MEMORANDUM

No 04/2011

Terminal units in DEA: Definition and Determination

Vladimir Krivonozhko
Finn R. Førsund
Andrey V. Lychev

ISSN: 0809-8786

Department of Economics
University of Oslo

This series is published by the
University of Oslo
Department of Economics

P. O.Box 1095 Blindern
N-0317 OSLO Norway
Telephone: + 47 22855127
Fax: + 47 22855035
Internet: http://www.sv.uio.no/econ
e-mail: econdep@econ.uio.no

In co-operation with
The Frisch Centre for Economic Research

Gaustadalleén 21
N-0371 OSLO Norway
Telephone: +47 22 95 88 20
Fax: +47 22 95 88 25
Internet: http://www.frisch.uio.no
e-mail: frisch@frisch.uio.no

Last 10 Memoranda

No 03/11	André K. Anundsen, Tord S. H. Krogh, Ragnar Nymoen, Jon Vislie *Overdeterminacy and endogenous cycles: Trygve Haavelmo's business cycle model and its implications for monetary policy*
No 02/11	Nils Chr. Framstad *Portfolio Separation Properties of the Skew-Elliptical Distributions*
No 01/11	Karine Nyborg , Tao Zhang *Is corporate social responsibility associated with lower wages?*
No 22/10	Rebecca Graziani, Nico Keilman *The sensitivity of the Scaled Model of Error with respect to the choice of the correlation parameters: A Simulation Study*
No 21/10	Jennifer L. Castle, Jurgen A. Doornik , David F. Hendry, Ragnar Nymoen *Testing the Invariance of Expectations Models of Inflation*
No 20/10	Erik Biørn *Identifying Trend and Age Effects in Sickness Absence from Individual Data: Some Econometric Problems*
No 19/10	Michael Hoel, Svenn Jensen *Cutting Costs of Catching Carbon* *Intertemporal effects under imperfect climate policy*
No 18/10	Hans Jarle Kind, Tore Nilssen, Lars Sørgard *Price Coordination in Two-Sided Markets: Competition in the TV Industry*
No 17/10	Vladimir Krivonozhko, Finn R. Førsund and Andrey V. Lychev *A Note on Imposing Strong Complementary Slackness Conditions in DEA*
No 16/10	Halvor Mehlum and Karl Moene *Aggressive elites and vulnerable entrepreneurs* *- trust and cooperation in the shadow of conflict*

Previous issues of the memo-series are available in a PDF® format at:
http://www.sv.uio.no/econ/forskning/publikasjoner/memorandum

TERMINAL UNITS IN DEA:

DEFINITION AND DETERMINATION

by

Vladimir Krivonozhko,

Institute for Systems Analysis, Russian Academy of Sciences,
Moscow

Finn R. Førsund,

Department of Economics, University of Oslo

Andrey V. Lychev

Accounts Chamber of the Russian Federation, Moscow

Abstract: Applications of the DEA models show that inadequate results may arise in some cases, two of these inadequacies being: a) too many efficient units may appear in some DEA models; b) a DEA model may show an inefficient unit from the point of view of experts as an efficient one. The purpose of this paper is to identify units that may unduly become efficient. The concept of a terminal unit is introduced for such units. A method for improving the adequacy of DEA models based on terminal units is suggested, and an example shown based on a real-life data set for Russian banks.

Keywords: Terminal units; DEA; Efficiency; Weight restrictions; Domination cones

JEL classifications: C44, C61, C67, D24

1. Introduction

After a decade of applications of DEA analysis, originating in Farrell (1957) and Farrell and Fieldhouse (1962) and generalised and put into the linear programming format we use today by Charnes et al. (1978) (hereafter called the CCR model), it was recognised that results both concerning efficiency scores and shape of the frontier production function, on which Farrell efficiency measures are based, were not always adequate when confronted with expert knowledge of the units to which DEA was applied. Types of inadequacies discussed in the literature have been that too many efficient units may appear in some DEA models, a DEA model may show an inefficient unit from the point of view of experts as an efficient one, too many zeros appear as solutions for the multipliers (weights), and units are not properly enveloped.

The first attempts in the literature (Thompson et al., 1986; Dyson and Thanassoulis, 1988) to restrict the estimation of the frontier function and consequently the efficiency scores, took two different types of failings as their point of departure. The problem of Thompson et al. (1986) was that the number of units under investigation was so small (only six) that all but one of the units was rated efficient using conventional DEA. In order to increase the discrimination restrictions on the so-called weights (appearing in the dual solution if the primal model is the envelopment model formulated in the space of inputs and outputs) were enforced. This approach was followed up in Thompson et al. (1990), and Charnes et al. (1989, 1990), the latter papers introducing the cone-ratio approach of basing the shape of the frontier on a few efficient units selected by experts by restricting the weights (multipliers) to be within cones in the dual space.

Dyson and Thanassoulis (1988) were taking a different tack. They were preoccupied with the consequence of zero weights (or weights of value ε where ε is a non-Archimedean number) leading to “some DMUs being assessed only on a small subset of their inputs and outputs, while their remaining inputs and outputs are all but ignored (p. 563).” Restriction on weights should be based on expert opinion, but the purpose was to eliminate zero weights, and not to reduce the efficiency of units being 100 % efficient within the conventional DEA model.

The development of this literature is reviewed in Allen et al. (1997) and Pedraja-Chaporro et al. (1997). An interesting new line of introducing restrictions directly in the input – output space and not in the dual space started with Bessent et al. (1988) and Lang et al. (1995) of extending the faces. This was followed up in Thanassoulis and Allen (1998) by explicitly reducing the number of zero weight for inefficient observations by introducing new unobserved units based on expert opinions using units called anchor units as point of departure. A formal attempt to define anchor units and to introduce ways of finding them was done in Allen and Thanassoulis (2004) in the case of constant returns to scale and a single input. This definition was generalised in Bougnol and Dulá (2009) to multiple inputs and outputs and variable returns to scale. An elaborate algorithm for finding anchor points was introduced. The empirical applications gave the somewhat surprising result that almost all extreme efficient units are in fact anchor points. The situation of some zeros for weights seems to be the normal situation for DEA applications. However, their algorithms may produce units that are just usual efficient units (vertices) in DEA models.

Edvardsen et al. (2000) suggested an empirical witty method for discovering "suspicious" units; they call them "exterior units". However, their method cannot discover all suspicious units.

An elegant and subtle approach was proposed in the DEA area to deal with the problems of inadequacies of the DEA models. This approach is based on incorporating domination cones (Yu, 1974) in DEA models. A number of outstanding papers were devoted to substantiation, development and applications of domination cones to DEA models (Brockett et al., 1997; Charnes et al., 1989; Charnes et al., 1990; Thompson et al., 1997; Wei et al., 2008; Yu et al., 1996). Cones are usually determined in the dual space of multipliers.

It is rather difficult, however, for a manager (the decision-maker) to determine cones in the multipliers space that is dual to the space of inputs and outputs where a production possibility set is constructed (Cooper et al., 2000). For this very reason only two particular DEA models with cones are widely used in practice at present: the assurance region model and the cone-ratio model (Cooper et al., 2000).

The purpose of this paper is to identify units that may unduly become efficient by making use of a new concept; a terminal unit. The plan of the paper is to go into the background in Section 2, using key elements form the cone-ratio approach developed in Charnes et al. (1990), based on the Banker et al. (1984) model of variable returns to scale, establishing necessary definitions. The main results are presented in Section 3, including the definition of a terminal unit and illustrating its difference from the term anchor unit and using domination cones to establish that terminal production units exist if some production units become inefficient if cones are inserted in the model. Some numerical experiments on data for Russian banks are carried out in Section 4, showing how to find a terminal unit and how to use experts to indicate an artificial efficient unit using a visual interactive graphical technique. Section 5 concludes and offer ideas for further research.

2. Background

Consider a set of n observations of actual production units (X_j, Y_j), $j = 1, \ldots, n$, where the vector of outputs $Y_j = (y_{1j}, \ldots, y_{rj}) \geq 0$, $j = 1, \ldots, n$, is produced from the vector of inputs $X_j = (x_{1j}, \ldots, x_{mj}) \geq 0$. The production possibility set T is the set $\{(X, Y) \mid \textit{the outputs } Y \geq 0 \textit{ can be produced from the inputs } X \geq 0\}$.

The model in Banker et al. (1984) exhibiting variable returns to scale (hereafter termed the BCC model) that is a primal input-oriented model can be written in the form

$$\min \theta$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} X_j \lambda_j + S^- = \theta X_o, \\
&\sum_{j=1}^{n} Y_j \lambda_j - S^+ = Y_o, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, \quad j = 1, \ldots, n, \\
&s_k^- \geq 0, \quad k = 1, \ldots, m, \\
&s_i^+ \geq 0, \quad i = 1, \ldots, r,
\end{aligned} \tag{1a}$$

where $X_j = (x_{1j},\dots,x_{mj})$ and $Y_j = (y_{1j},\dots,y_{rj})$ represent the observed inputs and outputs of production units $j = 1,\dots,n$, $S^- = (s_1^-,\dots,s_m^-)$ and $S^+ = (s_1^+,\dots,s_r^+)$ are vectors of slack variables. In this primal model the efficiency score θ of production unit (X_o, Y_o) is found; (X_o, Y_o) is any unit from the set of production units (X_j, Y_j), $j = 1,\dots,n$.

Notice that we do not use an infinitesimal constant ε (a non-Archimedean quantity) explicitly in the DEA models, since we suppose that each model is solved in two stages in order to separate efficient and weakly efficient units.

The dual multiplier form of the BCC model (1a) is expressed as

$$\max\,(u^T Y_0 - u_0)$$

subject to

$$\begin{gathered} u^T Y_j - v^T X_j - u_0 \le 0, \quad j = 1,\dots,n \\ v^T X_0 = 1, \\ v_k \ge 0, \quad k = 1,\dots,m, \quad u_i \ge 0, \quad i = 1,\dots,r \end{gathered} \tag{1b}$$

where (v, u, u_0) is a vector of dual variables, $v \in E^m$, $u \in E^r$, u_0 is an unconstrained scalar variable associated with the convexity constraint.

The BCC primal output-oriented model can be written in the following form

$$\max \eta$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} X_j \lambda_j + S^- = X_0, \\ &\sum_{j=1}^{n} Y_j \lambda_j - S^+ = \eta Y_0, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \ge 0, \quad j = 1,\dots,n, \\ &s_k^- \ge 0, \quad k = 1,\dots,m, \\ &s_i^+ \ge 0, \quad i = 1,\dots,r. \end{aligned} \tag{1c}$$

The dual multiplier form of the BCC output-oriented model (1c) is written in the form

$$\min\,(v^T X_0 + u_0)$$

subject to

$$u^T Y_j - v^T X_j - u_0 \le 0\,,\quad j = 1,\ldots,n \qquad \text{(1d)}$$

$$u^T Y_0 = 1,$$

$$v_k \ge 0\,,\; k = 1,\ldots,m,\; u_i \ge 0\,,\; i = 1,\ldots,r\,,$$

where (v,u,u_0) is a vector of dual variables, $v \in E^m$, $u \in E^r$, u_0 is a scalar variable associated with the convex constraint (the same symbols for dual variables are used as for models (1b)).

Definition 1. (Cooper et al. 2000). Unit $(X_o,Y_o) \in T$ is called efficient with respect to the input-oriented BCC model if and only if any optimal solution of (1a) satisfies: a) $\theta^* = 1$, b) all slacks s_k^-, s_i^+, $k = 1,\ldots,m$, $i = 1,\ldots,r$ are zero.

We denote the set of efficient points of T with respect to the input-oriented BCC model by $Eff_I T$.

If the first condition (a) in Definition 1 is satisfied, then unit (X_o,Y_o) is called input weakly efficient with respect to the BCC input-oriented model. We denote the set of these weakly efficient points by $WEff_I T$. In the DEA literature (Banker and Thrall, 1992; Seiford and Thrall, 1990) this set is also called the input boundary.

Definition 2. (Cooper et al. 2000). Unit $(X_o,Y_o) \in T$ is called efficient with respect to the output-oriented BCC model if and only if any optimal solution of (1c) satisfies: a) $\eta^* = 1$, b) all slacks s_k^-, s_i^+, $k = 1,\ldots,m$, $i = 1,\ldots,r$ are zero.

Let us denote the set of efficient points with respect to the output-oriented BCC model by $Eff_O T$.

If the first condition in Definition 2 is satisfied, then unit (X_o,Y_o) is called output weakly efficient with respect to the BCC model. We denote the set of these weakly efficient points by $WEff_O T$. In the DEA literature (Banker and Thrall 1992; Seiford and Thrall 1990), this set is also called the output boundary.

In DEA theory (Cooper et al. 2000), it was established that sets $Eff_I T$ and $Eff_O T$ coincide. So, let $Eff\;T$ denote both $Eff_I T$ and $Eff_O T$.

Definition 3. Production unit $(X',Y') \in T$ is weakly Pareto efficient if and only if there is no $(X,Y) \in T$ such that $X < X'$ and $Y > Y'$. We denote the set of weakly Pareto efficient units by $WEff_P T$.

Krivonozhko et al. (2005) have proved that the following relations hold:

$$Eff\ T \subseteq WEff_I T \cap WEff_O T, \quad WEff_I T \cup WEff_O T \subseteq WEff_P T = Bound\ T,$$

where the boundary of T is designated as $Bound\ T$.

The production possibility set T_B for the BCC model can be written in the form (Banker et al., 1984)

$$T_B = \left\{ (X,Y) \;\middle|\; \sum_{j=1}^{n} X_j \lambda_j \le X, \sum_{j=1}^{n} Y_j \lambda_j \ge Y, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \ge 0, j = 1,\dots,n \right\}. \quad (2)$$

It was shown in the DEA scientific literature (see, Krivonozhko et al., 2009) that the BCC model can approximate any DEA model from a large family of DEA models. For this reason, we consider the BCC model as a basic model in our exposition.

3. Main results

The main idea of incorporating domination cones in DEA models is to reduce the domain of multipliers. For this purpose, additional constraints on multipliers are incorporated in the DEA models.

In the assurance region method, constraints on the multipliers are added to the CCR model in the following manner; see Charnes et al. (1990),

$$\begin{aligned} l_{1i} \le \frac{v_i}{v_1} \le k_{1i}, \quad (i = 2,\dots,m), \\ L_{1s} \le \frac{u_s}{u_1} \le K_{1s}, \quad (s = 2,\dots,r), \end{aligned} \qquad (3)$$

where l_{1i}, k_{1i}, L_{1s}, K_{1s} are given low and upper bounds on the ratios of multipliers.

However, relations (3) cannot describe all possible polyhedral cones in space E^{m+r}.

Assertion 1. *There exist polyhedral cones in multidimensional space* E^{m+r} *that cannot be described by relations (3).*

Thus, formulas (3) describe only some subset of possible polyhedral cones in multidimensional space E^{m+r}. The next model enables one to use more general form of domination cones in the DEA models.

The dual multiplier form of the cone-ratio model is expressed as (Charnes et al., 1989; 1990; Yu et al., 1996)

$$\max\,(uY_o - u_o)$$

subject to

$$\begin{gathered} v^T X_o = 1, \\ -v^T X_j + u^T Y_j - u_o \le 0, \quad j = 1,\dots,n, \\ u \in U, \quad v \in V, \end{gathered} \tag{4a}$$

where variables $v \in E^m$, $u \in E^r$, $u_o \in E^1$ and $U \subseteq E^r_+$, $V \subseteq E^m_+$ are given polyhedral cones.

The primal problem of (4a) is written as

$$\min \theta$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} X_j \lambda_j - \theta X_o \in V^*, \\ &Y_o - \sum_{j=1}^{n} Y_j \lambda_j \in U^*, \\ &\sum_{j=1}^{n} \lambda_j = 1, \\ &\lambda_j \ge 0, \quad j = 1,\dots,n, \end{aligned} \tag{4b}$$

where V^* and U^* are negative polar cones of sets V and U, respectively.

In practice, polyhedral cones U and V are constructed as follows: a) some excellent units are chosen from the point of view of experts; b) averages of the optimal multipliers u_i^*, v_i^* are computed for every excellent unit $i \in Ex$. Vectors u_i^*, v_i^*, $i \in Ex$ form polyhedral cones U and V.

Cones U and V reduce the feasible domain of multipliers, while the feasible domains of inputs, see Fig. 1, and outputs, see Fig. 2, are expanding.

Now, we make an attempt to reveal the causes of inadequacies in DEA models.

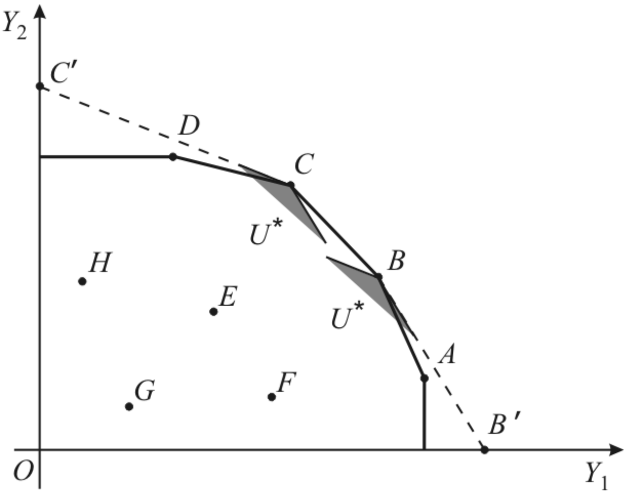


Fig. 1. Transformation of the frontier in the output subspace in the cone-ratio method

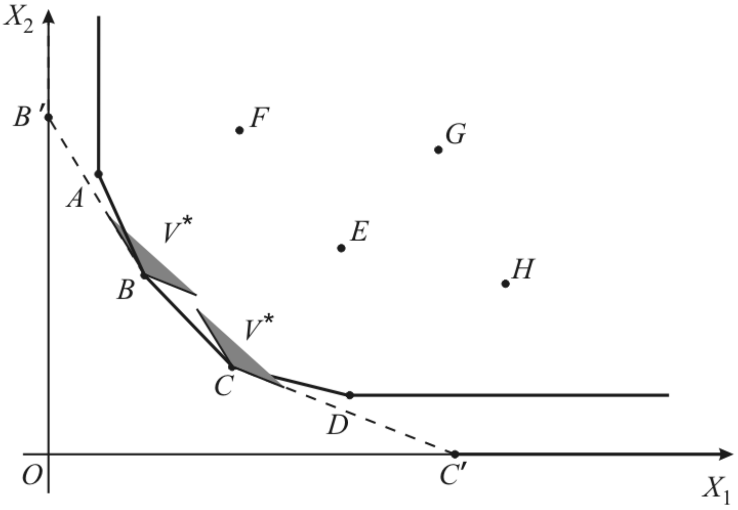


Fig. 2. Transformation of the frontier in the input subspace in the cone-ratio method

Assumption. *Let the cone-ratio method allows one to reduce the number of "suspicious" production units, i.e. the units that have 100% efficiency score, however, these units should be inefficient from the point of view of experts.*

Production possibility set T_B (2) is a convex polyhedral set. According to the classical theorems of Goldman (1956) and Motzkin (1936) any convex polyhedral set can be represented as a vector sum of convex combination of vertices and the non-negative linear combination of vectors (rays).

Definition 4. *We call an efficient (vertex) unit terminal unit if an infinite edge is going out from this unit.*

Definition 5. *We call a face* $\Gamma \subset WEff_I T \cup WEff_O T$ *of set* T_B *a terminal face if this face contains an infinite edge.*

Then the following assertion can be proved if Assumption above is valid.

Theorem 1. *If some production units become inefficient as a result of inserting cones in the BCC model (1), then it is necessary that there exist terminal production units among such inefficient units.*

Proof. See the proof of Theorem 4 in Appendix A for the more general case.

Observe that the results of theorem 1 was proved for the cone-ratio model at first, since this model was thoroughly elaborated from theoretical and practical points of view in the scientific literature on the DEA models, see Charnes et al. (1990), Brockett et al. (1997) and Cooper et al. (2000). However, the cone-ratio model cannot cover all possible cases where inadequate results may appear in the DEA models. Therefore theorem 4 is proven in this paper for the generalized DEA model with domination cones.

Thus, Theorem 1 shows that terminal points are the first "suspicious" units which may cause inadequate results in the DEA models.

The following optimization models enable us to find terminal units or units belonging to terminal faces of the production possibility set. Let *EF* designate the set of observed efficient units of the BCC model (1). For this purpose two types of models are solved for every efficient unit $q \in EF$.

Problem $P_i(q)$ ($i = 1,\ldots,m$)

$$\begin{gathered} \max J_{1i} = \eta \\ \sum_{j=1}^{n} X_j \lambda_j \le X_q + \tau d_i, \\ \sum_{j=1}^{n} Y_j \lambda_j \ge \eta Y_q, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \ge 0, \tau \ge 0, \end{gathered} \tag{5}$$

where $d_i = (0, \ldots, 1, \ldots, 0) \in E^m$, the unity is in ith position.

Theorem 2. *If in problem (5) the optimal value* $J_{1i}^* = 1$*, then unit* (X_q, Y_q) *is a terminal one or belongs to a terminal face* $\Gamma \subset WEff_O T$.

Proof. Under any $\tau \geq 0$ ray $(X_q + \tau d_i, Y_q)$ is a feasible subset for production possibility set T_B (2) due to monotonicity of T_B . It follows from model (5) that if $J_{1i}^* = 1$, then this ray belongs to the set $WEff_O T_B$. Hence this ray is an unbounded edge or belongs to an unbounded face of set T_B.

This completes the proof.

The following models determine infinite edges emanating along direction g_i, where $g_i = (0, \ldots, 1, \ldots, 0) \in E^r$ (the unity is in i th position).

Problem $R_i(q)$ $(i = 1, \ldots, r)$

$$\begin{gathered} \min J_{2i} = \theta \\ \sum_{j=1}^{n} X_j \lambda_j \leq \theta X_q, \\ \sum_{j=1}^{n} Y_j \lambda_j \geq Y_q - \tau g_i, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \geq 0, \tau \geq 0. \end{gathered} \tag{6}$$

Theorem 3. *If in problem (6) the optimal value* $J_{2i}^* = 1$*, then unit* (X_q, Y_q) *is a terminal one or belongs to a terminal face* $\Gamma \subset WEff_I T$.

Proof. Under any $\tau \geq 0$ ray $(X_q, Y_q - \tau g_i)$ belongs to the production possibility set T_B due to monotonicity of set T_B. It follows from (6) that if $J_{2i}^* = 1$, then this ray belongs to the set $WEff_I T_B$. Hence this ray is an unbounded edge or belongs to an unbounded face of set T_B.

This completes the proof.

Thus, models (5) and (6) enable us to reveal terminal units or efficient units belonging to unbounded faces.

Bougnol and Dulá (2009) determined an anchor point as an efficient vertex belonging to an unbounded face of set T_B. They proposed algorithms for discovering anchor points. However, their algorithms may produce units that are just usual efficient units (vertices) in the DEA models. Moreover, such units are not suitable as points of departure for considering improving the frontier. Indeed, consider the following example. In Figure 3, a two-inputs/one-output BCC model is depicted. Units A, B, C, D, E are the observed efficient production units that determine set T_B. Units M, A, B, C, L form the face of set T_B. This face belongs to the orthant X_1OY. Hence efficient unit B belongs to the unbounded face. However, unit B is just a common vertex of the BCC model. Increasing an input or decreasing an output of unit B generates a new inefficient point that does not belong to set $(WEff_I T \cup WEff_O T)$. This unit cannot be used to reduce the DEA-inefficient part of the production possibility set, see Allen and Thanassolius (2004), hence this unit is not suitable for the frontier improvement, what contradicts to the notion of an anchor point.

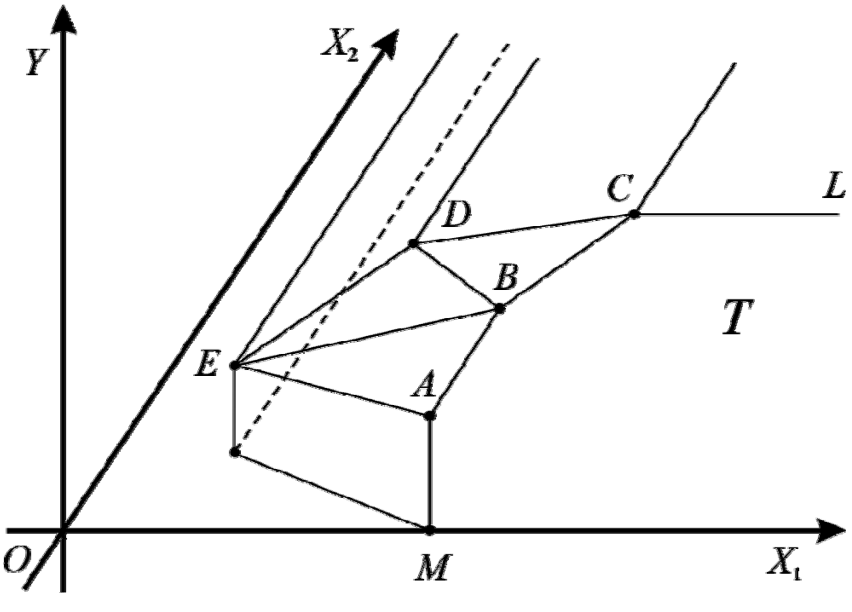
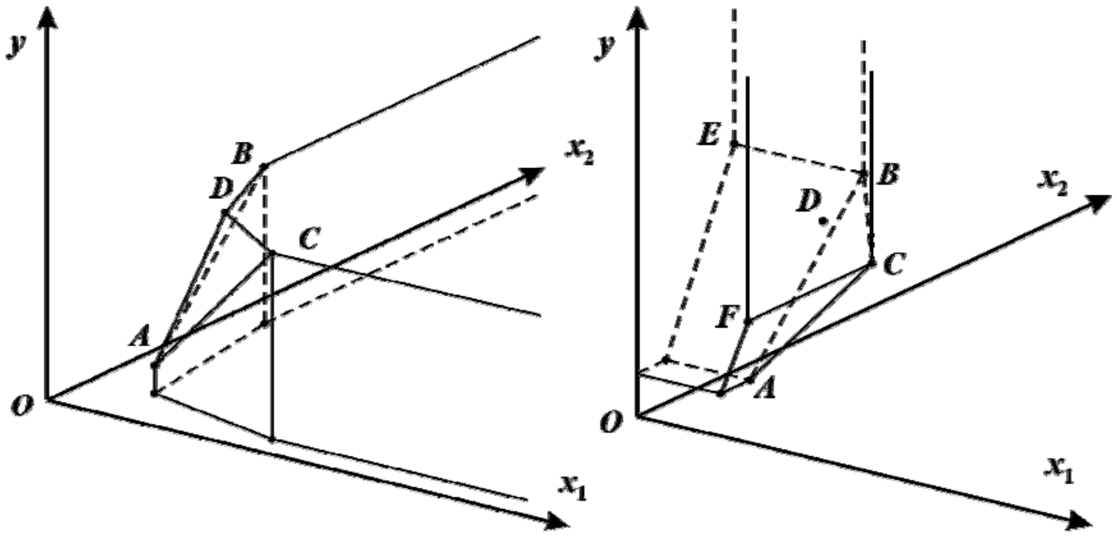


Fig. 3. Unit B is an anchor point, but not a terminal point

Edvardsen et al. (2008) suggested an empirical method for discovering “suspicious” units; they call them “exterior units”. However, their method cannot discover all suspicious units. Indeed, consider the following Figure 4, panel (a). The BCC model is determined by units A, B, C, D. Again consider the BCC model with the same units, but now y is an input variable, x_1 and x_2 are output variables, see Figure 4, panel (b). Point D is not efficient in this case. Hence this point is not



(a) (b)

Fig. 4. Point D is not an exterior unit in the three-dimensional BCC model

exterior. At the same time this point is a “suspicious” unit, since it may cause inadequacy in the DEA model.

However, the cone-ratio model (3) cannot help in every case where suspicious units appear in the DEA models. Figure 5 depicts a three-dimensional BCC model.

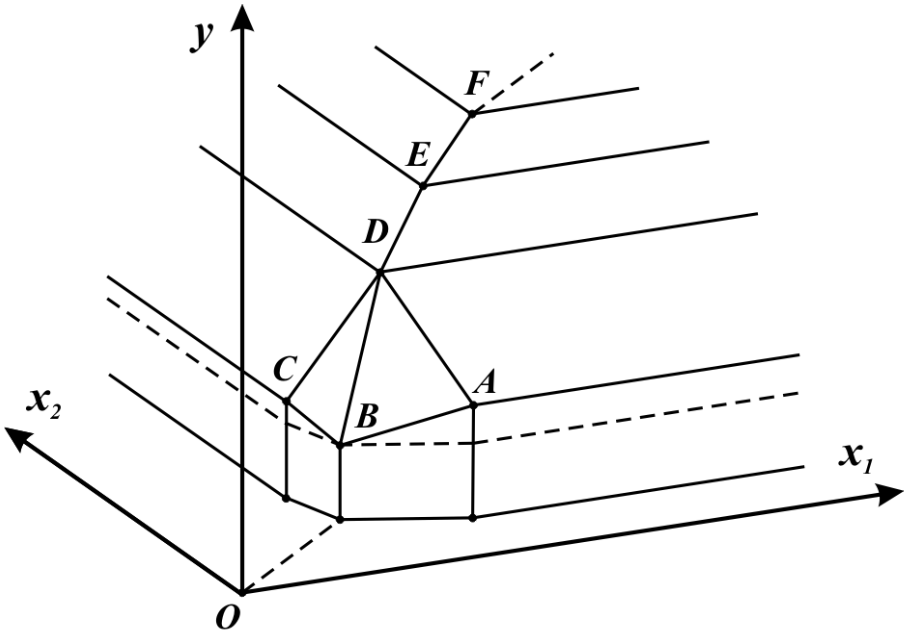


Fig. 5. Three-dimensional BCC model

Points A-F (efficient units) determine the production possibility set T_B. Point B is a terminal unit. However, it is impossible to transform the frontier with the help of cones U and V in such a way that terminal point B would be inefficient, see Fig. 6.

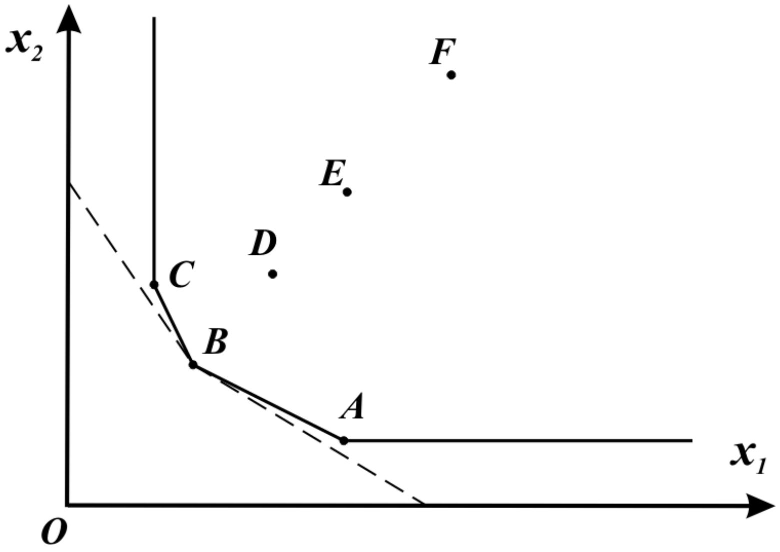


Fig. 6. Input isoquant for unit B

Only simultaneous transformation of the frontier in the space of inputs and outputs enables one to make suspicious unit B inefficient, see Fig. 7.

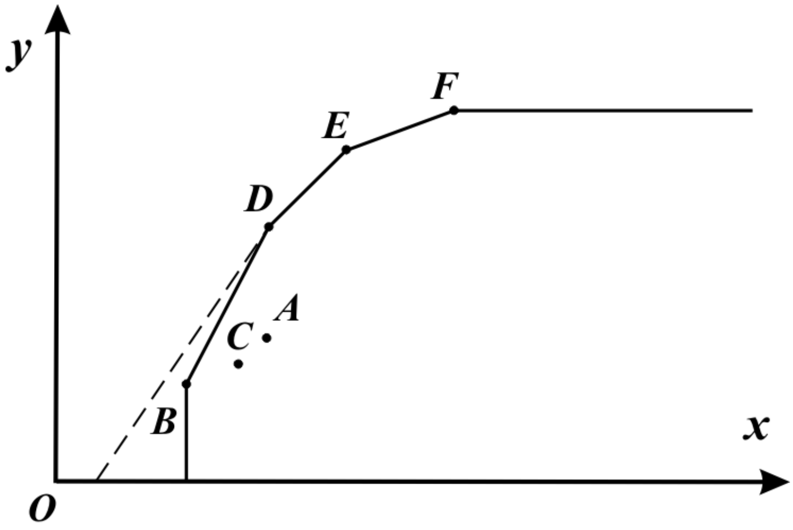


Fig. 7. Production function for unit B

Yu et al. (1996) proposed the following generalized DEA (GDEA) model that unifies and extends most the well-known DEA models based on using domination cones (see, e.g. Yu, 1974) in their constraint sets.

$$\max(u^T Y_0 - \delta_1 u_0)$$

subject to

$$\begin{aligned} & v^T \overline{X} - u^T \overline{Y} + u_0 \delta_1 e^T \in K, \\ & v^T X_0 = 1, \\ & \begin{pmatrix} v \\ u \end{pmatrix} \in W, \quad \delta_1 \delta_2 (-1)^{\delta_3} u_0 \geq 0, \\ & v \in E^m, u \in E^r, u_0 \in E^1. \end{aligned} \tag{7a}$$

The optimization dual problem to (7a) is written in the form (Yu et al, 1996):

$$\min \theta$$

subject to

$$\begin{aligned} & \begin{pmatrix} \overline{X}\lambda - \theta X_0 \\ -\overline{Y}\lambda + Y_0 \end{pmatrix} \in W^*, \\ & \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ & \lambda \in -K^*, \quad \lambda_{n+1} \geq 0, \quad \theta \in E^1, \end{aligned} \tag{7b}$$

where $\overline{X} = (X_1, \ldots, X_n)$ is an $m \times n$ matrix, $X_j = (x_{1j}, \ldots, x_{mj}) \geq 0$ is the input vector for the *jth* production unit $j = 1, \ldots, n$; $\overline{Y} = (Y_1, \ldots, Y_n)$ is an $r \times n$ matrix, $Y_j = (y_{1j}, \ldots, y_{rj}) \geq 0$ is the output vector for the *jth* production unit $j = 1, \ldots, n$. Parameters $\delta_1, \delta_2, \delta_3$ are binary ones assuming only the values 0 and 1. Vector e is determined as $e = (1, \ldots, 1) \in E^n$. Sets $W \subseteq E^{m+r}$ and $K \subseteq E^n$ are the closed convex cones, where E^{m+r} and E^n are Euclidean spaces of the dimensions $(m+r)$ and n, respectively. W^* and K^* are the negative polar cones (Charnes et al, 1989; Yu et al, 1996) of sets W and K, respectively. It is usually assumed in the DEA models that the polyhedral cones $W \subseteq E_+^{m+r}$ and $K \subseteq E_+^n$ and $\operatorname{int} W \neq \varnothing$ and $\operatorname{int} K \neq \varnothing$, then we get $W^* \neq \varnothing$ and $K^* \neq \varnothing$.

Theorem 4. *If some efficient units in model (1) become inefficient in model (7) as a result of inserting cones in the BCC model, then it is necessary that there exist terminal production units among such inefficient units.*

Proof. See Appendix A.

It is rather difficult for a manager (expert) to determine cones in the multipliers space that is dual to the space of inputs and outputs where a production possibility set is constructed. For this very reason it is difficult to use the GDEA model in practice.

Krivonozhko et al. (2009) proposed a model that is more general than the GDEA model, on the one hand, as it covers situations that the GDEA model cannot describe. On the other hand, this model enables one to construct step-by-step any model from a large family of the DEA models by incorporating artificial units and rays in the space of inputs and outputs in the BCC model, which makes the process of model construction visible and more understandable.

The production possibility set of this model is written in the form

$$\begin{aligned} T_G = \Bigg\{ (X,Y) \;\Bigg|\; & X \geq \sum_{j=1}^{n} X_j \lambda_j + \sum_{i \in I} D_i \mu_i + \sum_{k \in J} A_k \rho_k , \\ & Y \leq \sum_{j=1}^{n} Y_j \lambda_j + \sum_{i \in I} G_i \mu_i + \sum_{k \in J} B_k \rho_k , \ \sum_{j=1}^{n} \lambda_j + \sum_{i \in I} \mu_i = 1, \\ & \lambda_j \geq 0,\ j = 1, \ldots, n,\ \mu_i \geq 0,\ i \in I,\ \rho_k \geq 0,\ k \in J \Bigg\}. \end{aligned} \tag{8}$$

where (D_i, G_i), $i \in I$, I is a set of artificial production units, (A_k, B_k), $k \in J$, J is a set of vectors (rays) added to the model.

Figure 8 shows the transformation of the frontier of the two-dimensional BCC model with the help of artificial units and rays. In the figure, cone Q is formed by artificial rays, point B' is an artificial unit.

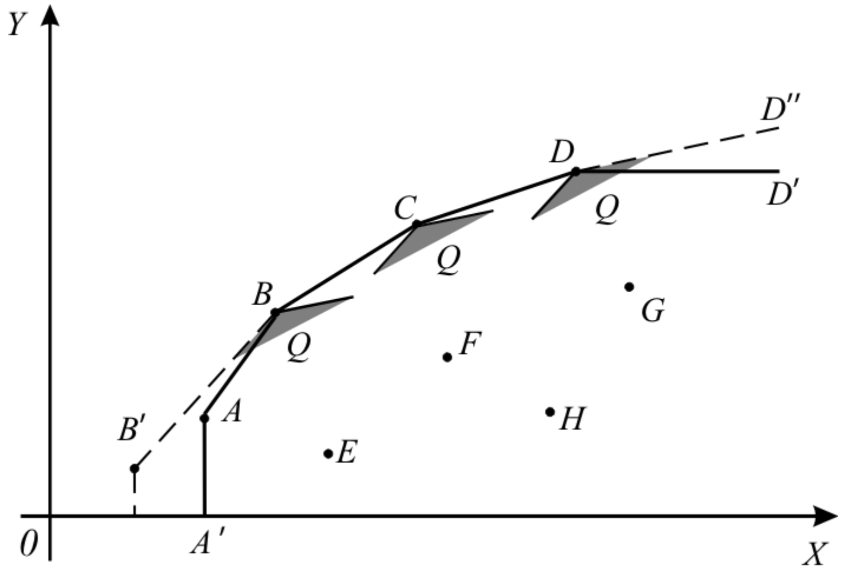


Fig. 8. Transformation of the frontier with the help of artificial units and rays

In addition to problem (4) and (5), we can also discover terminal (suspicious) production units with the help of constructions of two-dimensional and three-dimensional sections of the frontier.

Define three-dimensional affine subspace in space $E^{m+r`}$ as

$$\mathrm{Pl}(X_o,Y_o,d_1,d_2,d_3)=(X_o,Y_o)+\alpha d_1+\beta d_2+\gamma d_3, \tag{9}$$

where $(X_o,Y_o)\in T_B$, α , β and γ are any real numbers, directions $d_1, d_2, d_3 \in E^{m+r}$ are not parallel to each other.

Next, define intersections of the frontier with three-dimensional affine subspace

$$\begin{aligned}\mathrm{Sec}(X_o,Y_o,d_1,d_2,d_3)=\{(X,Y) \mid (X,Y)\in \mathrm{Pl}(X_o,Y_o,d_1,d_2,d_3) \cap \\ \cap\, WEff_P\, T(X,Y);\ d_1,d_2,d_3 \in E^{m+r}\},\end{aligned} \tag{10}$$

where $WEff_PT$ is a set of weakly Pareto-efficient points. Krivonozhko et al. (2005) have proved that set $WEff_PT$ coincides with the boundary of T_B (2).

By choosing different directions d_1, d_2 and d_3 we can construct various two-dimensional and three-dimensional sections going through point (X_o,Y_o) and cutting the frontier. Parametric optimization algorithms for construction of sections of the type (9) are described in detail by Krivonozhko et al. (2004) and Volodin et al. (2004).

Moreover, thanks to our package FrontierVision, one can add to the DEA model any artificial units and rays on the computer screen interactively.

Assertion 2. *There always exists a section (9) that reveals any terminal unit and/or efficient units belonging to an unbounded face.*

However, the specific section may not reveal some terminal units. In the three-dimensional BCC model, see Figure 5, unit B is a terminal one. In Figure 6, unit B does not look like a terminal one. The section in Figure 7 reveals this unit as a terminal point.

Generally speaking, a two-dimensional section of the type (9) consists mainly of a number of segments and two rays. The first and the last vertices in the chain of segments are usually terminal units.

4. Computational results

We validated our results by computational experiments on the efficiency analysis of Russian banks. We took the data from 920 Russia bank's financial accounts for January 1 of 2009, where we use the following inputs and outputs for the BCC output-oriented model:

Inputs: working assets; time liabilities; demand liabilities.

Outputs: equity capital; liquid assets; fixed assets.

Max, min and mean statistics for banks are shown in Table 1.

Table 1. Data for banks Russia 2008

Variables	**Mean**	**St. deviation**	**Min**	**Max**
Outputs				
Liquid assets, ths rubles	4279490	30304201	73	717402532
Equity capital, ths rubles	2205806	23572632	423	632286730
Fixed assets, ths rubles	608481	7414069	42	221058541
Inputs				
Demand liabilities, ths rubles	11318997	140641585	0	4184548095
Time liabilities, ths rubles	18289244	162725433	1	4213176749
Working assets, ths rubles	24587080	230385425	0	6233536293

The data were financial accounts of Russian banks for the year 2008. Remember that this year was the first year of the world crisis. It was important at that time for financial experts to have reliable tools for forecasting the behavior of financial institutes and for warning about possible bankruptcies.

Figure 9 represents the dependence of the number of units on the range of efficiency scores according to the BCC output model (1c, 1d).

Notice that the number of efficient banks is very low; only 42 units out of 920. The majority of banks have efficiency scores less than 50%.

Financial experts expressed doubt about the efficiency of some banks. For example, Figure 10 presents a cut of the frontier in a six-dimensional space by the two-dimensional plane for bank A; certainly we use legends instead of real names of

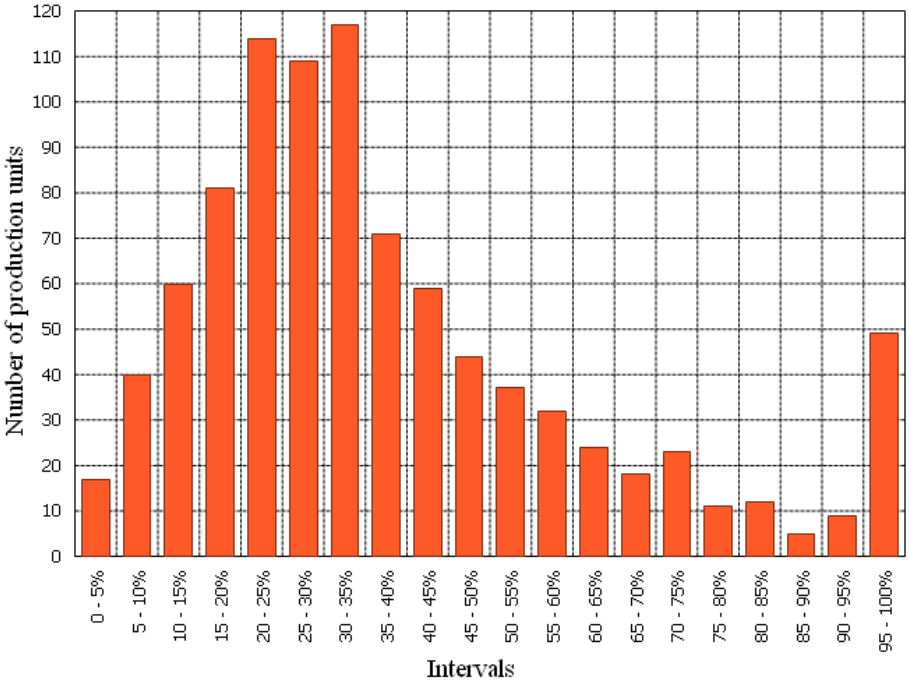


Fig. 9. Distribution of efficiency scores before the frontier transformation

banks. The directions of the plane are determined by two inputs: demand liabilities and working assets.

The scale is such that point $(1, 1)$ in the figure corresponds to bank A. According to the BCC model bank A is 100% efficient. However experts did not agree with this evaluation, and they were right, since bank A was bankrupted in six months. In fact point A is a typical terminal unit, since unbounded edges go out from this unit. However, Figure 10 cannot help us to improve the frontier.

For this purpose we should use another section. Figure 11 shows a cut of the frontier in a six-dimensional space by the two-dimensional plane for bank A. The horizontal axis in the figure is determined by input vector X_o and the vertical axis corresponds to output vector Y_o of bank A, respectively. The scale is such that point $(1, 1)$ corresponds to bank A. The solid line outlines the production function (the cut of the production possibility set) of the model. The balls in the figure denote projections of some other banks on the two-dimensional plane. Again, according to the model, bank A is efficient, which contradicts experts' opinion. However Figure 11 can help us to improve the frontier. Experts were asked to insert an artificial efficient unit on the screen by phrasing the question how much outputs should be expected form an efficient unit using the observed inputs of unit A (implicitly assuming a proportional increase of the outputs). This artificial unit is denoted by B. In the figure, the dotted line together with the solid line after it shows the frontier of the modified model.

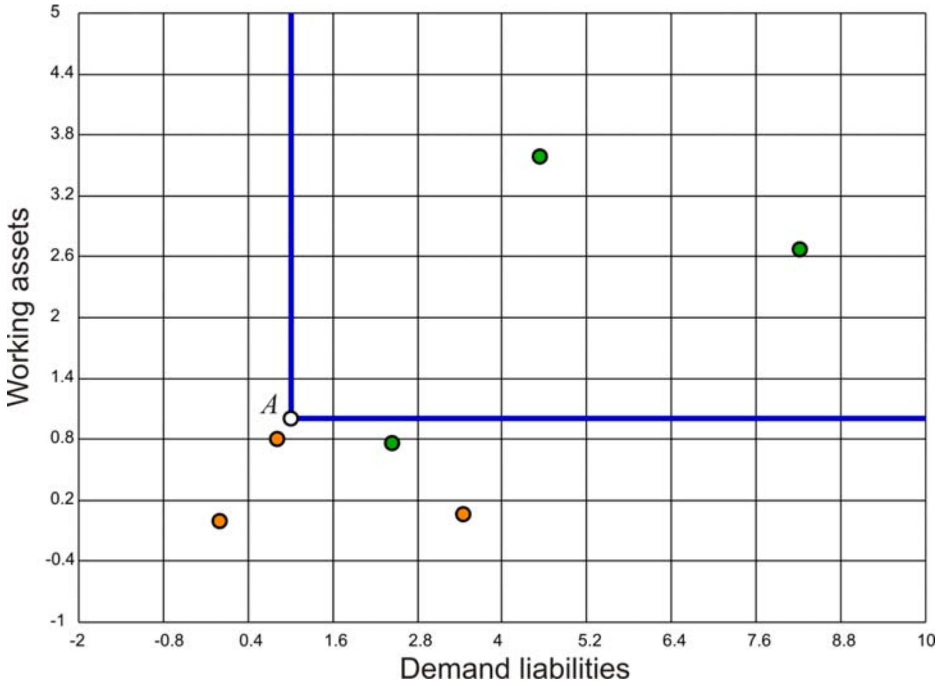


Fig. 10. Input isoquant for bank A

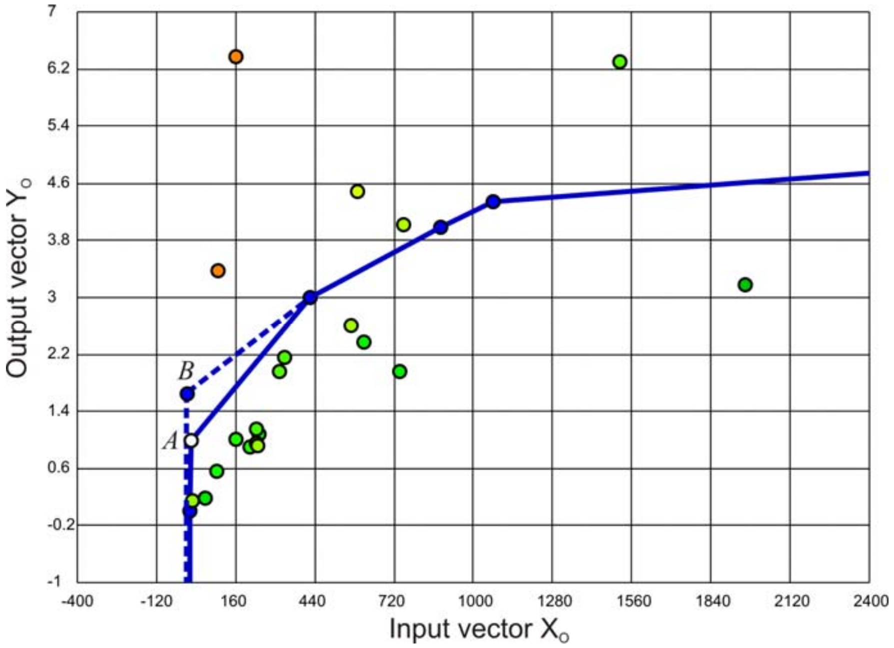


Fig. 11. Production function for bank A

After the frontier transformation the efficiency score of bank A became 48.3%.Some other banks also changed their efficiency scores after inserting artificial unit in the model. Table 2 shows efficiency scores of some banks, which were bankrupted during six months, in the BCC model before and after frontier transformation.

After the second run of the model, the experts recognized the modelling results to be adequate and reliable.

Table 2. Changes of efficiency scores after the frontier transformation

Name	Efficiency score before frontier transformation, in %	Efficiency score after frontier transformation, in %	Date of bankruptcy
C	1,79	1,50	30.03.2009
D	46,34	41,87	16.04.2009
E	50,65	50,65	23.04.2009
F	23,17	23,17	23.04.2009
G	41,68	41,68	07.05.2009
H	11,49	9,16	21.05.2009
I	53,46	53,46	09.06.2009
J	21,1	21,1	18.06.2009
A	100	48,36	25.06.2009
K	7,43	6,35	25.06.2009

We have presented an investigation for only one terminal unit, but demonstrated that the choice of terminal units as units that should be investigated using expert information worked out satisfactorily; reducing the efficient unit to an inefficient one and also reducing several other units' scores and improving the realism of the results. However, the working out of a more formal procedure for eliciting expert help in providing more realistic efficient units based on terminal units is still to be done.

5. Conclusions

In this paper, we proposed tools for discovering units which may cause inadequate results in the DEA models. It was shown that terminal units constitute “suspicious” points in the first place. If the graph of intersection of the frontier with a two-dimensional plane is constructed, then the first and the last vertices of the graph are usually terminal units. However, it is not necessarily condition that terminal units may cause inadequate results in the DEA models, such units may be quite normal efficient points. Only experts in the specific area can evaluate the adequacy of efficiency scores of terminal units.

Terminal units arise because non-countable (continuous) production possibility set T is determined on the basis of a finite number of production units; some of these units turn out to be terminal ones. The gap of derivatives may take place at these points. For example, the left-hand side scale elasticity takes infinite value, and the right-hand side scale elasticity takes zero value at some terminal points, see Førsund et al. (2007).

Let us remember that Farrell (1957) introduced artificial units at infinity in order to smooth his model, see also Førsund et al. (2009).

In the second place, units belonging to unbounded faces and not being terminal units are also “suspicious” points. These units may also cause inadequate results in the DEA models. However our computational experience shows that the number of such units in real-life data sets is very small in comparison with the number of terminal units.

We also propose how to deal with inadequacies in the DEA models with the help of incorporating artificial units and rays interactively on the screen of the computer by experts into some BCC model. This makes the DEA models more adequate and adjustable.

Only one case of eliciting information from experts suggesting artificial units that should be efficient based on terminal units has been shown. However, several other experiments were carried out, as indicated above. Carrying out systematic experiments on new real-life datasets will be attempted in a further research, together with developing more formal procedures for eliciting information from experts on the activities under investigation constructing artificial efficient units based on terminal units. The use of interactive graphical representation in a space of sufficiently reduced dimensions to be readily understandable and recognizable by experts seems a promising approach.

Appendix A

Proof of Theorem 4. Consider efficient unit (X_1,Y_1) in model (1) under evaluation, that remains efficient in model (7) after inserting domination cones, and efficient unit (X_2,Y_2) in model (1) that becomes inefficient in model (7) after inserting domination cones. Assume, at first, that $K = E_+^n$, this implies that cone K coincides with the first non-negative orthant.

If unit (X_2,Y_2) is a terminal point, then the theorem is proved. Assume that unit (X_2,Y_2) is not a terminal unit.

Let (v_1,u_1) be optimal dual solution in model (7) and (v_2,u_2) be optimal dual solution in model (1) for units (X_1,Y_1) and (X_2,Y_2), respectively. It is known that dual optimal solution (v_1,u_1) is an orthogonal vector to some face of the frontier at point (X_1,Y_1) (see Cooper et al., page 120, theorem 5.1). Notice that we do not write here the third part u_0 of the dual vector, because we need only an orthogonal vector for our purpose.

Dual optimal solution (v_1,u_1) for problem (7) is also optimal solution for problem (1) since the inclusion of cones in model (1) may only decrease the feasible set of dual variables (Yu et al., 1996).

Denote vector of dual variables by $w=(v,u)\in E^{m+r}$. Next, let $\|w\|_2$ be the quadratic norm of vector w.

Consider the following linear programming problem with a parameter in the objective function (see Dantzig 1997, 2003)

$$\begin{aligned}
&\max\left[\bar{u}_1+\alpha(\bar{u}_2-\bar{u}_1)\right]^T Y-\left[\bar{v}_1+\alpha(\bar{v}_2-\bar{v}_1)\right]^T X\\
&X\geq\sum_{j=1}^{n}X_j\lambda_j,\\
&Y\leq\sum_{j=1}^{n}Y_j\lambda_j,\\
&\sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\geq 0,
\end{aligned}\tag{A.1}$$

here

$$\bar{u}_1 = u_1 / \|w_1\|_2 ,\ \bar{u}_2 = u_2 / \|w_2\|_2 ,\ \bar{v}_1 = v_1 / \|w_1\|_2 ,\ \bar{v}_2 = v_2 / \|w_2\|_2 , \tag{A.2}$$

where $w_1 = (v_1, u_1)$ and $w_2 = (v_2, u_2)$.

According to the theory of linear programming with a parameter in the objective function when α is increasing, the optimal solution of problem (A.1) moves along the frontier from one face to another face.

It follows from (A.2), that some components of $\bar{w}_1 = (\bar{v}_1, \bar{u}_1)$ will be greater than the corresponding components of $\bar{w}_2 = (\bar{v}_2, \bar{u}_2)$ and vice versa. Some components of vector $\bar{w}_3 = \left(\bar{v}_1 + \alpha^*(\bar{v}_2 - \bar{v}_1), \bar{u}_1 + \alpha^*(\bar{u}_2 - \bar{u}_1)\right)$ will be equal to zero under some $\alpha^* > 1$.

Since vector $\bar{w}_3$ is orthogonal to some face $\Gamma \subset T_B$, this face contains at least one unbounded edge and one terminal point.

Points of face Γ are not efficient in model (7) with domination cones. Let us dwell on this in detail.

According to the assumption, unit (X_1, Y_1) is efficient in model (1) and in model (7). Optimal dual variables $w_1 = (v_1, u_1)$ are associated with efficient unit (X_1, Y_1) in both model (1) and model (3). Hence vectors w_1, $\bar{w}_1$ belongs to some cone W_1 that is included in problem (1) in order to get problem (7a). Unit (X_2, Y_2) is efficient in problem (1) and inefficient in problem (7). Hence vector $w_2 = (v_2, u_2)$ associated with unit (X_2, Y_2) does not belong to cone W_1.

Since vectors w_2, $\bar{w}_2$ and cone W_1 are convex sets, we can construct a hyperplane

$$(\beta, w) = b , \tag{A.3}$$

where $\beta \in E^{m+r}$, $w \in E^{m+r}$, b is a scalar, that separates these two vectors and cone W_1, or, in other words,

$$(\beta, w) < b \text{, under } w \in W_1, \tag{A.4}$$

$$(\beta, w_2) > b .$$

It follows from (A.4), that

$$(\beta, w_3) = \beta^T \left[w_1 - \alpha^* w_1 + \alpha^* w_2 \right] > -(\alpha^* - 1)\beta^T w_1 + \alpha^* w_2 > -(\alpha^* - 1)\beta^T b + \alpha^* b = b$$

under $\alpha^* > 1$.

Hence vectors w_3 and $\overline{w_3} = w_3 / \|w_3\|_2$ do not belong to cone W_1.

Thus, points of face Γ associated with dual vector w_3 are inefficient in model (3), and among units of face Γ there exist terminal points.

Now, let $K \subseteq E^n_+$. The inclusion of cone K in model (7) may only expand the production possibility set T_B, therefore the number of efficient units in model (1) that become inefficient in model (7) may only increased.

This completes the proof.

References

Allen R., Athanassopoulos A., Dyson R. G., Thanassoulis E. (1997) Weight restrictions and value judgments in data envelopment analysis: evolution, development and future directions. *Annals of Operations Research* 73: 13-34.

Allen R., Thanassoulis E. (2004) Improving envelopment in data envelopment analysis. *European Journal the Operational Research* 154(2): 363–379.

Banker R. D., Charnes, A., Cooper, W. W. (1984) Some models for estimating technical and scale efficiency in data envelopment analysis. *Management Science* 30: 1078–1092.

Bessent A., Bessent W., Elam J., Clark T. (1988) Efficiency frontier determination by constrained facet analysis. *Operations Research* 36(5): 785-796.

Bougnol M.-L., Dulá J. H. (2009) Anchor points in DEA. *European Journal the Operational Research* 192(2): 668–676.

Brockett P. L., Charnes A., Cooper W. W., Huang Z. M., Sun D. B. (1997) Data transformations in DEA cone ratio envelopment approaches for monitoring bank performance. *European Journal the Operational Research* 98: 250–268.

Charnes A., Cooper W. W. and Rhodes E. (1978): “Measuring the efficiency of decision making units,” *European Journal of Operations Research* 2, 429-444.

Charnes A., Cooper W.W., Huang Z.M., Sun D.B. (1990) Polyhedral cone-ratio DEA Models with an Illustrative Application to Large Commercial Banks. *Journal of Econometrics* 46: 73–91.

Charnes A., Cooper W. W., Wei Q. L., Huang Z. M. (1989) Cone ratio data envelopment analysis and multi-objective programming. *International Journal of Systems Science* 20: 1099–1118.

Cooper W.W., Seiford L.M., Tone K. (2000) Data Envelopment Analysis. Kluwer Academic Publishers: Boston.

Dantzig G. B., Thapa M. N. (1997) Linear Programming 1: Introduction. Springer-Verlag: New York.

Dantzig G. B., Thapa M. N. (2003) Linear Programming 2: Theory and Extensions. Springer-Verlag: New York.

Dyson R. G., Thanassoulis E. (1988) Reducing weight flexibility in data envelopment analysis. *Journal of the Operational Research Society* 39 (6): 563-576.

Edvardsen D.J, Førsund F.R., Kittelsen S.A.C. (2008) Far out or alone in the crowd: a taxonomy of peers in DEA. *Journal of Productivity Analysis* 29: 201–210.

Farrell M. J. (1957) The measurement of productive efficiency. *Journal of the Royal Statistical Society*, *Series A (General)* 120 (III), 253-281(290).

Farrell M. J. and Fieldhouse M. (1962) Estimating efficient production functions under increasing returns to scale. *Journal of the Royal Statistical Society*, *Series A (General)* 125 (2), 252-267.

Førsund F. R., Hjalmarsson L., Krivonozhko V. E., Utkin O. B. (2007) Calculation of scale elasticities in DEA models: direct and indirect approaches. *Journal of Productivity Analysis* 28: 45–56.

Førsund F. R., Kittelsen S. A. C., Krivonozhko V. E. (2009) Farrell revisited – Visualizing properties of DEA production frontiers. *Journal of the Operational Research Society* 60: 1535–1545.

Goldman A. J. (1956) Resolution and Separation Theorems for Polyhedral Convex Sets. In: Kuhn H.W. and Tucker A.W. (eds). Linear inequalities and related systems. Princeton University Press, New Jersey, 41–51.

Krivonozhko V. E., Utkin O. B., Volodin A. V., Sablin I. A., Patrin M. (2004) Constructions of economic functions and calculations of marginal rates in DEA using parametric optimization methods. *Journal of the Operational Research Society* 55: 1049–1058.

Krivonozhko V. E., Utkin, O. B., Volodin A. V., Sablin I. A. (2005) About the structure of boundary points in DEA. *Journal of the Operational Research Society* 56: 1373–1378.

Krivonozhko V. E., Utkin O. B., Safin M. M., Lychev A. V. (2009) On some generalization of the DEA models. *Journal of the Operational Research Society* 60: 1518–1527.

Lang P., Yolalan O.R., Kettani O. (1995) Controlled envelopment by face extension in DEA. *Journal of the Operational Research Society* 46(4): 473-491.

Motzkin T. S. (1936) Beiträge zur Theorie der linearen Ungleichungen. Dissertation, Basel, Jerusalem.

Pedraja-Chaparro F., Salinas-Jimenez J., Smith P. (1997) On the role of weight restrictions in data envelopment analysis. *Journal of Productivity Analysis* 8: 215-230.

Thanassoulis E., Allen R. (1998) Simulating weight restrictions in data envelopment analysis by means of unobserved DMUs. *Management Science* 44(4): 586-594.

Thompson R. G., Brinkman E. S., Dharmapala P. S., Gonzalez-Lima M. D., Thrall R. M. (1997) DEA/AR profit rations and sensitivity of 100 large U.S. banks. *European Journal of Operational Research* 98: 213–229.

Thompson R. G., Langemeier L. N., Lee C.-H., Lee E., Thrall R. M. (1990) The role of multiplier bounds in efficiency analysis with application to Kansas farming. *Journal of Econometrics* 46: 93-108.

Thompson R. G., Singleton Jr F. R., Thrall R. M., Smith B. A. (1986) Comparative site evaluation for locating a high-energy physics lab in Texas. Interfaces 16(6): 35-49.

Volodin A. V., Krivonozhko V. E., Ryzhikh D. A., Utkin O. B. (2004) Construction of Three-Dimensional Sections in Data Envelopment Analysis by Using Parametric Optimization Algorithms *Computational Mathematics and Mathematical Physics* 44: 589–603.

Wei Q., Yan H., Xiong L. (2008) A bi-objective generalized data envelopment analysis model and point-to-set mapping projection. *European Journal of Operational Research* 190: 855–876.

Yu P. L. (1974). Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. *Journal of Optimization Theory and Applications* 14: 319–377.

Yu G., Wei Q., Brockett P. (1996). A generalized data envelopment analysis model: a unification and extension of existing methods for efficiency analysis of decision making units. *Annals of Operations Research* 66: 47–89.