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Coordinating under incomplete information*

Geir B. Asheim† Seung Han Yoo‡

September 29, 2007

Abstract

We show that, in a minimum effort game with incomplete information where player types are independently drawn, there is a largest and smallest Bayesian equilibrium, leading to the set of equilibrium payoffs (as evaluated at the interim stage) having a lattice structure. Furthermore, the range of equilibrium payoffs converges to those of the deterministic complete information version of the game, in the limit as the incomplete information vanishes. This entails that such incomplete information alone cannot explain the equilibrium selection suggested by experimental evidence.

Keywords and Phrases: Minimum effort games, Coordination games, Incomplete information

JEL Classification Numbers: C72

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1 Introduction

In a *minimum effort game* (Bryant, 1983; van Huyck et al., 1990; Legros and Matthews, 1993; Vislie, 1994; Hvide, 2001), players simultaneously exert efforts in order to produce a public good,[1] *with the output being determined by the player exerting the minimum effort.* Since no player wishes to exert more effort than the minimum effort of his opponents, such a game has a continuum of (pure strategy) Nash-equilibria that are Pareto-ranked.

While it might seem natural to restrict attention to the unique Pareto-dominant equilibrium, experimental evidence (see van Huyck et al., 1990) does not seem to support this argument. Subsequently, Carlsson and Ganslandt (1998) and Anderson et al. (2001) have provided a theoretical foundation for van Huyck et al.'s results by introducing noise in the players' effort choice, by letting their strategic choices translate into efforts with the addition of noise terms ("trembles").

Both Carlsson and Ganslandt (1998) and Anderson et al. (2001) indicate that such noise may be interpreted as or motivated by uncertainty about the objective functions of the players.[2] Hence, it is of interest to pose the following question: If each player's uncertainty about the effort of his opponent is not due to trembles, but to a small amount of *incomplete information* about their motivation (e.g., their willingness to pay for the public good, or their cost of contributing effort), will a similar equilibrium selection be obtained? We show in this paper that this is not the case: *Introducing incomplete information without trembles in the action choices does not reduce the set of equilibrium payoff profiles.*

[1] Although we will interpret output as a public good throughout this paper, an equivalent interpretation is that output is a private good divided among the players by a linear sharing rule.

[2] Carlsson and Ganslandt (1998, pp. 23–24) write: "The noise may also result from slightly imperfect information about the productivity of the different agents' efforts . . . ", while Anderson et al. (2001, p. 181) motivate their approach by suggesting that "[e]ven in experimental set-ups, in which money payoff can be precisely stated, there is still some recidual haziness in the players' actual payoffs, in their perceptions of the payoffs, . . . ".

We establish that, in the minimum effort game with incomplete information where player types are independently drawn, there is a largest and smallest Bayesian equilibrium, leading to the set of equilibrium payoff profiles (as evaluated at the interim stage) having a lattice structure. Hence, there is a unique Bayesian equilibrium that is weakly preferred to any other Bayesian equilibrium, for all types of each player. Moreover, any Bayesian equilibrium is weakly preferred to the unique Bayesian equilibrium where all players exert minimum effort, for all types of each player. The range of equilibrium payoffs converges to those of the deterministic complete information version of the game, in the limit as the incomplete information vanishes. This entails that such incomplete information alone cannot explain the equilibrium selection suggested by experimental evidence.

van Damme (1991, Chapter 5) analyze finite normal form games "in which each player, although knowing his own payoff function exactly, has only imprecise information about the payoff functions of his opponents", referring to them as *disturbed games*. He shows that, under certain conditions, only perfect equilibria of an undisturbed game can be approximated by equilibria of disturbed games, as the disturbances go to 0. The minimum effort game has infinite action sets and hence is outside the class studied by van Damme (1991). Still, we may note that the (pure strategy) Nash equilibria of the minimum effort game, which all can be approximated in a similar manner, are strict and thus pass any test of strategic stability.

The information structure of this paper differs from those in *global games*. In Carlsson and Ganslandt (1998) and Anderson et al. (2001), players' noise terms are independent. So, the exact counterpart of their models with incomplete information must be one in which player types are independently drawn. However, in global games — as originally modeled by Carlsson and van Damme (1993) and generalized by Frankel et al. (2003) — player types are correlated.

Our paper belongs to a large class of games with strategic complementarities, so-called *supermodular games*. Supermodular games were first introduced by Topkis (1979) and further explored by Vives (1990) and Milgrom and Roberts (1990). For games with incomplete information, existence of pure Bayesian equilibria is shown by Vives (1990) for games that are supermodular in actions; by Athey (2001) for games that satisfy a single crossing condition; and recently by Van Zandt and Vives (2007) for games where (a) actions are strategic complements, (b) there is complementarity between actions and types, and (c) interim beliefs are increasing in type with respect to first-order stochastic dominance. Our analysis echoes Vives (1990) and Van Zandt and Vives (2007) by showing the existence of a largest and a smallest Bayesian equilibrium.

We start by introducing the minimum effort game in Section 2, before illustrating incomplete information in Section 3 through the case with two players and two types for each player. We then turn to the analysis of the general n-player case with a continuum of types in Sections 4 and 5. We offer concluding remarks in Section 6, and collect the proofs and some intermediate results in an appendix.

2 The minimum effort game

Consider a coordination game, with $I = \{1, 2, ..., n\}$ ($n \geq 2$) as the player set, and $[0, \infty)$ as the action set for each player i. Player i's *action*, e_i, is interpreted as effort. The players' efforts are chosen simultaneously. Denote by b_i player i's benefit coefficient. The *payoff* function for player i is given as

$$b_i g\left(\min\left\{e_1, ..., e_n\right\}\right) - ce_i\,,$$

where $g(\min\{e_1, ..., e_n\})$ is the outcome and c is the constant marginal cost of effort. Hence, the outcome is a function g of the minimum effort. We assume throughout this paper that c is positive and that $g : [0, \infty) \to \mathbb{R}$ satisfies $g(0) = 0$, $g'(\cdot) > 0$, $g''(\cdot) < 0$, $g'(e) \to \infty$ as $e \to 0$, and $g'(e) \to 0$ as $e \to \infty$.

Note that the benefit coefficients, b_i, $i \in I$, allow for heterogeneity between the players, by endowing them with different willingness to pay for the public good. However, by writing the payoff function as

$$g\left(\min\left\{e_1, ..., e_n\right\}\right) - \frac{c}{b_i}e_i \;=\; g\left(\min\left\{e_1, ..., e_n\right\}\right) - c\frac{e_i}{b_i}\,,$$

it is apparent that the analysis of this paper remains unchanged if we instead interpret the heterogeneity as different costs of contributing effort, or different productivity of effort.

Our assumptions on $g(\cdot)$ entails that for any $b > 0$, there is a unique effort level

$$\bar{e}(b) := \arg\max_e bg(e) - ce$$

determined by $bg'(\bar{e}(b)) = c$. Furthermore, the function $\bar{e} : (0, \infty) \to [0, \infty)$ is continuous and increasing. The interpretation is that player i will choose to exert $\bar{e}(b_i)$ if he believes that his effort will be minimal and hence determine the outcome.

With complete information about the benefit coefficients it is straightforward to show that $e = (e_1, \ldots, e_n)$ is a (pure strategy) Nash equilibrium if and only if, for all $i \in I$, $e_i = e^*$ for some $e^* \in [0\,,\ \bar{e}(\min\{b_1, \ldots, b_n\})]$. Furthermore, if $0 \leq e' < e'' \leq \bar{e}(\min\{b_1, \ldots, b_n\})$, then it holds for all $i \in I$ that

$$b_i g\left(e'\right) - ce' < b_i g\left(e''\right) - ce''\,.$$

This shows that with complete information the minimum effort game has a continuum of Nash-equilibria that are Pareto-ranked. In particular, with homogeneous players (i.e., $b_i = b$ for all $i \in I$), the range of equilibrium payoffs is given by

$$[0\,,\ bg(\bar{e}(b)) - c\bar{e}(b)]\ .$$

3 Illustrating incomplete information: Two types

Before turning to the general analysis of incomplete information in Section 4, it is instructive to illustrate incomplete information in the simplest setting, with two players and two types for each player, since the basic structure of the analysis carries over to the more general case.

The *type* of each player i corresponds to his benefit coefficient b_i, which may take the values in the set $\{b_L, b_H\}$, with $0 < b_L < b_H$. The types of each player is private information and is i.i.d., being b_H with probability P and b_L with probability $1 - P$.

A *strategy* for each player i is a function $s_i : \{b_L, b_H\} \to [0, \infty)$. A strategy profile (s_1, s_2) is a Bayesian equilibrium if, for each $i \in \{1, 2\}$,

$$s_i(b_L) = \arg\max_{e\in[0,\infty)} u\left(e, s_j, b_L\right) \tag{1}$$

$$s_i(b_H) = \arg\max_{e\in[0,\infty)} u\left(e, s_j, b_H\right) \ , \tag{2}$$

where, for $k = L$, H,

$$u\left(e_i, s_j, b_k\right) := Pb_k g\left(\min\left\{e_i, s_j(b_H)\right\}\right) + (1 - P)\, b_k g\left(\min\left\{e_i, s_j(b_L)\right\}\right) - ce_i \, .$$

To investigate the range of equilibria payoffs in this simple incomplete information setting, consider the following uniquely determined effort levels,

$$\begin{aligned} e_L &:= \bar{e}(b_L) \\ e_H &:= \arg\max_e Pb_H g(e) - ce \, , \end{aligned}$$

and consider the strategy $\bar{s}$ defined by,

$$\begin{aligned} \bar{s}(b_L) &:= e_L \\ \bar{s}(b_H) &:= \max\{e_L, e_H\} \, . \end{aligned}$$

The following result shows that the strategy $\bar{s}$ provides an upper bound on equilibrium effort.

Proposition 1 *Any Bayesian equilibrium $s = (s_1, s_2)$ satisfies that for every player i, $0 \leq s_i(b_L) \leq \bar{s}(b_L)$ and $0 \leq s_i(b_H) \leq \bar{s}(b_H)$.*

Our main result of this section establishes the existence of a largest and smallest Bayesian equilibrium and shows that the set of Bayesian equilibrium payoff profiles (as evaluated at the interim stage) has a lattice structure.

Proposition 2 *(i) The symmetric strategy profile $s = (s_1, s_2)$ where for every player i, $s_i = \underline{s}$, with $\underline{s}$ defined by $\underline{s}(b_k) = 0$ for $k = L$, H, is a Bayesian equilibrium.*

(ii) *The symmetric strategy profile* $s = (s_1, s_2)$ *where for every player* i, $s_i = \bar{s}$, *is a Bayesian equilibrium.*

(iii) *If* $s = (s_1, s_2)$ *is a Bayesian equilibrium, then, for* $i \in \{1, 2\}$ *and* $k = L, H$,

$$0 = u(\underline{s}(b_k), \underline{s}, b_k) \le u(s_i(b_k), s_j, b_k) \le u(\bar{s}(b_k), \bar{s}, b_k)\,.$$

(iv) *For* $i \in \{1, 2\}$ *and* $k = L, H$, *if* u *satisfies*

$$0 = u(\underline{s}(b_k), \underline{s}, b_k) \le u \le u(\bar{s}(b_k), \bar{s}, b_k)\,,$$

then there exists a Bayesian equilibrium $s = (s_1, s_2)$ *such that* $u(s_i(b_k), s_j, b_i) = u$.

Proposition 2 entails that, in this simple version of the minimum effort game with incomplete information, the range of equilibrium payoffs converges to those of the deterministic complete information version of the game, in the limit as the incomplete information vanishes, by having b_L and b_H converge to a common benefit coefficient b. In the next two sections we show that this result carries over to the minimum effort game with a continuum of types.

4 Incomplete information with a continuum of types

In the incomplete information version of the minimum effort game with a continuum of types, the type b_i of each player i is drawn independently from an absolutely continuous CDF $F : B \to [0, 1]$, where $B = [\,\underline{b}\,,\, \bar{b}\,]$ denotes the set of types, with $0 < \underline{b} < \bar{b}$. A *strategy* $s_i : B \to [0, \infty)$ for each player i is a measurable function, with S_i denoting i's strategy set. Write $b_{-i} := (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$, $\Omega := B^{n-1}$, $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, and $S_{-i} := S_1 \times \cdots \times S_{i-1} \times S_{i+1}, \times \cdots \times S_n$. Define $\Phi : \Omega \to [0, 1]$ by

$$\Phi(b_{-i}) := F(b_1) \times \cdots \times F(b_{i-1}) \times F(b_{i+1}) \times \cdots \times F(b_n)\,.$$

Then the payoff of a player of type $b_i \in B$ can be written as

$$u(e_i, s_{-i}, b_i) := b_i G(e_i, s_{-i}) - ce_i\,,$$

where

$$G\left(e_i, s_{-i}\right) := \int_{\Omega} \min\{g(e_i), g\left(\min_{j \neq i}\left\{s_j\left(b_j\right)\right\}\right)\} d\Phi\left(b_{-i}\right)\,.$$

If a player of type b_i believes that his effort will be minimal and hence determine the outcome, he will choose to exert $\bar{e}(b_i)$. However, when playing with opponents whose strategies are given by s_{-i}, type b_i will choose an effort in $[0,\, \bar{e}(b_i)]$, since other players, following their strategies, may choose efforts smaller than $\bar{e}(b_i)$ and determine the outcome if type b_i exerts $\bar{e}(b_i)$. The following proposition shows that each type b_i of player i has a unique best response

$$\beta(s_{-i})(b_i) := \arg\max_e u(e, s_{-i}, b_i)\,,$$

which is an element of $[0,\, \bar{e}(b_i)]$ for each b_i, and which is a continuous and non-decreasing function of b_i.

Proposition 3 *For every $s_{-i} \in S_{-i}$, the following holds. Each type b_i of player i has a unique best response $\beta(s_{-i})(b_i)$. Furthermore, $\beta(s_{-i})$ is a continuous and non-decreasing function of b_i.*

A strategy profile $s = (s_1, \ldots, s_n)$ is a *Bayesian equilibrium*, if,

$$\text{for each type } b_i \text{ of every player } i, s_i(b_i) = \beta(s_{-i})(b_i)\,.$$

It follows from Proposition 3 that $s_i(\cdot)$ is a continuous and non-decreasing function if s_i is part of a Bayesian equilibrium.

To investigate the range of equilibrium payoffs under incomplete information, consider the strategy $\bar{s} : B \to [0, \infty)$ defined by

$$\bar{s}(b_i) := \sup\{e \mid \exists b \leq b_i \text{ satisfying } F(b) < 1 \text{ s.t. } e(b) = e\}\,,$$

where $e : \{b \in B \mid F(b) < 1\} \to [0, \infty)$ is defined by

$$e(b) := \arg\max_e bg(e)(1 - F(b))^{n-1} - ce\,.$$

By the assumptions on $g(\cdot)$ it follows that, for each $b \in B$ satisfying $F(b) < 1$, $e(b)$ is uniquely determined by $bg'(e(b))(1 - F(b))^{n-1} = c$. The following result conveys the importance of the strategy $\bar{s}$.

Proposition 4 *Any Bayesian equilibrium $s = (s_1, \ldots, s_n)$ satisfies that, for each type b_i of every player i, $0 \leq s_i(b_i) \leq \bar{s}(b_i)$.*

Our main result of this section establishes the existence of a largest and smallest Bayesian equilibrium and shows that the set of Bayesian equilibrium payoff profiles (as evaluated at the interim stage) has a lattice structure.

Proposition 5 *(i) The symmetric strategy profile $s = (s_1, \ldots, s_n)$ where for every player i, $s_i = \underline{s}$, with $\underline{s}$ defined by $\underline{s}(b_i) = 0$ for each type b_i, is a Bayesian equilibrium.*

(ii) The symmetric strategy profile $s = (s_1, \ldots, s_n)$ where for every player i, $s_i = \bar{s}$, is a Bayesian equilibrium.

(iii) If $s = (s_1, \ldots, s_n)$ is a Bayesian equilibrium, then, for each type b_i of every player i,

$$0 = u\big(\underline{s}(b_i), \underbrace{(\underline{s}, \ldots, \underline{s})}_{n-1 \text{ times}}, b_i\big) \leq u\big(s_i(b_i), s_{-i}, b_i\big) \leq u_i\big(\bar{s}(b_i), \underbrace{(\bar{s}, \ldots, \bar{s})}_{n-1 \text{ times}}, b_i\big) .$$

(iv) For each type b_i of every player i, if

$$0 = u\big(\underline{s}(b_i), \underbrace{(\underline{s}, \ldots, \underline{s})}_{n-1 \text{ times}}, b_i\big) \leq u \leq u\big(\bar{s}(b_i), \underbrace{(\bar{s}, \ldots, \bar{s})}_{n-1 \text{ times}}, b_i\big) ,$$

then there exists a Bayesian equilibrium $s = (s_1, \ldots, s_n)$ such that $u_i\big(s_i(b_i), s_{-i}, b_i\big) = u$.

In the next section we show that the range of equilibrium payoffs converges to those of the deterministic complete information version of the game, in the limit as the incomplete information vanishes.

5 Vanishing incomplete information

Given some $b > 0$, consider two sequences $\{\underline{b}^m\}_{m=1}^{\infty}$ and $\{\bar{b}^m\}_{m=1}^{\infty}$ satisfying

$$\underline{b}^m < \underline{b}^{m+1} < b < \bar{b}^{m+1} < \bar{b}^m \text{ for all } m \in \mathbb{N} \quad \text{and} \quad \lim_{m\to\infty}\underline{b}^m = b = \lim_{m\to\infty}\bar{b}^m \,.$$

Write $B_m := [\underline{b}^m, \bar{b}^m]$ for each $m \in \mathbb{N}$, implying that $B_m \supset B_{m+1}$ for all $m \in \mathbb{N}$ and $\cap_{m=1}^{\infty} B_m = \{b\}$.

For each $m \in \mathbb{N}$, construct a incomplete information minimum effort game where the type b_i of each player i is drawn independently from an absolutely continuous CDF $F_m : B_m \to [0, 1]$. A strategy $s_i : B_m \to [0, \infty)$ of each player i is a measurable function. Denote $\Omega_m := B_m^{n-1}$, and define $\Phi_m : \Omega_m \to [0, 1]$ by

$$\Phi_m(b_{-i}) := F_m(b_1) \times \cdots \times F_m(b_{i-1}) \times F_m(b_{i+1}) \times \cdots \times F_m(b_n) \,.$$

Then the payoff of an agent of type $b_i \in B_m$ can be written as

$$u_m(e_i, s_{-i}, b_i) := b_i G_m(e_i, s_{-i}) - ce_i \,,$$

where

$$G_m\left(e_i, s_{-i}\right) := \int_{\Omega_m} \min\{g(e_i), g\left(\min_{j\neq i}\left\{s_j\left(b_j\right)\right\}\right)\} d\Phi_m\left(b_{-i}\right) \,.$$

Let the strategy $\bar{s}^m : B_m \to [0, \infty)$ be defined by

$$\bar{s}^m(b_i) := \sup\{e \mid \exists b \le b_i \text{ satisfying } F_m(b) < 1 \text{ s.t. } e^m(b) = e\} \,,$$

where $e^m : \{b \in B_m \mid F_m(b) < 1\} \to [0, \infty)$ is defined by

$$e^m(b) := \arg\max_e bg(e)(1 - F_m(b))^{n-1} - ce \,.$$

Note that Propositions 4 and 5 (ii)–(iv) apply to the strategy $\bar{s}^m(\cdot)$, and Proposition 5 (i) applies to the strategy $\underline{s}^m$ defined by $\underline{s}^m(b_i) = 0$ for each type b_i.

Since the sequence $\{B_m\}_{m=1}^{\infty}$ converges to a singleton set containing only the benefit coefficient b, the sequence of incomplete information games converges, in the limit as $m \to \infty$, to a complete information game where all players have a common benefit coefficient b. The following result shows that the range of equilibrium payoffs converges to those of the deterministic complete information version of the game, in the limit as the incomplete information vanishes.

Proposition 6 *Consider the sequence of incomplete information minimum effort games determined by the sequence of supports* $\{B_m\}_{m=1}^{\infty}$ *and the sequence of CDFs* $\{F_m\}_{m=1}^{\infty}$. *Then*

$$\lim_{m\to\infty} u_m(\bar{s}^m(\underline{b}^m), s_{-i}, b_i) = bg(\bar{e}(b)) - c\bar{e}(b) = \lim_{m\to\infty} u_m(\bar{s}^m(\bar{b}^m), s_{-i}, b_i)\,.$$

Proposition 6 entails that small uncertainty about the payoffs of the opponents in the minimum effort game with a continuum of types does not result in equilibrium selection, provided that player types are independently drawn. This shows that such an incomplete information version of the minimum effort game does not lead to the the equilibrium selection results obtained by Carlsson and Ganslandt (1998) and Anderson et al. (2001) through their versions of the minimum effort game, where the players' strategic choices translate into efforts with the addition of noise terms.

6 Concluding remarks

Equilibrium selection in the minimum effort game has been studied since van Huyck et al. (1990) obtained their experimental evidence. Both Carlsson and Ganslandt (1998) and Anderson et al. (2001) obtain results consistent with the experiment evidence by introducing noise. Neither Carlsson and Ganslandt (1998) nor Anderson et al. (2001) study incomplete versions of the minimum effort game, since in their formulations the action choices of the players do not depend on private information.

In the present paper we endow the players with private information about their payoff functions, where player types are independently drawn. We establish that such incomplete information alone does not lead to equilibrium selection in the minimum effort game. This means that one should be cautious in interpreting the results of Carlsson and Ganslandt (1998) and Anderson et al. (2001) in terms of incomplete information about the payoff functions of the opponents.

Appendix: Proofs

Proof of Proposition 1. Assume that s is a Bayesian equilibrium. Since the effort set is $[0,\infty)$, it remains to be shown that $s_i(b_L) \leq \bar{s}(b_L)$ and $s_i(b_H) \leq \bar{s}(b_H)$.

Suppose that $s_i(b_L) > \bar{s}(b_L)$. However, then, for any opponent strategy s_j,

$$\begin{aligned}
& Pb_L g\left(\min\left\{s_i(b_L), s_j(b_H)\right\}\right) + (1-P)\, b_L g\left(\min\left\{s_i(b_L), s_j(b_L)\right\}\right) - cs_i(b_L) \\
& - \left[Pb_L g\left(\min\left\{\bar{s}(b_L), s_j(b_H)\right\}\right) + (1-P)\, b_L g\left(\min\left\{\bar{s}(b_L), s_j(b_L)\right\}\right) - c\bar{s}(b_L)\right] \\
\leq\ & Pb_L g\left(s_i(b_L)\right)) + (1-P)\, b_L g\left(s_i(b_L)\right) - cs_i(b_L) \\
& - \left[Pb_L g\left(\bar{s}(b_L)\right) + (1-P)\, b_L g\left(\bar{s}(b_L)\right) - c\bar{s}(b_L)\right] \\
=\ & b_L g\left(s_i(b_L)\right) - cs_i(b_L) - \left[b_L g\left(\bar{s}(b_L)\right) - c\bar{s}(b_L)\right] \ < \ 0\,,
\end{aligned}$$

where the weak inequality follows since g is increasing and the strict inequality follows since, by the definition of $\bar{s}(b_L)$ and the property that g is strictly concave, $b_L g(e) - ce$ is a decreasing function of e for $e > \bar{s}(b_L) = e_L$. This contradicts by (1) that (s_i, s_j) in a Bayesian equilibrium, for any $s_i(b_H)$, and shows that $s_i(b_L) \leq \bar{s}(b_L)$.

Suppose that $s_i(b_H) > \bar{s}(b_H)$. Then it follows from the definition of $\bar{s}(b_H)$ and the first part of the proof that, for any opponent strategy s_j that might be part of a Bayesian equilibrium, it holds that $s_j(b_L) \leq \bar{s}(b_L) \leq \bar{s}(b_H) < s_i(b_H)$. This implies the equality below,

$$\begin{aligned}
& Pb_H g\left(\min\left\{s_i(b_H), s_j(b_H)\right\}\right) + (1-P)\, b_H g\left(\min\left\{s_i(b_H), s_j(b_L)\right\}\right) - cs_i(b_H) \\
& - \left[Pb_H g\left(\min\left\{\bar{s}(b_H), s_j(b_H)\right\}\right) + (1-P)\, b_H g\left(\min\left\{\bar{s}(b_H), s_j(b_L)\right\}\right) - c\bar{s}(b_h)\right] \\
=\ & Pb_H g\left(\min\left\{s_i(b_H), s_j(b_H)\right\}\right) + (1-P)\, b_H g\left(s_j(b_L)\right) - cs_i(b_H) \\
& - \left[Pb_H g\left(\min\left\{\bar{s}(b_H), s_j(b_H)\right\}\right) + (1-P)\, b_H g\left(s_j(b_L)\right) - c\bar{s}(b_H)\right] \\
\leq\ & Pb_H g\left(s_i(b_H)\right) - cs_i(b_H) \ - \ \left[Pb_H g\left(\bar{s}(b_H)\right) - c\bar{s}(b_H)\right] \ < \ 0\,,
\end{aligned}$$

while the weak inequality follows since g is increasing and the strict inequality follows since, by the definition of $\bar{s}(b_H)$ and the property that g is strictly concave, $Pb_H g(e) - ce$ is a decreasing function of e for $e > \bar{s}(b_H) \geq e_H$. This contradicts by (2) that (s_i, s_j) in a Bayesian equilibrium, for any $s_i(b_L)$, and shows that $s_i(b_H) \leq \bar{s}(b_H)$. ■

Proof of Proposition 2. *Part (i).* Assume that $s_j = \underline{s}$. Then clearly $u\left(e, s_j, b_L\right)$ and $u\left(e, s_j, b_H\right)$ are decreasing in e for all $e \geq 0$, establishing the result by (1) and (2).

Part (ii). Assume that $s_j = \bar{s}$. By Proposition 1 and (1) and (2), it is sufficient to show that $u\left(e, s_j, b_L\right)$ is increasing in e for all $e \leq \bar{s}(b_L)$, and $u\left(e, s_j, b_H\right)$ is increasing in e for all $e \leq \bar{s}(b_H)$. This follows from the definition of $\bar{s}$.

Part (iii). We have that $0 = u(\underline{s}(b_k), \underline{s}, b_k) \leq u(s_i(b_k), s_j, b_k)$, since $u(0, s_j, b_L) = 0$ and $u(0, s_j, b_H) = 0$, independently of s_j. Hence, each type of player i can always ensure himself a non-negative payoff by setting $e_i = 0$. To show that $u(s_i(b_k), s_j, b_k) \leq u(\bar{s}(b_k), \bar{s}, b_k)$ for each $k = L$, H, note that $u(e_i, s_j, b_k)$ is non-decreasing in both $s_j(b_L)$ and $s_j(b_H)$. Hence, by Proposition 1, $u(s_i(b_L), s_j, b_L)$ and $u(s_i(b_H), s_j, b_H)$ are maximized for fixed $s_i(b_L)$ and $s_i(b_H)$ by setting $s_j = \bar{s}$. Moreover, given $s_j = \bar{s}$, it follows from part (ii) that $u(e_i, s_j, b_L)$ is maximized by setting $e_i = \bar{s}(b_L)$, and $u(e_i, s_j, b_H)$ is maximized by setting $e_i = \bar{s}(b_H)$.

Part (iv). For all $e \in [0, \bar{s}(b_H)]$, let s^e be given by

$$s^e(b_L) := \min\{e_L, e\}$$
$$s^e(b_H) := e\,.$$

Then, for any $e \in [0, \bar{s}(b_H)]$, $u(e', s^e, b_L)$ is increasing in e' for all $e' \leq s^e(b_L)$, and $u(e', s^e, b_H)$ is increasing in e' for all $e' \leq s^e(b_H)$. Moreover, $u(e', s^e, b_L)$ is decreasing in e' for all $e' \geq s^e(b_L)$, and $u(e', s^e, b_H)$ is decreasing in e' for all $e' \geq s^e(b_H)$. Hence, by (1) and (2), (s^e, s^e) is a symmetric Bayesian equilibrium. Furthermore, $u(s^e(b_L), s^e, b_L)$ and $u(s^e(b_H), s^e, b_H)$ are continuous functions of e, with $u(s^0(b_L), s^0, b_L) = u(s^0(b_H), s^0, b_H) = 0$, and, for $k = L$, H, $u(s^{\bar{s}(b_H)}(b_k), s^{\bar{s}(b_H)}, b_k) = u(\bar{s}(b_k), \bar{s}, b_k)$. This establishes part (iv). ■

Turn now to the case with a continuum of types considered in Sections 4 and 5. Let

$$B_{-i}(b_i) := \{b_{-i} \in \Omega \mid b_j \geq b_i \text{ for every } j \neq i\}$$

denote the set of opponent type profiles such that each opponent type j, $b_j \geq b_i$, and let

$$A(e_i, s_{-i}) := \{b_{-i} \in \Omega \mid s_j(b_j) \geq e_i \text{ for every } j \neq i\}$$

denote the set of opponent type profiles having the property that no opponent exert an effort less than e_i when their strategy profile is given by s_{-i}. Then the function $G(e_i, s_{-i}) := \int_\Omega \min\{g(e_i), g(\min_{j\neq i}\{s_j(b_j)\})\} d\Phi(b_{-i})$ can be written as

$$G(e_i, s_{-i}) = \int_{A(e_i,s_{-i})} g(e_i)\, d\Phi(b_{-i}) + \int_{\Omega\setminus A(e_i,s_{-i})} g(\min_{j\neq i}\{s_j(b_j)\})\, d\Phi(b_{-i})\,.$$

As a function of e_i, G has the following properties.

Lemma 1 *For every $s_{-i} \in S_{-i}$, the following holds. (i) G is a continuous function of e_i. (ii) If $e_i' < e_i''$, then $0 \leq G(e_i'', s_{-i}) - G(e_i', s_{-i}) \leq g(e_i'') - g(e_i')$. (iii) If $G(e_i', s_{-i}) < G(e_i'', s_{-i})$, then, for every $\lambda \in (0, 1)$,*

$$G(\lambda e_i' + (1-\lambda)e_i'', s_{-i}) > \lambda G(e_i', s_{-i}) + (1-\lambda)G(e_i'', s_{-i})\,.$$

Proof. *(i)* Fix e_i and let $\varepsilon > 0$. Since g is continuous, there exists $\delta > 0$ such that $|g(e_i') - g(e_i)| < \varepsilon$ for all e_i' satisfying $|e_i' - e_i| < \delta$. This in turn implies that, for all $(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)$,

$$|\min\{g(e_i'), g(\min_{j\neq i}\{e_j\})\} \; - \; \min\{g(e_i), g(\min_{j\neq i}\{e_j\})\}| < \varepsilon$$

for all e_i' satisfying $|e_i' - e_i| < \delta$. This in turn implies that, for fixed s_{-i},

$$\begin{aligned} &|G(e_i', s_{-i}) - G(e_i, s_{-i})| \\ &\quad = \Big| \int_\Omega \min\{g(e_i'), g(\min_{j\neq i}\{s_j(b_j)\})\} d\Phi(b_{-i}) \\ &\qquad\qquad - \int_\Omega \min\{g(e_i), g(\min_{j\neq i}\{s_j(b_j)\})\} d\Phi(b_{-i}) \Big| \; < \; \varepsilon \end{aligned}$$

for all e_i' satisfying $|e_i' - e_i| < \delta$. This shows that G is a continuous function of e_i.

(ii) Let $e_i' < e_i''$, implying that, for fixed s_{-i}, $A(e_i', s_{-i}) \supseteq A(e_i'', s_{-i})$. Hence, it follows from the definition of G that

$$\begin{aligned} G(e_i'', s_{-i}) - G(e_i', s_{-i}) \; &= \int_{A(e_i'', s_{-i})} (g(e_i'') - g(e_i'))\, d\Phi(b_{-i}) \\ &+ \int_{A(e_i', s_{-i})\setminus A(e_i'', s_{-i})} (g(\min_{j\neq i}\{s_j(b_j)\}) - g(e_i'))\, d\Phi(b_{-i}) \; . \end{aligned}$$

Since g is increasing and $g(\min_{j\neq i}\{s_j(b_j)\}) \leq g(e_i'')$ on $\Omega\setminus A(e_i'', s_{-i})$ and $A(e_i',\, s_{-i}) \subseteq \Omega$, we have that $0 \leq G(e_i'', s_{-i}) - G(e_i', s_{-i}) \leq g(e_i'') - g(e_i')$.

(iii) Assume $G(e_i', s_{-i}) < G(e_i'', s_{-i})$, and fix $\lambda \in (0, 1)$. Write $e_i := \lambda e_i' + (1-\lambda)e_i''$. Since G is non-decreasing, we have that $e_i' < e_i < e_i''$, implying that, for fixed s_{-i}, $A(e_i', s_{-i}) \supseteq A(e_i, s_{-i}) \supseteq A(e_i'', s_{-i})$. It follows from the definition of G that

$$\begin{aligned} G(e_i'', s_{-i}) - G(e_i, s_{-i}) \; &= \int_{A(e_i'', s_{-i})} (g(e_i'') - g(e_i))\, d\Phi(b_{-i}) \\ &+ \int_{A(e_i, s_{-i})\setminus A(e_i'', s_{-i})} (g(\min_{j\neq i}\{s_j(b_j)\}) - g(e_i))\, d\Phi(b_{-i}) \; , \\ G(e_i, s_{-i}) - G(e_i', s_{-i}) \; &= \int_{A(e_i, s_{-i})} (g(e_i) - g(e_i'))\, d\Phi(b_{-i}) \\ &+ \int_{A(e_i', s_{-i})\setminus A(e_i, s_{-i})} (g(\min_{j\neq i}\{s_j(b_j)\}) - g(e_i'))\, d\Phi(b_{-i}) \; . \end{aligned}$$

Hence,

$$
\begin{aligned}
&G(e_i, s_{-i}) - [\lambda G(e_i', s_{-i}) + (1-\lambda) G(e_i'', s_{-i})] \\
&= \lambda \int_{A(e_i,s_{-i})} (g(e_i) - g(e_i'))\, d\Phi\,(b_{-i}) + (1-\lambda) \int_{A(e_i'',s_{-i})} (g(e_i) - g(e_i''))\, d\Phi\,(b_{-i}) \\
&\quad + \lambda \int_{A(e_i',s_{-i}) \setminus A(e_i,s_{-i})} (g\,(\min_{j\neq i} \{s_j\,(b_j)\}) - g(e_i'))\, d\Phi\,(b_{-i}) \\
&\quad + (1-\lambda) \int_{A(e_i,s_{-i}) \setminus A(e_i'',s_{-i})} (g(e_i) - g\,(\min_{j\neq i} \{s_j\,(b_j)\}))\, d\Phi\,(b_{-i}) \\
&= \int_{A(e_i'',s_{-i})} [g(e_i) - (\lambda g(e_i') + (1-\lambda) g(e_i''))]\, d\Phi\,(b_{-i}) \\
&\quad + \int_{A(e_i,s_{-i}) \setminus A(e_i'',s_{-i})} [g(e_i) - (\lambda g(e_i') + (1-\lambda) g\,(\min_{j\neq i} \{s_j\,(b_j)\}))]\, d\Phi\,(b_{-i}) \\
&\quad + \lambda \int_{A(e_i',s_{-i}) \setminus A(e_i,s_{-i})} (g\,(\min_{j\neq i} \{s_j\,(b_j)\}) - g(e_i'))\, d\Phi\,(b_{-i}) \ .
\end{aligned}
$$

Since g is strictly concave, $e_i' < e_i''$, and $\lambda \in (0, 1)$, we have that

$$
0 \ < \ g(e_i) - (\lambda g(e_i') + (1-\lambda) g(e_i'')) \ . \tag{A1}
$$

Furthermore, since $g\,(\min_{j\neq i} \{s_j\,(b_j)\}) \leq g(e_i'')$ on $\Omega \setminus A(e_i'', s_{-i})$, (A1) implies that

$$
0 \ < \ g(e_i) - (\lambda g(e_i') + (1-\lambda) g\,(\min_{j\neq i} \{s_j\,(b_j)\}))
$$

on $\Omega \setminus A(e_i'', s_{-i})$. Hence, if $A(e_i, s_{-i})$ has non-zero measure, we have established that

$$
G(e_i, s_{-i}) - [\lambda G(e_i', s_{-i}) + (1-\lambda) G(e_i'', s_{-i})] > 0 \,.
$$

Moreover, this is trivially the case if $A(e_i, s_{-i})$ has zero measure, because then $G(e_i', s_{-i}) < G(e_i, s_{-i}) = G(e_i'', s_{-i})$. ■

Proof of Proposition 3. *Unique best response.* By Lemma 1(i), $u(e, s_{-i}, b_i) = b_i G(e_i, s_{-i}) - ce_i$ attains a local maximum on $[0, \bar{e}(b_i)]$. The strict concavity of $g(\cdot)$ and Lemma 1(ii) imply that any such local maximum is also a global maximum:

$$
0 > b_i g(e_i) - ce_i - (b_i g(\bar{e}(b_i)) - c\bar{e}(b_i)) \geq b_i G(e_i, s_{-i}) - ce_i - (b_i G(\bar{e}(b_i), s_{-i}) - c\bar{e}(b_i))
$$

if $e_i > \bar{e}(b_i)$. Suppose that there exist e_i' and e_i'', with $0 \leq e_i' < e_i'' \leq \bar{e}(b_i)$, satisfying

$$
b_i G(e_i', s_{-i}) - ce_i' = b_i G(e_i'', s_{-i}) - ce_i'' = \max_e b_i G(e, s_{-i}) - ce \,.
$$

Since $c > 0$, we must have $b_i\big(G(e_i'', s_{-i}) - G(e_i', s_{-i})\big) = c(e_i'' - e_i') > 0$, implying that $G(e_i', s_{-i}) < G(e_i'', s_{-i})$. However, then Lemma 1(iii) implies that

$$\begin{aligned} & b_i G(\lambda e_i' + (1-\lambda) e_i'', s_{-i}) - c\,(\lambda e_i' + (1-\lambda) e_i'') \\ & > \lambda\,(b_i G(e_i', s_{-i}) - c e_i') + (1-\lambda)\,(b_i G(e_i'', s_{-i}) - c e_i'') \\ & = \max\nolimits_e b_i G(e, s_{-i}) - ce\,, \end{aligned}$$

which contradicts that e_i' and e_i'' are best responses. Hence, $\beta(s_{-i})(b_i) := \arg\max_e b_i G(e, s_{-i}) - ce$ exists and is unique.

$\beta(s_{-i})$ is continuous. Suppose that $\beta(s_{-i})$ is not a continuous function of b_i. Then there exists a sequence $\{b_i^m\}_{m=1}^\infty$ such that $b_i^m \to b_i^0$ and $\beta(s_{-i})(b_i^m) \nrightarrow \beta(s_{-i})(b_i^0)$ as $m \to \infty$. Since, for all m, $\beta(s_{-i})(b_i^m) \in [0, \bar{e}(\bar{b})]$ (cf. the first part of the proof), there exists a subsequence $\{\tilde{b}_i^m\}_{m=1}^\infty$ satisfying $\tilde{b}_i^m \to b_i^0$ and $\tilde{e}_i^m \to \tilde{e}_i^0 \neq e_i^0$ as $n \to \infty$, where we write $e_i^0 := \beta(s_{-i})(b_i^0)$ and, for all m, $\tilde{e}_i^m := \beta(s_{-i})(\tilde{b}_i^m)$. The definition of $\beta(s_{-i})$ implies that the following inequalities are satisfied for all m:

$$\begin{aligned} \tilde{b}_i^m G(\tilde{e}_i^m, s_{-i}) - c\tilde{e}_i^m \ &\geq\ \tilde{b}_i^m G(e_i^0, s_{-i}) - c e_i^0 \\ b_i^0 G(e_i^0, s_{-i}) - c e_i^0 \ &\geq\ b_i^0 G(\tilde{e}_i^m, s_{-i}) - c\tilde{e}_i^m\,. \end{aligned}$$

Since G is a continuous function of e_i, by taking limits and keeping in mind that $\tilde{b}_i^m \to b_i^0$ and $\tilde{e}_i^m \to \tilde{e}_i^0 \neq e_i^0$ as $m \to \infty$, we now obtain that

$$b_i^0 G(\tilde{e}_i^0, s_{-i}) - c\tilde{e}_i^0 \ =\ b_i^0 G(e_i^0, s_{-i}) - c e_i^0 \ =\ \max\nolimits_e b_i^0 G(e, s_{-i}) - ce\,,$$

where $\tilde{e}_i^0 \neq e_i^0$. This contradicts that $\beta(s_{-i})(b_i^0)$ is unique and shows that $\beta(s_{-i})$ is a continuous function of b_i.

$\beta(s_{-i})$ is non-decreasing. Let $b_i' < b_i''$, and write $e_i' := \beta(s_{-i})(b_i')$ and $e_i'' := \beta(s_{-i})(b_i'')$. The definition of $\beta(s_{-i})$ implies the following inequalities:

$$\begin{aligned} b_i' G(e_i', s_{-i}) - c e_i' \ &\geq\ b_i' G(e_i'', s_{-i}) - c e_i'' \qquad \text{(A2)} \\ b_i'' G(e_i'', s_{-i}) - c e_i'' \ &\geq\ b_i'' G(e_i', s_{-i}) - c e_i'\,. \end{aligned}$$

Hence,

$$(b_i'' - b_i')\,[G(e_i'', s_{-i}) - G(e_i', s_{-i})] \ \geq\ 0\,.$$

Since G is a non-decreasing function of e, this implies that $G(e_i', s_{-i}) = G(e_i'', s_{-i})$ if $e_i' > e_i''$. However, $e_i' > e_i''$ and $G(e_i', s_{-i}) = G(e_i'', s_{-i})$ contradicts (A2). Hence, $e_i' \leq e_i''$, showing that $\beta(s_{-i})$ is a non-decreasing function of b_i. ■

The observation that $s_i(\cdot)$ is a continuous and non-decreasing function if s_i is part of a Bayesian equilibrium can be applied to show the following useful result.

Lemma 2 *Any Bayesian equilibrium satisfies*

(i) $G(e_i, s_{-i}) - G(e', s_{-i}) \leq \big(g(e_i) - g(e')\big)(1 - F(b'))^{n-1}$ *whenever* $e' < e_i$ *and* $b' \leq \sup\big(\{b \mid s_j(b) < e' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$*, and*

(ii) $G(e'', s_{-i}) - G(e_i, s_{-i}) \geq \big(g(e'') - g(e_i)\big)(1 - F(b''))^{n-1}$ *whenever* $e_i < e''$ *and* $b'' \geq \sup\big(\{b \mid s_j(b) < e'' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$.

Proof. *Part (i).* Assume $e' < e_i$ and $b' \leq \big(\sup\{b \mid s_j(b) < e' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$. Since $s_j(\cdot)$ is non-decreasing for all j, the existence of $k \neq i$ such that $s_k(b_k) \geq e'$ and $\underline{b} \leq b_k < b'$ would imply that b_k is an upper bound for $\{b \mid s_j(b) < e' \text{ for all } j \neq i\} \cup \{\underline{b}\}$ and thus contradict that $b' \leq \sup\big(\{b \mid s_j(b) < e' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$. Hence, for all $j \neq i$, $s_j(b_j) \geq e'$ implies $b_j \geq b'$; i.e., $A(e', s_{-i}) \subseteq B_{-i}(b')$. It now follows from the definition of G that

$$\begin{aligned}
& G(e_i, s_{-i}) - G(e', s_{-k}) \\
= & \int_{A(e_i, s_{-i})} \big(g(e_i)) - g(e')\big) d\Phi(b_{-i}) \\
& + \int_{A(e', s_{-i}) \setminus A(e_i, s_{-i})} \big(g(\min_{j \neq i}\{s_j(b_j)\}) - g(e')\big) d\Phi(b_{-i}) \\
\leq & \int_{A(e', s_{-i})} \big(g(e_i) - g(e')\big) d\Phi(b_{-i}) \\
\leq & \int_{B_{-i}(b')} \big(g(e_i) - g(e')\big) d\Phi(b_{-i}) \\
= & \big(g(e_i) - g(e')\big)(1 - F(b'))^{n-1},
\end{aligned}$$

since $g(\min_{j \neq i}\{s_j(b_j)\}) \leq g(e_i)$ on $\Omega \setminus A(e_i, s_{-i})$.

Part (ii). Assume $e_i < e''$ and $b'' \geq \sup\big(\{b \mid s_j(b) < e'' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$. Since $s_j(\cdot)$ is non-decreasing and continuous for all j, the existence of $k \neq i$ such that $s_k(b_k) < e''$ and $b_k \geq b''$ would imply that b'' is not an upper bound for $\{b \mid s_j(b) < e'' \text{ for all } j \neq i\} \cup \{\underline{b}\}$ and thus contradict that $b'' \geq \sup\big(\{b \mid s_j(b) < e'' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$. Hence, for all $j \neq i$, $b_j \geq b''$ implies $s_j(b_j) \geq e''$; i.e., $B_{-i}(b'') \subseteq A(e'', s_{-i})$. It now follows from the definition of

G that

$$\begin{aligned}
& G(e'', s_{-i}) - G(e_i, s_{-k}) \\
= & \int_{A(e'',s_{-i})} \big(g(e'')) - g(e_i)\big) d\Phi(b_{-i}) \\
& + \int_{A(e_i,s_{-i})\setminus A(e'',s_{-i})} \big(g(\min_{j\neq i}\{s_j(b_j)\}) - g(e_i)\big) d\Phi(b_{-i}) \\
\geq & \int_{A(e',s_{-i})} \big(g(e'') - g(e_i)\big) d\Phi(b_{-i}) \\
\geq & \int_{B_{-i}(b'')} \big(g(e'') - g(e_i)\big) d\Phi(b_{-i}) \\
= & \big(g(e'') - g(e_i)\big)(1 - F(b''))^{n-1},
\end{aligned}$$

since $g(\min_{j\neq i}\{s_j(b_j)\}) \geq g(e_i)$ on $A(e_i, s_{-i})$. ■

Proof of Proposition 4. Assume that s is a Bayesian equilibrium. Since the effort set is $[0,\infty)$, it remains to be shown that for each type b_i of every player i, $s_i(b_i) \leq \bar{s}(b_i)$.

Part 1. First, we show this for $\underline{b}$; i.e., for every player i, $s_i(\underline{b}) \leq \bar{s}(\underline{b})$. Suppose to the contrary that there exists i such that $s_i(\underline{b}) > \bar{s}(\underline{b})$. From Lemma 2 (ii),

$$G(s_i(\underline{b}), s_{-i}) - G(\bar{s}(\underline{b}), s_{-i}) \leq g(s_i(\underline{b})) - g(\bar{s}(\underline{b})).$$

Hence,

$$\begin{aligned}
& u(s_i(\underline{b}), s_{-i}, \underline{b}) - u(\bar{s}(\underline{b}), s_{-i}, \underline{b}) \\
= \;& \underline{b}G(s_i(\underline{b}), s_{-i}) - cs_i(\underline{b}) - [\underline{b}G(\bar{s}(\underline{b}), s_{-i}) - c\bar{s}(\underline{b})] \\
\leq \;& \underline{b}g(s_i(\underline{b})) - cs_i(\underline{b}) - [\underline{b}g(\bar{s}(\underline{b})) - c\bar{s}(\underline{b})] \\
= \;& \underline{b}g(s_i(\underline{b}))(1-F(\underline{b}))^{n-1} - cs_i(\underline{b}) - [\underline{b}g(\bar{s}(\underline{b}))(1-F(\underline{b}))^{n-1} - c\bar{s}(\underline{b})] \; < \; 0\,.
\end{aligned}$$

The second equality follows since $F(\underline{b}) = 0$, while the strict inequality follows since $g(\cdot)$ is strictly concave and $\bar{s}(\underline{b}) = e(\underline{b})$. This contradicts that s_i can be played in a Bayesian equilibrium if $s_i(\underline{b}) > \bar{s}(\underline{b})$.

Part 2. Second, we show this for all types in $(\underline{b}, \bar{b}]$; i.e., for for each type $b_i \in (\underline{b}, \bar{b}]$ of every player i, $s_i(b_i) \leq \bar{s}(b_i)$. Suppose to the contrary that there exists $b' \in \big(\underline{b}, \bar{b}\big]$ and i such that $s_i(b') > \bar{s}(b')$. We divide this part into two cases; one case where there is a unique player k maximizing $s_j(b')$ over all $j \in I$, and another case where there are more than one player maximizing $s_j(b')$ over all $j \in I$.

Case 1: $s_k(b') > \max\{\max_{j\neq k}\{s_j(b')\}, \bar{s}(b')\}$. Choose any e_k satisfying

$$\max\{\max_{j\neq k}\{s_j(b'), \bar{s}(b')\} \; < \; e_k \; < \; s_k(b')\,.$$

Then $b' \leq \sup\{b \mid s_j(b) < e_k \text{ for all } j \neq k\}$, and it follows from Lemma 2 that

$$G(s_k(b'), s_{-k}) - G(e_k, s_{-k}) \;\leq\; \big(g(s_k(b')) - g(e_k)\big)(1 - F(b'))^{n-1}\,.$$

Hence,

$$\begin{aligned} & u(s_k(b'), s_{-k}, b') - u(e_k, s_{-k}, b') \\ = \;& b'G(s_k(b'), s_{-k}) - cs_k(b') - [b'G(e_k, s_{-k}) - ce_k] \\ \leq \;& b'g(s_k(b'))(1 - F(b'))^{n-1} - cs_k(b') - \big[b'g(e_k)(1 - F(b'))^{n-1} - ce_k\big] \;<\; 0\,. \end{aligned}$$

The strict inequality follows since $-cs_k(b') + ce_k < 0$ if $F(b') = 1$, and it follows since $g(\cdot)$ is strictly concave and

$$s_k(b') > e_k > \max\{\max_{j \neq k}\{s_j(b')\}, \bar{s}(b')\} \geq e(b')$$

if $F(b') < 1$. This contradicts that s_k can be played in a Bayesian equilibrium if $s_k(b') > \max\{\max_{j\neq k}\{s_j(b')\}, \bar{s}(b')\}$.

Case 2: $K := \arg\max_{j\in I} s_j(b')$ is not a singleton and $s_i(b') > \bar{s}(b')$ if $i \in K$. It follows from Proposition 3 that, for each $i \in K$, there exists

$$b_i'' := \min\{b_i \mid s_i(b_i) = s_i(b')\}$$

Let $b'' := \min\{b_i'' \mid i \in K\}$. It follows from Case 1 that there exist at least two players $i \in K$ for which $b_i'' = b''$. Let k denote one of these. Note that $s_k(b'') = s_k(b') > \bar{s}(b') \geq \bar{s}(b'') \geq \bar{s}(\underline{b})$. It follows from Part 1 that $b'' > \underline{b}$.

Consider a sequence $\{e^m\}_{m=1}^{\infty}$ such that $\bar{s}(\underline{b}) < e^m < e^{m+1} < s_k(b'')$ for each $m \in \mathbb{N}$ and $e^m \to s_k(b'')$ as $m \to \infty$. Let for each $m \in \mathbb{N}$,

$$b^m := \sup\{b \mid s_j(b) < e^m \text{ for all } j \neq k\}\,;$$

i.e., $b > b^m$ is equivalent to the existence of $j \neq k$ with $s_j(b_j) \geq e^m$ and $b_j < b$. Since $\bar{s}(\underline{b}) \geq s_i(\underline{b})$ for all i (cf. Part 1 of this proof), $s_i(\cdot)$ is continuous (cf. Proposition 3 and the definition of a Bayesian equilibrium), and the fact that $\max_{j\neq k} s_j(b'') = s_k(b'')$, it follows that (i) $\underline{b} < b^m < b''$, and (ii) $b^m \to b''$ as $m \to \infty$.

For each $m \in \mathbb{N}$ it now follows from Lemma 2 that

$$G(s_k(b''), s_{-k}) - G(e^m, s_{-k}) \;\leq\; \big(g(s_k(b'')) - g(e^m)\big)(1 - F(b^m))^{n-1}\,.$$

Hence,

$$\begin{aligned}
& u(s_k(b''), s_{-k}, b'') - u(e^m, s_{-k}, b'') \\
= \; & b''G(s_k(b''), s_{-k}) - cs_k(b'') - [b''G(e^m, s_{-k}) - ce^m] \\
\leq \; & b''g(s_k(b''))(1 - F(b^m))^{n-1} - cs_k(b'') - \big[b''g(e^m)(1 - F(b^m))^{n-1} - ce^m\big].
\end{aligned}$$

To show that this difference is negative for large m, note first that if $b'' > \sup\{b \mid F(b) < 1\}$, then there exists $M \in \mathbb{N}$ such that $F(e^M) = 1$ and $u(s_k(b''), s_{-k}, b'') - u(e^m, s_{-k}, b'') \leq -cs_k(b'') + ce^M < 0$. Otherwise, $F(e^m) < 1$ for all $m \in \mathbb{N}$, and we can let, for each $m \in \mathbb{N}$, $e^*(b^m)$ be defined by

$$e^*(b^m) := \arg\max_e b''g(e)(1 - F(b^m))^{n-1} - ce\,.$$

By the assumptions on $g(\cdot)$ it follows that, for each $m \in \mathbb{N}$, $e^*(b^m)$ is uniquely determined by $b''g'(e^*(b^m))(1 - F(b^m))^{n-1} = c$. Since F is absolutely continuous, we have from the strict concavity of $g(\cdot)$ and the definition of $e(\cdot)$ that $e^*(b^m) \to e(b'')$ as $m \to \infty$. Hence, $s_k(b'') > e^M > e^*(b^M) > e(b'')$ for sufficiently large $M \in \mathbb{N}$, since $s_k(b'') > \bar{s}(b'') \geq e(b'')$ and $e^m \to s_k(b'')$ as $m \to \infty$. Therefore,

$$\begin{aligned}
& u(s_k(b''), s_{-k}, b'') - u(e^M, s_{-k}, b'') \\
\leq \; & b''g(s_k(b''))(1 - F(b^M))^{n-1} - cs_k(b'') - \big[b''g(e^M)(1 - F(b^M))^{n-1} - ce^M\big] \; < \; 0
\end{aligned}$$

by the definition of $e^*(b^M)$ and the strict concavity of $g(\cdot)$. This contradicts that s_k can be played in a Bayesian equilibrium if $K := \arg\max_{j \in I} s_j(b')$ is not a singleton and $s_i(b') > \bar{s}(b')$ if $i \in K$. ■

Proof of Proposition 5. *Part (i).* Assume that $s_j = \underline{s}$ for every $j \neq i$. Then $G(e, s_{-i}) = 0$ for all $e \geq 0$, which clearly implies that, for all $b_i \in [\underline{b}, \bar{b}]$, $u(e, s_{-i}, b_i)$ is decreasing in e for all $e \geq 0$, establishing the result by Proposition 4.

Part (ii). Assume that $s_j = \bar{s}$ for every $j \neq i$. By Proposition 4 it is sufficient to show that, for all $b_i \in [\underline{b}, \bar{b}]$, $u(e', s_{-i}, b_i) < u(e'', s_{-i}, b_i)$ if $e' < e'' \leq \bar{s}(b_i)$.

Since F is absolutely continuous, the properties of $g(\cdot)$ and the definition of $e(\cdot)$ entail that (a) $e(\cdot)$ is continuous and (b) $e(b_i) \to 0$ as $b_i \uparrow \sup\{b \mid F(b) < 1\}$. The definition of $\bar{s}(\cdot)$ now implies that, for each $b_i \in [\underline{b}, \bar{b}]$, there exists b'' satisfying $\underline{b} \leq b'' \leq b_i$ and $F(b'') < 1$ such that $e(b'') = \bar{s}(b'') = \bar{s}(b_i)$. Hence, since $\bar{s}(\cdot)$ is non-decreasing and $s_j = \bar{s}$ for every $j \neq i$, we have that $b'' \geq \sup\big(\{b \mid s_j(b) < e'' \text{ for all } j \neq i\} \cup \{\underline{b}\}\big)$ if $e'' \leq \bar{s}(b_i)$. Hence, if

$e' < e'' \leq \bar{s}(b_i)$, Lemma 2 implies that

$$G(e'', s_{-i}) - G(e', s_{-i}) \;\geq\; \big(g(e'') - g(e')\big)(1 - F(b''))^{n-1} \;>\; 0,$$

where the strict inequality follows since $g(\cdot)$ is increasing and $F(b'') < 1$. By the definition of $e(\cdot)$ and the strict concavity of $g(\cdot)$,

$$\begin{aligned} & b''G(e'', s_{-i}) - ce'' - \big[b''G(e's_{-i}) - ce'\big] \\ \geq\;\; & b''g(e'')(1 - F(b''))^{n-1} - ce'' - \big[b''g(e')(1 - F(b''))^{n-1} - ce'\big] \;>\; 0\,. \end{aligned}$$

Since $b_i \geq b''$ and $G(e'', s_{-i}) > G(e's_{-i})$, this implies that

$$\begin{aligned} & u(e'', s_{-i}, b_i) - u(e', s_{-i}, b_i) \\ =\;\; & b_iG(e'', s_{-i}) - ce'' - \big[b_iG(e's_{-i}) - ce'\big] \\ \geq\;\; & b''G(e'', s_{-i}) - ce'' - \big[b''G(e's_{-i}) - ce'\big] \;>\; 0 \end{aligned}$$

which establishes that $u(e', s_{-i}, b_i) < u(e'', s_{-i}, b_i)$ if $e' < e'' \leq \bar{s}(b_i)$.

Part (iii). We have that, for each type b_i of every player i,

$$0 = u\big(\underline{s}(b_i), \underbrace{(\underline{s}, \ldots, \underline{s})}_{n-1 \text{ times}}, b_i\big) \leq u\big(s_i(b_i), s_{-i}, b_i\big)$$

since, for each b_i, $u(0, s_{-i}, b_i) = 0$, independently of s_{-i}. Hence, each type b_i of player i can always ensure himself a non-negative payoff by setting $e_i = 0$. To show that, for each type b_i of every player i,

$$u\big(s_i(b_i), s_{-i}, b_i\big) \leq u\big(\bar{s}(b_i), \underbrace{(\bar{s}, \ldots, \bar{s})}_{n-1 \text{ times}}, b_i\big)\,,$$

note that, for each b_i, the definition of u and Proposition 4 imply that $u(s_i(b_i), s_{-i}, b_i)$ is maximized for fixed s_i over the set of opponent Bayesian equilibrium strategies by setting $s_j = \bar{s}$ for all $j \neq i$. Moreover, given $s_j = \bar{s}$ for all $j \neq i$, it follows from part (ii) that, for each b_i, $u(e_i, s_j, b_i)$ is maximized by setting $e_i = \bar{s}(b_i)$.

Part (iv). For all $e \in [0, \bar{s}(\bar{b})]$, let s^e be given by

$$s^e(b_i) := \min\{\bar{s}(b_i), e\}\,.$$

for all $b_i \in [\underline{b}, \bar{b}\,]$. Then, for any $e \in [0, \bar{s}(\bar{b})]$, it follows from part (i) that $u(e', s_{-i}, b_i)$ with $s_j = s^e$ for all $j \neq i$ reaches a local maximum on $[0, e]$ at $s^e(b_i)$ and is decreasing in e' for all $e' \geq e$. Hence, $(s_1, \ldots, s_n)$ with $s_i = s^e$ for all $i \in I$ is a symmetric Bayesian equilibrium.

Furthermore, for all b_i, $u(s^e, s_{-i}, b_i)$ with $s_j = s^e$ for all $j \neq i$ is a continuous function of e, with

$$0 \;=\; u\big(\underline{s}(b_i), \underbrace{(\underline{s}, \ldots, \underline{s})}_{n-1 \text{ times}}, b_i\big) \;=\; u(s^0(b_i), \underbrace{(s^0, \ldots, s^0)}_{n-1 \text{ times}}, b_i)\,,$$

$$u(s^{\bar{s}(\bar{b})}, \underbrace{(s^{\bar{s}(\bar{b})}, \ldots, s^{\bar{s}(\bar{b})})}_{n-1 \text{ times}}, b_i) \;=\; u\big(\bar{s}(b_i), \underbrace{(\bar{s}, \ldots, \bar{s})}_{n-1 \text{ times}}, b_i\big)\,.$$

This establishes part (iv). ■

Proof of Proposition 6. For each $m \in \mathbb{N}$, it follows from the definition of $e^m : \{b \in B_m \mid F_m(b) < 1\} \to [0, \infty)$ that

$$e^m(\underline{b}^m) \;=\; \bar{e}(\underline{b}^m) \tag{A3}$$

$$e^m(b) \;\leq\; \bar{e}(b) \;\leq\; \bar{e}(\bar{b}^m) \text{ for all } b \in B_m \text{ such that } F_m(b) < 1\,, \tag{A4}$$

keeping in mind that $(1 - F_m(b))^{n-1} \leq 1$ for all $b \in B_m$ and $\bar{e}(\cdot)$ is an increasing function. It follows from (A3) and (A4) and the definition of $\bar{s}^m : B_m \to [0, \infty)$ that

$$\bar{e}(\underline{b}^m) \;\leq\; \bar{s}^m(b_i) \;\leq\; \bar{e}(\bar{b}^m) \text{ for all}\, b_i \in B_m\,. \tag{A5}$$

Since $\bar{e}(\cdot)$ is a continuous function and $\lim_{m\to\infty}\underline{b}^m = b = \lim_{m\to\infty}\bar{b}^m$, we have that

$$\lim_{m\to\infty}\bar{e}(\underline{b}^m) \;=\; \bar{e}(b) \;=\; \lim_{m\to\infty}\bar{e}(\bar{b}^m)\,,$$

which in combination with (A5) entails that

$$\lim_{m\to\infty}\bar{s}^m(\underline{b}^m) \;=\; \bar{e}(b) \;=\; \lim_{m\to\infty}\bar{s}^m(\bar{b}^m)\,.$$

The result now follows from the properties of the payoff functions u_m, $m \in \mathbb{N}$, and the CDFs Φ_m, $m \in \mathbb{N}$. ■

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