MEMORANDUM

No 15/2010

Leader, Or Just Dominant? The Dominant-Firm Model Revisited

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ISSN: 0809-8786

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Leader, Or Just Dominant?
The Dominant-Firm Model Revisited

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August 6, 2010

Abstract

I revisit the dominant-firm model and discuss its implicit assumption of a sequential move structure. I argue that a simultaneous move structure is often more reasonable and derive an alternative formulation of the model based on this approach.

Keywords: dominant firm, monopoly with fringe, sequential moves

JEL Codes: D42, L12

Acknowledgements: I am grateful to Gavin Reid, Tore Nilssen and seminar participants at the University of Oslo and Norwegian School of Economics and Business Administration for valuable comments and discussions.

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1 Introduction

In the dominant-firm model - also called partial monopoly and monopoly with fringe - the two extremes of monopoly and perfect competition are brought together to form a simple, yet very useful and popular, tool for analysing aspects of imperfect competition and strategic interaction.[1] However, as it is commonly formulated, the model is based on (implicit) assumptions - that the dominant firm both has a first-mover advantage and an ability to commit - that are not warranted, often unreasonable and rarely relevant.

Schenzler, Siegfried and Thweatt (1992, 173) tracks the history of what they call the 'Static Equilibrium Dominant Firm Price Leadership Model' back to Forchheimer (1908) (see also Reid, 1977, 1979, and Dimand and Dimand, 1996). Other leading contributors to the development of the model were Nichol (1930), Zeuthen (1930), Stackelberg (1934) and Stigler (1940). As Schenzler, Siegfried and Thweatt tells it: *'The story begins with Karl Forchheimer shortly after the turn of the century, and ends with a comprehensive presentation of the static equilibrium model by George Stigler in 1940, the form in which it has been used ever since.'*[2]

Stigler showed the supply curve of the fringe (the horisontal sum of their marginal costs) as increasing and, at each price, subtracted the fringe's quantity supplied from market demand to obtain the dominant firm's residual demand function. The marginal revenue of the dominant firm was then derived from its residual demand function, and the intersection of marginal revenue and marginal costs yielded its profit-maximising quantity. Given this quantity, market price was found from the residual demand function, which again determined the quantity of the fringe as well as total market quantity.[3]

Seen from the standpoint of the perfectly competitive benchmark, the model introduces two principally different elements. The first element concerns market power; the dominant firm is able to affect market price and takes this into account when making its supply decision. The second element concerns influence on competitor behaviour;

[1] For a recent example, see Tardiff and Weismann (2009) who extends the dominant-firm model to multi-market participation.

[2] The interest in price leadership did not end there, but the subsequent discussions tended to concentrate on empirical evidence and practical relevance of such behaviour; see eg. Stigler (1947), Markham (1951) and Bain (1960).

[3] As a member of the Department of Economics at the University of Oslo, I am obliged to point out that Ragnar Frisch lectured on a basically identical model to that of Stigler - which Frisch called Production Monopoly With Atomistic Outsider Group (in Norwegian, "produksjonsmonopol med atomistisk outsidergruppe") as early as 1932-33. Frisch's lecture notes were printed and made available to his students (Frisch, 1933, 1941), but the theory was never presented to the wider academic community.

the dominant firm foresees the fringe's response to alternative prices and takes this into account when making its decision. Alternatively, seen from the standpoint of complete monopoly, the model exposes the monopolist to competition, but also provides the monopolist with the ability to take account of how price affects competitors.

The market-power and influence-competitor incentives work in opposite directions. On the one hand, the incentive to exercise market power tends to reduce output and increase price. On the other hand, the incentive to play on competitors' supply response tends to increase output and reduce price. As it turns out, while the influence-competitor incentive counteracts the market-power incentive, the latter dominates; the resulting market outcome involves levels of output and price that lie between those of perfect competition and complete monopoly.

Although the dominant-firm model is generally viewed as static, the assumption that the dominant firm takes the supply response of the fringe into account introduces an essentially dynamic feature. For the fringe to respond in the assumed manner, it would have to observe the supply decision of the dominant firm, or at least the resulting market price. There is consequently an implicit sequentiality in the interaction between the dominant firm and the fringe, with the dominant firm moving first and the fringe subsequently following. Furthermore, there is an implicit element of commitment. While the decision of the dominant firm is *ex ante* optimal (in the sense of maximising profits), it is not optimal *ex post*: given the supply decision of the fringe, the dominant firm would want to adjust its original decisions if it could. In other words, while fringe firms are making a best response to the decisions of their competitors, the dominant firm is not.

For the study of strategic market behaviour - and, in particular, the interaction between a dominant firm and its smaller competitors - the assumed sequentiality of decisions is not warranted. It is perfectly possible, and indeed simpler, to analyse the model under the assumption that the dominant firm takes the supply of the fringe as given. The fundamental nature of results would be the same, with the dominant firm being partly, but not fully, able to raise price towards monopoly levels due to competition from the fringe.

Also, moving away from sequentiality and commitment would often make the model more reasonable. In many real-life applications, firms do have the opportunity to adjust their decisions in response to the behaviour of their competitors; this is particular true if decisions concern prices. In such cases, it would make more sense to base the analysis on a set up in which firms make best-response decisions at equilibrium. It is only in those rare events that (i) decisions are irreversible and (ii) the dominant firm has a first-mover advantage that assumptions of sequentiality and commitment are relevant.

In order to analyse these issues, one needs a framework that allows for disentangling and making explicit the various assumptions alluded to above. This may be done by seeing the dominant-firm model as the limiting outcome of a game between a large firm and a group of smaller competitors, an approach corresponding to that sometimes used to provide a foundation for the model of perfect competition. Establishing such a framework makes it possible to analyse the implications of the various assumptions; in particular, it leads to an alternative formulation of the dominant-firm model that would often seem more reasonable.[4]

In the next section, I present the standard formulation of the dominant-firm model by means of an example. In the following section, I show how the model may be seen as the limiting outcome of a sequential oligopoly game between a dominant firm and a group of smaller competitors. In the subsequent section, I present a simultaneous-move version of the previous game and demonstrate that in the limit it leads to an alternative formulation of the dominant-firm model. Then follows two sections containing a comparison of the two formulations as well as a more general discussion of their relative merits. The final section contains my conclusion.

2 The Standard Dominant-Firm Model

In this section, the standard formulation of the dominant-firm model is presented by means of a parameterised example. While it would certainly be possible to undertake the entire analysis based on more general assumptions, such a generalisation would not seem to add much to the issues that are under consideration here; indeed, the fundamental insights would seem to generalise in a straightforward manner. The parameterised formulation has the advantage of simplifying the analysis, as well as allowing for closed-form solutions and exact characterisations of equilibrium outcomes.

In this formulation, a dominant firm and a fringe produce homogeneous goods and supply a market with linear demand. The (indirect) demand function is given by

$$P = 1 - X, \tag{1}$$

where P is market price and X is aggregate output.

[4]Hence, I beg to differ with Stigler (1940, 522) on the solution of the monopoly with fringe: *'before embarking on the classical problem of duopoly, it may be permissible to restate a case in which there is no doubt concerning the solution - that of the dominant firm (also known as partial monopoly)'*.

The cost of the dominant firm is quadratic in output and so its profit is given by

$$\pi_M = PX_M - \frac{d_M}{2} X_M^2, \tag{2}$$

where X_M is the output of the dominant firm and d_M is a non-negative constant. Note that marginal cost of the dominant firm at output X_M equals $d_M X_M$.

Correspondingly, the profit of the fringe - represented here by a single representative unit - is

$$\pi_F = PX_F - \frac{d_F}{2} X_F^2, \tag{3}$$

where X_F is the output of the fringe and d_F is a non-negative constant (possibly, but not necessarily, equal to d_M). Marginal cost of the fringe is $d_F X_F$.

The fringe takes price as given and the first-order condition for profit maximisation is therefore the familiar condition that price equals marginal cost:

$$P = d_F X_F. \tag{4}$$

From the first-order condition (4), we find the supply function of the fringe to be

$$X_F(P) = \frac{P}{d_F}. \tag{5}$$

Then, from (5) and (1), we derive the (inverse) net-demand function facing the dominant firm:

$$P = \frac{d_F}{1 + d_F}[1 - X_M]. \tag{6}$$

It follows that the dominant firm's profits may be written as

$$\pi_M = \frac{d_F}{1 + d_F}[1 - X_M] X_M - \frac{d_M}{2} X_M^2. \tag{7}$$

The first-order condition for the dominant firm's profit maximisation becomes

$$\frac{d_F}{1 + d_F}[1 - 2X_M] - d_M X_M = 0, \tag{8}$$

implying that the equilibrium output of the dominant firm is

$$X_M^{ME} = \frac{d_F}{2d_F + d_M + d_F d_M}. \tag{9}$$

From (4), (6) and (9), it then follows that the equilibrium output of the fringe is given by

$$X_F^{ME} = \frac{d_F + d_M + d_F d_M}{[1 + d_F][2d_F + d_M + d_F d_M]}. \tag{10}$$

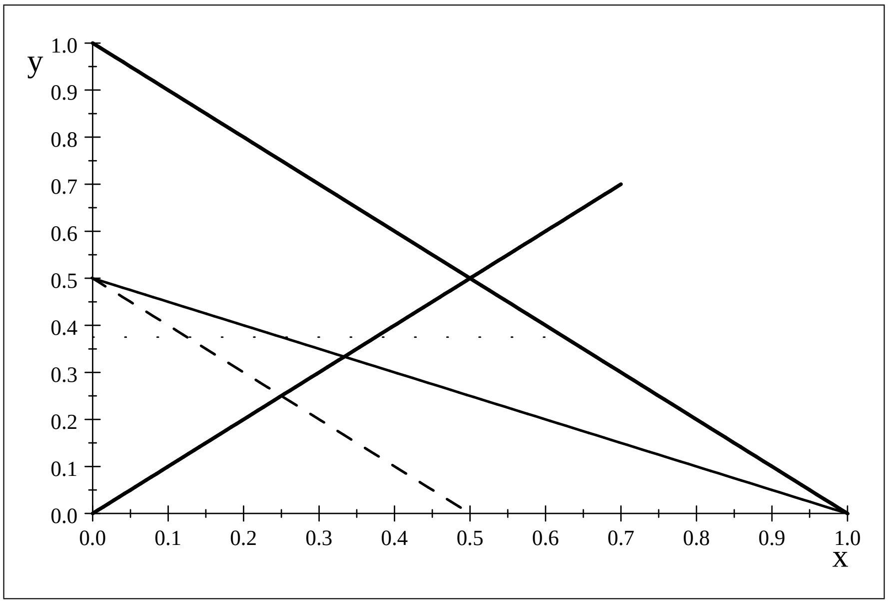
Comparing (9) and (10), we find that output of the dominant firm is less than that of the fringe if and only if $d_M > \frac{d_F^2}{1+d_F}$; in the case when cost functions are identical, i.e. $d_F = d_M$, the dominant firm supplies less than the fringe. Note that, from (4), (6) and (8), irrespective of the relation between outputs, marginal cost of the fringe always exceeds that of the dominant firm, i.e. $d_F X_F > d_M X_M$, and therefore overall costs of producing aggregate output $X = X_F + X_M$ is not minimised at equilibrium.

Equilibrium values of aggregate output and market price become

$$X^{ME} = \frac{2d_F + d_M + d_F d_M + d_F^2}{[1 + d_F]\,[2d_F + d_M + d_F d_M]} \tag{11}$$

$$P^{ME} = \frac{d_F\,[d_F + d_M + d_F d_M]}{[1 + d_F]\,[2d_F + d_M + d_F d_M]} \tag{12}$$

The figure below illustrates the model for the case in which the dominant firm and the fringe face symmetric cost functions. Net demand (the thin line) is found by subtracting fringe supply (the thick, up-ward sloping line) from total demand (the thick, downward-sloping line). Marginal revenue (the dashed thin line) is found from net demand. The profit maximising output of the dominant firm equalises marginal revenue and marginal cost (the thick, upward-sloping line) and the corresponding market price is given by the net demand curve at this level of output. The supply of the fringe equals the difference between total and net demand at this price.



Market equilibrium, $d_F = d_M = 1$.

Equilibrium aggregate output is decreasing, and equilibrium market price is increasing, in the parameter d_F. At the extreme, when $d_F \longrightarrow \infty$, we obtain the complete monopoly outcome where only the dominant firm produces:

$$X^M = X_M^M = \frac{1}{2 + d_M}, \tag{13}$$

$$P^M = \frac{1 + d_M}{2 + d_M}. \tag{14}$$

At the other extreme, when $d_M \longrightarrow \infty$, we have a (perfectly competitive) outcome where only the fringe produces

$$X^F = X_F^F = \frac{1}{1+d_F}, \tag{15}$$

$$P^F = \frac{d_F}{1+d_F}. \tag{16}$$

We may compare the dominant-firm model with a model in which all firms act as price takers. In this latter case of perfect competition, the equilibrium outcome is given by

$$X_F^{PC} = \frac{d_M}{d_F + d_M + d_F d_M} \tag{17}$$

$$X_M^{PC} = \frac{d_F}{d_F + d_M + d_F d_M} \tag{18}$$

$$X^{PC} = \frac{d_F + d_M}{d_F + d_M + d_F d_M} \tag{19}$$

$$P^{PC} = \frac{d_F d_M}{d_F + d_M + d_F d_M} \tag{20}$$

Comparing (12) with (14) and (20), and (11) with (13) and (19), we find that the equilibrium price and aggregate output in the dominant-firm model lie between those of perfect competition and complete monopoly; that is, $P^{PC} < P^{ME} < P^M$ and $X^{PC} > X^{ME} > X^M$.

3 A Game-Theoretic Formulation

We now turn to a game-theoretic formulation with an equilibrium outcome that approaches that of the dominant-firm model when fringe firms become sufficiently numerous (and small).

The modelling approach corresponds to that used to demonstrate that the perfectly competitive outcome may be seen as a limiting case of oligopoly when the number of firms increases towards infinity. In this approach, one needs to make assumptions to ensure that aggregate supply remains bounded when the number of firms increases beyond bounds; moreover, one needs to make assumptions to preserve the asymmetry between the dominant firm and individual fringe firms.[5] Given that firms supply homogenous goods, the assumptions needed to ensure both asymmetry and boundedness must be imposed on production technology or inputs. Here, we assume that costs differ between the dominant

[5] While asymmetry is here taken as exogenous, asymmetry may also arise endogenously, as in Ghemawat (1990), Besanko and Doraszelski (2004) and Eső, Nocke and White (2007).

firm and fringe firms, and that marginal costs of individual fringe firms increase with the number of such firms; one interpretation of this assumption is that there exists some essential input in limited supply of which the dominant firm controls a larger share than other firms.[6] In the Appendix, I demonstrate that the same results may be obtained in a model where the supply side consists of a number of identical plants of which the dominant firm controls a certain fraction and the rest are operated independently.[7]

Suppose $N+1$ firms play a sequential game. First Firm M chooses output and subsequently, after observing Firm M's choice, the other N firms choose output simultaneously. The Subgame Perfect Equilibrium of this game is found by means of backwards induction.

The profits of firm n, $n = 1, ..., N$, are

$$\pi_n = P x_n - \frac{d_F N}{2} x_n^2. \tag{21}$$

where x_n is the output of firm n. Marginal cost at x_n equals $d_F N x_n$.

Taking the output of competitors as given, firm n maximises (21) subject to (1). The first-order condition for this profit-maximisation problem becomes

$$1 - X - x_n - d_F N x_n = 0, \tag{22}$$

where $X = X_M + \sum_{n=1}^{N} x_n$ is aggregate output.

At the symmetric equilibrium all fringe firms produce the same amount, denoted x_F, so the first-order condition (22) reduces to

$$1 - X_M - [N + 1 + d_F N]\, x_F = 0. \tag{23}$$

From this expression, we find how the output of fringe firms relates to that of Firm M:

$$x_F = \frac{1 - X_M}{N + 1 + d_F N}. \tag{24}$$

Firm M's profits are given by (2) as before. Firm M maximises profits subject to the response of the fringe (24) and demand (1). The first-order condition for this problem may be written

$$1 - N\frac{1 - X_M}{N + 1 + d_F N} - X_M - \left[1 - \frac{N}{N + 1 + d_F N}\right] X_M - d_M X_M = 0, \tag{25}$$

[6] This is in the spirit of Perry and Porter (1985); indeed, the assumed cost functions are essentially the same as theirs.

[7] Ino and Kawamori (2009) consider a set up in which one firm, as a result of innovation, has a cost advantage over other firms: in the limit, when the number of other firms go to infinity, as a Stackelberg leader the innovating firm, depending on the size of the cost advantage, either acts competitively or occupies the entire market.

implying that the equilibrium output of Firm M is

$$X_M^{SE} = \frac{1 + d_F N}{2\left[1 + d_F N\right] + d_M \left[N + 1 + d_F N\right]}. \tag{26}$$

It follows that each fringe firm supplies

$$x_F^{SE} = \frac{1 + d_F N + d_M \left[N + 1 + d_F N\right]}{\left[N + 1 + d_F N\right]\left\{2\left[1 + d_F N\right] + d_M \left[N + 1 + d_F N\right]\right\}} \tag{27}$$

and that the total supply of the fringe is

$$X_F^{SE} = N x_F^{SE} = \frac{N\left\{1 + d_F N + d_M \left[N + 1 + d_F N\right]\right\}}{\left[N + 1 + d_F N\right]\left\{2\left[1 + d_F N\right] + d_M \left[N + 1 + d_F N\right]\right\}}. \tag{28}$$

At equilibrium, total market output and market price are

$$X^{SE} = \frac{N\left[1 + d_F N\right] + \left[1 + d_F N + d_M N\right]\left[N + 1 + d_F N\right]}{\left[N + 1 + d_F N\right]\left\{2\left[1 + d_F N\right] + d_M \left[N + 1 + d_F N\right]\right\}} \tag{29}$$

$$P^{SE} = \frac{\left[1 + d_F N\right]\left\{1 + d_F N + d_M \left[N + 1 + d_F N\right]\right\}}{\left[N + 1 + d_F N\right]\left\{2\left[1 + d_F N\right] + d_M \left[N + 1 + d_F N\right]\right\}} \tag{30}$$

Note that for $N = 0$, we again have the complete monopoly outcome characterised in (13) and (14). The case in which $N = 1$ corresponds to the so-called Stackelberg model of duopoly. In order to consider the limit when N increases beyond bounds, we observe that the above equilibrium expressions may alternatively be written

$$X_M^{SE} = \frac{\frac{1}{N} + d_F}{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right]} \tag{31}$$

$$X_F^{SE} = \frac{\frac{1}{N} + d_F + d_M + d_M\left[\frac{1}{N} + d_F\right]}{\left[1 + \frac{1}{N} + d_F\right]\left\{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right]\right\}} \tag{32}$$

$$X^{SE} = \frac{2\left[\frac{1}{N} + d_F\right] + d_M + \left[\frac{1}{N} + d_F\right] d_M + \left[\frac{1}{N} + d_F\right]^2}{\left[1 + \frac{1}{N} + d_F\right]\left\{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right]\right\}} \tag{33}$$

$$P^{SE} = \frac{\left[\frac{1}{N} + d_F\right]\left\{\frac{1}{N} + d_F + d_M + \left[\frac{1}{N} + d_F\right] d_M\right\}}{\left[1 + \frac{1}{N} + d_F\right]\left\{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right]\right\}} \tag{34}$$

Then it is easy to see that, as $N \longrightarrow \infty$, we get

$$X_M^{SE} \longrightarrow \frac{d_F}{2d_F + d_M + d_F d_M} \tag{35}$$

$$X_F^{SE} \longrightarrow \frac{d_F + d_M + d_F d_M}{\left[1 + d_F\right]\left[2d_F + d_M + d_F d_M\right]} \tag{36}$$

$$X^{SE} \longrightarrow \frac{2d_F + d_M + d_F d_M + d_F^2}{\left[1 + d_F\right]\left[2d_F + d_M + d_F d_M\right]} \tag{37}$$

$$P^{SE} \longrightarrow \frac{d_F\left[d_F + d_M + d_F d_M\right]}{\left[1 + d_F\right]\left[2d_F + d_M + d_F d_M\right]} \tag{38}$$

The limiting expressions are identical to those derived for the equilibrium outcome of the standard formulation of the dominant-firm model. In other words, the standard formulation of the dominant-firm model may be seen as the limiting outcome of a sequential game in which the dominant firm moves first and the fringe subsequently follows.

4 A Simultaneous-Move Formulation

In this section, we consider a simultaneous-move version of the game laid out in the previous section. Here all firms, including Firm M, choose output simultaneously. It is shown that the limiting outcome of this game corresponds to an alternative formulation of the dominant-firm model.

Taking the output of its competitors as given, the first-order condition for Firm M's profit maximisation becomes

$$1 - X - X_M - d_M X_M = 0. \tag{39}$$

The corresponding first-order conditions for the other firms are as given above in (22).

Exploiting the symmetry of the fringe firms, the first-order conditions (39) and (22) collaps to the following system of equations

$$\begin{aligned} N x_F + [2 + d_M] X_M &= 1, & (40) \\ [N + 1 + d_F N] x_F + X_M &= 1, & (41) \end{aligned}$$

which has the solution

$$\begin{aligned} X_M^{SI} &= \frac{1 + d_F N}{2[1 + d_F N] + d_M [N + 1 + d_F N] + N}, & (42) \\ x_F^{SI} &= \frac{1 + d_M}{2[1 + d_F N] + d_M [N + 1 + d_F N] + N}. & (43) \end{aligned}$$

It follows that the total supply of the fringe is

$$X_F^{SI} = N x_F^{SI} = \frac{N[1 + d_M]}{2[1 + d_F N] + d_M [N + 1 + d_F N] + N}, \tag{44}$$

while aggregate output and market price are

$$\begin{aligned} X^{SI} &= \frac{1 + d_F N + N[1 + d_M]}{2[1 + d_F N] + d_M [N + 1 + d_F N] + N}, & (45) \\ P^{SI} &= \frac{1 + d_F N + d_M [1 + d_F N]}{2[1 + d_F N] + d_M [N + 1 + d_F N] + N}. & (46) \end{aligned}$$

Note that for $N = 0$, we again have the pure monopoly outcome characterised in (13) and (14). Rewriting the equilibrium expressions as

$$\begin{aligned} X_M^{SI} &= \frac{\frac{1}{N} + d_F}{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right] + 1} & (47) \\ X_F^{SI} &= \frac{1 + d_M}{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right] + 1} & (48) \\ X^{SI} &= \frac{\frac{1}{N} + d_F + 1 + d_M}{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right] + 1} & (49) \\ P^{SE} &= \frac{\frac{1}{N} + d_F + \left[\frac{1}{N} + d_F\right] d_M}{2\left[\frac{1}{N} + d_F\right] + d_M + d_M\left[\frac{1}{N} + d_F\right] + 1} & (50) \end{aligned}$$

we see that, as $N \longrightarrow \infty$,

$$X_M^{SI} \longrightarrow \frac{d_F}{2d_F + d_M + d_F d_M + 1} \tag{51}$$

$$X_F^{SI} \longrightarrow \frac{1 + d_M}{2d_F + d_M + d_F d_M + 1} \tag{52}$$

$$X^{SI} \longrightarrow \frac{1 + d_F + d_M}{2d_F + d_M + d_F d_M + 1} \tag{53}$$

$$P^{SI} \longrightarrow \frac{d_F [1 + d_M]}{2d_F + d_M + d_F d_M + 1} \tag{54}$$

This limiting outcome corresponds to the outcome of a model similar to that described in Section 2 above, with the only difference being that the dominant firm takes fringe supply as given.

To see this, note that the (inverse) net demand function facing the dominant firm now becomes

$$P = 1 - X_F - X_M, \tag{55}$$

where X_F is taken as given. It follows that the dominant firm's profits may be written

$$\pi_M = [1 - X_F - X_M] X_M - \frac{d_F}{2} X_M^2, \tag{56}$$

leading to the first-order condition for profit maximisation given by

$$1 - X_F - [2 + d_M] X_M = 0. \tag{57}$$

Together with conditions (4) and (1), condition (57) leads to the solution

$$X_M^{MI} = \frac{d_F}{2d_F + d_M + d_F d_M + 1} \tag{58}$$

$$X_F^{MI} = \frac{1 + d_M}{2d_F + d_M + d_F d_M + 1} \tag{59}$$

$$X^{MI} = \frac{1 + d_F + d_M}{2d_F + d_M + d_F d_M + 1} \tag{60}$$

$$P^{MI} = \frac{d_F [1 + d_M]}{2d_F + d_M + d_F d_M + 1} \tag{61}$$

In other words, if we start from a game in which firms choose strategies simultaneously the limiting outcome corresponds to a dominant-firm model in which the dominant firm takes the supply of the fringe as given.

Note that, in this formulation, the output of the dominant firm is less than that of the fringe if and only if $d_M > d_F - 1$; in the case when cost functions are identical, i.e. $d_F = d_M$, the dominant firm supplies less than the fringe.

Note also that total market output and market price fall between those of perfect competition and pure monopoly; that is, $P^{PC} < P^{MI} < P^M$ and $X^{PC} > X^{MI} > X^M$.

Moreover, total output is decreasing, and price is increasing, in d_F and the outcome approaches that of complete monopoly characterised by (13) and (14) as $d_F \longrightarrow \infty$.

5 Comparison

In this section, we compare and discuss the two alternative formulations of the dominant-firm model presented in previous sections.

We first observe that, in broad terms, the two formulations lead to similar results. On the one hand, due to the market power of the dominant firm, price exceeds that under perfect competition. On the other hand, due to the presence of the competitive fringe, price is less than that under complete monopoly. The impact of the fringe is increasing in the elasticity of its supply, which is determined by the slope of the marginal cost function d_F. The difference between outcomes of the two formulations may consequently be seen as quantitative rather than qualitative.

Direct calculations leads to the following results:

$$X^{MI} - X^{M} = \frac{[1+d_M]^2}{[2+d_M]\,[2d_F+d_M+d_Fd_M+1]} > 0 \tag{62}$$

$$X^{ME} - X^{MI} = \frac{d_F^2}{[1+d_F]\,[2d_F+d_M+d_Fd_M]\,[2d_F+d_M+d_Fd_M+1]} > 0 \tag{63}$$

$$X^{PC} - X^{ME} = \frac{d_F^3}{[1+d_F]\,[d_F+d_M+d_Fd_M]\,[2d_F+d_M+d_Fd_M]} > 0 \tag{64}$$

It follows that $X^M < X^{MI} < X^{ME} < X^{PC}$ and $P^M > P^{ME} > P^{MI} > P^{PC}$.

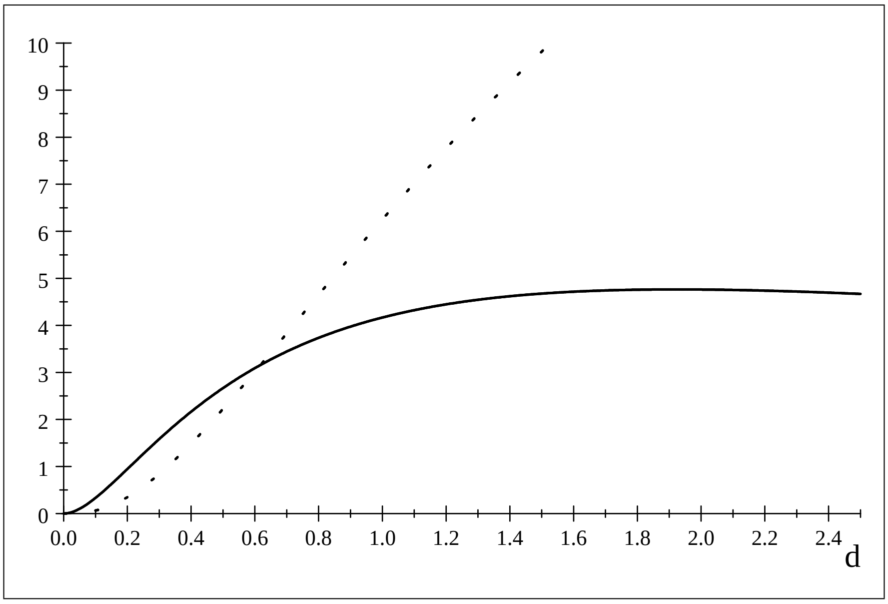
In other words, while both models lead to prices that exceed those corresponding to perfect competition, but fall below those of complete monopoly, aggregate output is higher and price lower when the dominant firm takes the supply response of the fringe into account. The reason is that when the dominant firm expects the fringe to reduce output following a reduction in price, the dominant firm has an incentive to raise output in order to increase its market share. Note that this incentive works in the opposite direction of the incentive following from the response of demand; taken in isolation, the demand response provides the dominant firm with an incentive to reduce output in order to secure a higher price. While the incentive resulting from demand response always dominates, the net incentive depends on the elasticity of the fringe's supply; in particular, the flatter is the fringe supply curve, the less is the incentive to reduce output.

The figure below contains plots of the percentage difference between aggregate outputs defined as, respectively,

$$\Delta_X^{EI} = \frac{X^{ME} - X^{MI}}{X^{ME}} = \frac{d_F^2}{[1+d_F][1+d_F+d_M][2d_F+d_M+d_Fd_M]} \tag{65}$$

$$\Delta_X^{PE} = \frac{X^{PC} - X^{ME}}{X^{PC}} = \frac{d_F^3}{[1+d_F][d_F+d_M][2d_F+d_M+d_Fd_M]} \tag{66}$$

as a function of d_F, for the case $d_M = 1$. The difference in aggregate output between the two formulations of the dominant-firm model Δ_X^{EI} increases with d_F for small values of d_F, but then falls off and disappears for sufficiently high values. The difference between the standard dominant-firm aggregate output and the perfectly competitive aggregate output Δ_X^{PE} increases for all values of d_F (it approaches $\frac{1}{3}$ or 33.3% in the limit). For small values of d_F (less than approx. 0.6), the difference between the two dominant-firm outcomes is greater than that between the standard dominant-firm outcome and the perfectly competitive outcome; for larger values of d_F, the relation is reversed.



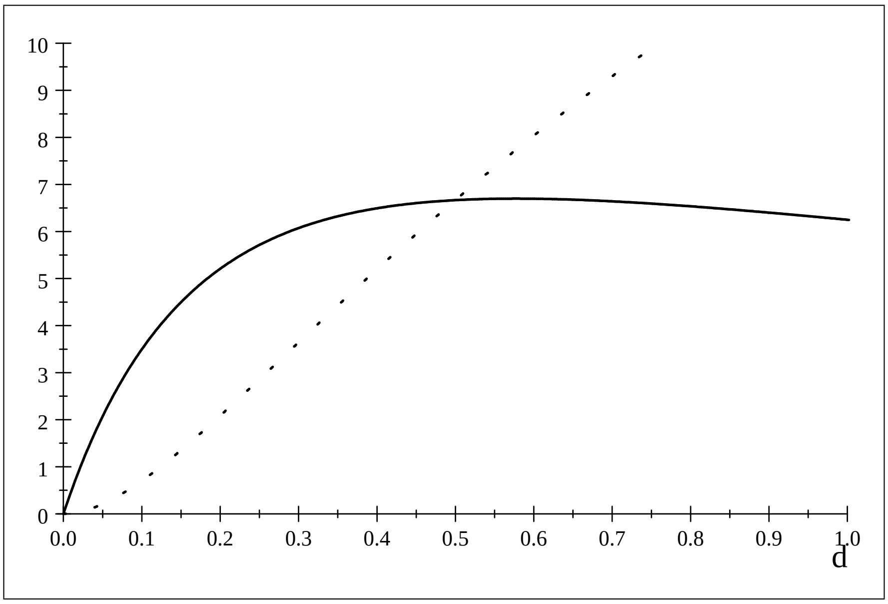
Δ_X^{EI} (solid line) and Δ_X^{PE} (dotted line) as functions of d_F; $d_M = 1$.

Alternatively, we may consider differences in market outcomes from the price side. The percentage differences in prices, given by,

$$\Delta_P^{EI} = \frac{P^{MI} - P^{ME}}{P^{ME}} = \frac{d_F}{[1+d_F][1+d_M][2d_F+d_M+d_Fd_M]} \tag{67}$$

$$\Delta_P^{PE} = \frac{P^{ME} - P^{PC}}{P^{PC}} = \frac{d_F^2}{d_M[1+d_F][2d_F+d_M+d_Fd_M]} \tag{68}$$

are illustrated in the figure below. The difference in prices between the two formulations of the dominant-firm model increases with d_F for small values of d_F, but then falls off and disappears for sufficiently high values (the maximum value of approx. 6.5 percent is reached at $d_F \approx 0.6$). The difference between the standard dominant-firm price and the perfectly competitive price increases for all values of d_F (it approaches $\{d_M [2 + d_M]\}^{-1}$ in the limit). For small values of d_F (less than approx. 0.5), the difference between the two dominant-firm outcomes is greater than that between the standard dominant-firm outcome and the perfectly competitive outcome; for larger values of d_F, the relation is reversed.

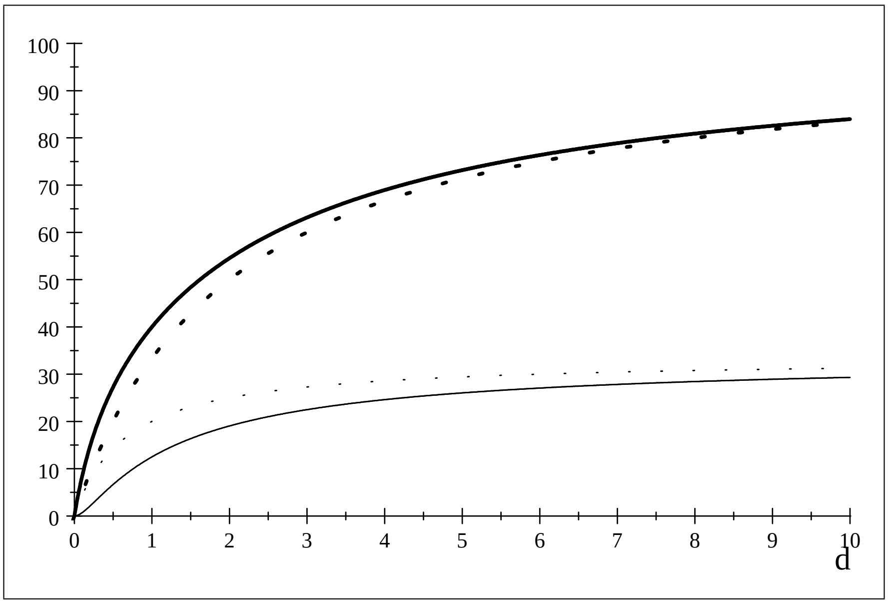


Δ_P^{EI} (solid line) and Δ_P^{PE} (dotted line) as functions of d_F; $d_M = 1$.

Comparing (9) and (58), we find that $X_M^{MI} < X_M^{ME}$; in other words, the dominant firm's output is less when the it takes the fringe's supply as given than when it moves first and plays on the fringe's supply response. Conversely, we find, comparing (10) and (59), that the output of the fringe is larger in the former than in the latter case; that is, $X_F^{MI} > X_F^{ME}$. Since $X^{MI} < X^{ME}$, it follows that the fringe has a larger market share, and the dominant firm therefore a lower market share, in the simultaneous-move formulation than in the sequential-move formulation of the dominant-firm model, i.e. $\alpha_M^{MI} \equiv \frac{X_M^{MI}}{X^{MI}} < \alpha_M^{ME} \equiv \frac{X_M^{ME}}{X_M}$.

For the two formulations, market share of the dominant firm, as well as price premia (given by the percentage with which market price exceeds the competitive level), are illustrated in the figure below. The market share of the dominant firm at which price exceeds the competitive level by 10 percent (where the commonly used SSNIP test would indicate market power) is 12.5 percent in the simultaneous-move formulation and 34.7 percent in the sequential-move formulation. At a market share of 50 percent for the dominant firm, which in EU competition law would be taken as a presumption of dominance, price exceeds the competitive price by 20.0 percent in the simultaneous-move formulation and

17.1 percent in the sequential-move formulation; at a market share of 70 percent, which in EU law would be taken as strong evidence of dominance, the corresponding numbers are 25.9 percent and 25.0 percent.



α_M^{ME} (solid, thick line), Δ_P^{PE} (solid, thin line), α_M^{MI} (dotted, thick line) and Δ_P^{PE} (dotted, thin line) as functions of d_F; $d_M = 1$.

Finally, in both formulations the deadweight welfare loss or inefficiency is due partly to a sub-optimal level of total market output and partly to a distorted allocation of this output between the fringe and the dominant firm. Both elements of inefficiency is larger in the simultaneous-move formulation. That the first element is larger follows from the observation that total market output is smaller in the simultaneous-move formulation. To see that the second element is larger also, observe that cost inefficiency due to distorted allocation of output is related to the difference in marginal costs between the fringe and the dominant firm, given by $d_F X_F - d_M X_M$. Since $X_F^{MI} > X_F^{ME}$ and $X_M^{MI} < X_M^{ME}$, it follows that the difference in marginal cost is greater in the simultaneous-move than in the sequential-move formulation of the model.

6 Discussion

The analysis in the previous section suggests that it makes a difference, at least quantitatively, what one assumes about the dominant firm's behaviour towards the fringe; in particular, the alternative formulation involves both a higher price and greater inefficiency. The question then is what assumptions are more reasonable.

From the game-theoretic foundations of the two models, this question becomes one of order of moves. The sequential-moves model implies not only that the dominant firm has a first-mover advantage, but also that it can commit not to adjust in response to the supply decisions of its competitors. The simultaneous-moves model involves no such commitment possibility but rather implies that the dominant firm makes a best response to its competitors' decisions. The question about order of moves corresponds to the question about the appropriateness of, respectively, the Stackelberg and Cournot models of oligopoly.

The sequential-move structure may make sense when strategies are difficult, or costly, to adjust. This may be the case, for example, for some types of capacity. Then, given that a firm already has invested in capacity, it may not find it optimal to adjust upon the entry of new competitors.

In other cases, strategies are easier to adjust. Then the sequential move structure leads to an outcome that may be seen as *ex post* inoptimal; given the opportunity to adjust, the dominant firm would want to do so.[8] For example, in the above setting the marginal profit of the dominant firm evaluated at the sequential equilibrium is negative, indicating that upon observing the actual supply decision of the fringe the dominant firm would want to reduce output and raise price. In such cases, the simultaneous-move structure makes more sense since at equilibrium all participants play best response to their opponents' strategies.

In this connection, it is tempting to cite Stigler (1940, 522-23), who says that a dominant-firm market form exists when:

> *'one firm sells such a large proportion of the commodity ... that the other (small) firms individually ignore any effect they may have on prices; and ... this dominant firm behaves passively, i.e., it sets the price and sells the remainder after the minor firms have sold all they wish at the ruling price'.*

If one were to take Stigler's description of the sequence of events literally (which is probably not intended), after *'the minor firms have sold all they wish at the ruling price'*, the dominant firm would want to sell *'the remainder'* only if this quantity corresponded to that derived in our simultaneous-move formulation of the model; with the standard

[8] von Stackelberg, who, in the words of Schenzler, Siegfried and Thweatt (1992), *'completed the [static equilibrium dominant firm price leadership] model in 1934 in the form of asymmetric duopoly'*, characterised the market as lacking a stable equilibrium.

sequential-move formulation - i.e. Stigler's own - upon observing the remaining demand, the dominant firm would have wanted to revise the price upwards and sell less.

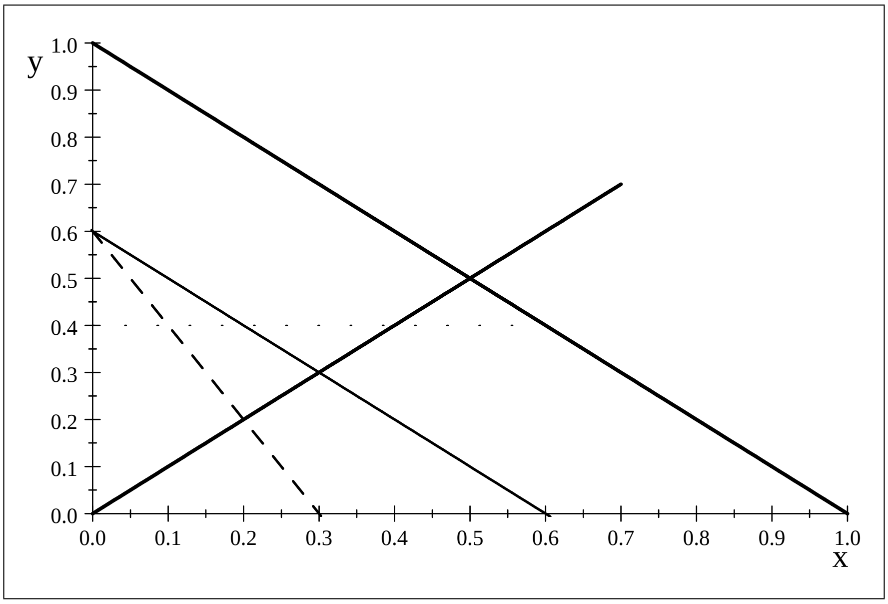
Attempts have been made to make the sequentiality of moves endogenous to the model. Deneckre and Kovenock (1992) provide a game-theoretic model of dominant-firm price leadership in a duopoly model of capacity-constrained price competition:

> *'We show that with efficiently rationed demand, and when capacities are in the range where the simultaneous move price-setting game yields a mixed-strategy solution, there are reasonable specifications of games of timing with ex-post inflexible prices in which the high-capacity firm becomes a price leader.'*

It is not clear how this result generalises. More importantly, the model does not solve the commitment problem; indeed, the assumption of *ex post* inflexible prices will in many applications be deemed unrealistic.

Some might argue that not taking account of the supply response of the fringe somehow makes the model less relevant as a tool for studing interaction between a dominant price-leading firms and its competitors. However, this is like saying that the Cournot model is less relevant than the Stackelberg model for studying oligopolistic interaction. It must be emphasised that in both formulations of the dominant-firm model the dominant firm effectively sets the price. Moreover, in both formulations the dominant firm is constrained by the presence of the fringe; price is lower than if the fringe were not present and it is lower the more elastic is fringe supply.

The latter point is illustrated in the figure below. Residual demand facing the dominant firm is found by subtracting the supply of the fringe from aggregate demand; here, however, at each price level one subtracts equilibrium fringe supply rather than the quantity corresponding to marginal cost of the fringe. Since fringe equilibrium supply is higher the lower is the marginal cost of the fringe, aggregate output is higher, and price lower, the more elastic is its supply.



Market equilibrium, alternative formulation, $d_F = d_M = 1$.

If the purpose of employing the dominant-firm model is to study the incentives of a firm with market power, but where this power is constrained by competition from other firms, it does seem unnessary to introduce additional elements of first-mover advantage and commitment.[9] This would typically be the case in competition or antitrust cases concerning dominant position, where dominance is generally taken as referring to market power as such, as exemplified by the view taken by the European Court of Law in case 27/76 United Brands versus the EU Commission (European Court of Law, 1978):

> *"The dominant position referred to in Article 86 relates to a position of economic strength enjoyed by an undertaking which enables it to prevent effective comptition being maintained on the relevant market by giving it the power to behave to an appreciable extent independently of its competitors, customers and ultimately of its consumers."*

Moreover, if the essential difference between firms in a given market is how their market power (i.e. ability to influence price) depends on their respective size or market share, there seems little reason to add additional distinguising features. Indeed, in order to concentrate on the main issue, one would want to abstract from such other considerations.

[9] Daughety (1990) discusses how move order affects the relationships between market concentration and market performance in oligopolistic industries; however, while he does point to examples of asymmetric behaviour, he does not provide any real theory of sequentiality of moves.

7 Conclusion

There is nothing in the underlying logic of the incomplete monopoly or dominant-firm idea that warrants a sequential-move structure. Consider for example the way Schenzler, Siegfried and Thweatt (1992, 171-2) present the theory:

> *'The dominant firm in this model is expected to behave as a price leader in anticipation that its smaller rivals will behave as passive price followers. Consequently, the dominant firm derives its demand for a homogenuous product as a residual by subtracting its rivals' supply from industry demand. It then maximises its profits by behaving as if it locates the output level where its marginal cost (MC) equals marginal revenue (MR) derived from its demand, i.e., like a monopolist. In this model the rivals do, in fact, behave as price-takers. Consequently, the expectations of all sellers are fulfilled and a stable equilibrium results.*
>
> *Equilibrium output in this market falls short of its competitive level, but exceeds the level that the dominant firm would offer for sale if it were a complete monopolist. In this situation the deadweight welfare loss is a weighted average of the efficiency loss of complete monopoly and of perfect competition (zero), the weights depending on the industry elasticity of demand, the aggregate supply elasticity of the dominant firm's rivals, and the market shares of the dominant firm and its rivals. These market shares, in turn, depend on the technologies and factor prices available to each firm, and the number of rivals in the competitive fringe. The essence of the model is that the monopolist's usual output restriction is mitigated by expanded output from the rivals (assuming increasing marginal costs) induced by the price leader's higher price. The monopolist accepts the burden of restricting output for the whole industry.'*

Indeed, not only is there nothing here that warrants a sequential-move structure, but there is something that suggests a simultaneous-move structure is more appropriate. In particular, the understanding that *'the expectations of all sellers are fulfilled and a stable equilibrium results'* decidedly points to an equilibrium concept based on best responses, as would result from a simultaneous-move formulation of the interaction between the dominant firm and the fringe.

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A A Plant-Based Formulation

This appendix contains an alternative game-theoretic foundation of the dominant-firm model. In this model the industry consists of a number of identical plants, some operated by the monopolist and the rest operated individually, by independent firms. We concentrate on the case in which all firms move simultaneously.

The profit at the plant n is given by

$$\pi_n = Px_n - \frac{d}{2}x_n^2, \tag{69}$$

where x_n is output, P is market price and d is a non-negative constant.

Plants $n = M+1, ..., N$ are operated individually. The first-order condition for profit maximum is

$$\frac{d\pi_n}{dx_n} = \frac{dP}{dx_n}x_n + P - dx_n = 0, \tag{70}$$

which reduces to

$$1 - X - [1+d]\, x_n = 0. \tag{71}$$

Plants $n = 1, ..., M$ are operated by a single company - the dominant firm. The first-order conditions for combined profit maximum of these plants is

$$\frac{d}{dx_n}\sum_{m=1}^{M}\pi_m = \sum_{m=1}^{M}\frac{dP}{dx_n}x_m + P - dx_n = 0, \tag{72}$$

which reduce to

$$1 - X - \sum_{m\neq n}^{M} x_m - [1+d]\, x_n = 0. \tag{73}$$

At the equilibrium where all fringe firms produce the same amount x_F, and the dominant firm produces the same amount at each plant x_M, the following system of equations determines equilibrium quantities:

$$1 - Mx_M - [N-M]\, x_F - [1+d]\, x_F = 0, \tag{74}$$

$$1 - Mx_M - [N-M]\, x_F - [M+d]\, x_M = 0. \tag{75}$$

This system has the solution

$$x_F = \frac{M+d}{[M+d]\,[N-M] + [2M+d]\,[1+d]}, \tag{76}$$

$$x_M = \frac{1+d}{[M+d]\,[N-M] + [2M+d]\,[1+d]}. \tag{77}$$

Note that output at the plant level is less at plants operated by the multi-plant dominant firm than at the plants operated by single-plant firms, i.e. $x_M < x_F$ when

$M > 1$. With $M = 1$, we have the symmetric Cournot outcome $x_F = x_M = \frac{1}{N+1+d}$. With $M = N$, we have the monopoly outcome $x_M = \frac{1}{2N+d}$.

From the above expressions we find

$$X_F = [N-M]\,x_F = \frac{[N-M][M+d]}{[M+d][N-M]+[2M+d][1+d]} \tag{78}$$

$$X_M = Mx_M = \frac{M[1+d]}{[M+d][N-M]+[2M+d][1+d]} \tag{79}$$

$$X = X_F + X_M = \frac{[N-M][M+d]+M[1+d]}{[M+d][N-M]+[2M+d][1+d]} \tag{80}$$

$$P = \frac{[M+d][1+d]}{[M+d][N-M]+[2M+d][1+d]} \tag{81}$$

Note that for given a number of plants, N, aggregate output is decreasing, and market price is increasing, in the number of plants operated by the dominant firm, M.

We want to consider the outcome when the total number of plants increases beyond bounds, given that the fraction of plants controlled by the dominant firm remains constant (to a first approximation) and that total output remains bounded.[10] In particular, let $M = \mu N$ and $d = \delta N$, which implies

$$X_F = \frac{[1-\mu][\mu+\delta]}{[\mu+\delta][1-\mu]+[2\mu+\delta]\left[\frac{1}{N}+\delta\right]} \tag{82}$$

$$X_M = Mx_M = \frac{\mu\left[\frac{1}{N}+\delta\right]}{[\mu+\delta][1-\mu]+[2\mu+\delta]\left[\frac{1}{N}+\delta\right]} \tag{83}$$

$$X = X_F + X_M = \frac{[1-\mu][\mu+\delta]+\mu\left[\frac{1}{N}+\delta\right]}{[\mu+\delta][1-\mu]+[2\mu+\delta]\left[\frac{1}{N}+\delta\right]} \tag{84}$$

$$P = \frac{[\mu+\delta]\left[\frac{1}{N}+\delta\right]}{[\mu+\delta][1-\mu]+[2\mu+\delta]\left[\frac{1}{N}+\delta\right]} \tag{85}$$

As $N \to \infty$, these expressions reduce to

$$X_F = \frac{[1-\mu][\mu+\delta]}{[\mu+\delta][1-\mu]+[2\mu+\delta]\,\delta} \tag{86}$$

$$X_M = \frac{\mu\delta}{[\mu+\delta][1-\mu]+[2\mu+\delta]\,\delta} \tag{87}$$

$$X = \frac{[1-\mu][\mu+\delta]+\mu\delta}{[\mu+\delta][1-\mu]+[2\mu+\delta]\,\delta} \tag{88}$$

$$P = \frac{[\mu+\delta]\,\delta}{[\mu+\delta][1-\mu]+[2\mu+\delta]\,\delta} \tag{89}$$

With $\mu = 0$, we have the perfectly competitive outcome, corresponding to a representative, price-taking firm with costs $\frac{\delta}{2}x^2$, where $X = \frac{1}{1+\delta}$ and $P = \frac{\delta}{1+\delta}$.

[10] Alternatively, we could hold the size of the monopolist constant (say, with $M = 1$), while the number of fringe firms increase.

With $\mu = 1$, we have the monopoly outcome, corresponding to a single firm with costs $\frac{\delta}{2}x^2$, where $X = \frac{1}{2+\delta}$ and $P = \frac{1+\delta}{2+\delta}$.

For $0 < \mu < 1$, we get an outcome corresponding to a model with one (representative) price-taking firm with costs $\frac{1}{2}\frac{\delta}{1-\mu}x^2$ and one firm with costs $\frac{1}{2}\frac{\delta}{\mu}x^2$ taking the competitor quantities as given but taking account of the impact of own output on price.

Note that even though underlying technologies are the same, (aggregate) costs differ between the dominant firm and the fringe (unless $\mu = \frac{1}{2}$).