \geq 0$. Now, by contradiction, if $(p_0^{(1)}, p_1(t)) = 0$ then $p_2(t) \neq 0$. But then, $p_2(s) \neq 0$, for s close to t. By the adjoint equation for $p_1(.)$, close to t, the two addends in the right hand side of the adjoint equation for $p_1(.)$ are nonnegative when not including the minus- signs (proved earlier) and with the term containing p_2 being nonzero. With $p_1(.) \geq 0$ (proved earlier), this gives $p_1(t) \neq 0$, a contradiction. So $(p_0^{(1)}, p_1(t)) \neq 0$. This, combined with $H_C \geq 0$ at t and $p_1(t) \geq 0$ gives $p_0^{(1)} = 1$. All conditions in Lemma 8 are then satisfied, so a Nash equilibrium exists.

An additional comment.
We can prove that for M_i large enough, the Nash equilibrium is a Nash equilibrium even if $[0, U_i]$ is replaced by $[0, \infty)$. Let us first prove that there exists a positive constant M_1^* such that if $M_1 \geq M_1^*$, then, for any y, for any optimal pair $(k_1(.), C(.))$, $k_1(.) \in X_*(y)$, we have $C(t) < M_1^* \leq M_1$ for all t.

Now, $|\partial\phi_i/\partial\kappa_j|$ is bounded by a constant Φ for $k_1, k_2 \in [\mu/2, K]$ independent of i, j. Note that by the adjoint equations, for any $s \in [0, T]$, $p_1(s) = A(s)p_1(T)$ for some positive constant $A(s)$ continuously dependent on s, in fact, by the existence of Φ, for two positive constants B and D, independent of $y(.)$ and $k_1(.)$, for all s, $Dp_1(T) \leq p_1(s) \leq Bp_1(T)$. First, choose a positive number q so large that $k_1^{**}(T) < k_1^T$, $\dot{k}_1^{**} = \max\{\phi_1(k_1^{**}, 0), \phi_1(k_1^{**}, K)\} - q$, $k_1^{**}(0) = k_0^1$. Choose $F > 1$ so large that $1/(2q^{1/2}) > (B/D)/(2(Fq)^{1/2})$, and let M_1 be any given number $\geq M_1^* := Fq$. Now, for any $k_1 \in X^1_*(y)$, $k_1(.) \leq k_1^{**}(.)$ if $C_1 \geq Fq \geq q$ for all t, so $k_1(T) < k_1^T$, which is not possible. So for some t, $C_1 < q$, and at this point $H_C \leq 0$, thus $1/(2q^{1/2}) \leq p(t)$, (above we showed $p_0^1 = 1$). If for some $M, C_1 = M \geq Fq$ for some t', then $H_c \geq 0$ at t', i.e. $1/(2(M)^{1/2}) \geq p(t')$. Now, $1/(2q^{1/2}) \leq p(t) \leq Bp_1(T) \leq Bp(t')/D \leq (B/D)/(2(M)^{1/2})$, which contradicts $M \geq Fq$ and the definition of F. Hence, $C_1(t) < M_1^* = Fq$ for all t.

For $M_i \geq Fq$, any Nash equilibrium is also a Nash equilibrium for $U_i = [0, \infty)$. To see this, note that for any given $y \in Y$, for any y - admissible pair $(k_1(.), C_1(.))$ in the original problem, by definition $\dot{k}_1(.)$ is integrable, so $C_1(.)$ is integrable, hence $C_1^M(.) := \min\{M, C_1(.)\} \to C_1(.)$ in L_1 when $M \to \infty$,

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is nonincreasing. Then any optimal $k_i(.) \in K^i_*(y)$ satisfies $k_i(t) > k_i^T/2$ for all t. If, by contradiction, $k_i(t') = k_i^T/2$, for some $t' < T$, let t' be smallest possible. Then for some $t < t'$, t' close to t', $\dot{k}_i(t) < 0$, $k_i(t) < k_i^T$. For some $t'' > t'$, it must be the case that $k_i(t'') = k_i(t)$, $\dot{k}_i(t'') \geq 0$ (we must reach k_1^T at $t = T$), hence $C_i(t) > C_i(t'')$ contradicting the maxium condition and monotonicity of $p(.)$. So any optimal $k_i(.) \in K^i_*(y)$ satisfies $k_i(t) > k^T/2$ for all t.

and the solution k_1^M (.) corresponding to C_1^M (.) satisfies $k_1^M(.) \geq k_1(.)$, so is admissible. The criterion values α_M corresponding to $C_1^M(.)$ converges to the criterion value α corresponding to $C_1(.)$, and $\alpha_M \leq$ the optimal value β $(= \beta(y))$ of the criterion for all $M = M_1$ when M_1 is $\geq Fq$, (then β does not depend on M_1). Hence, $\alpha = \lim_{M\to\infty} \alpha_M \leq \beta$.

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