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Farrell Revisited: Visualising the DEA Production Frontier

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FARRELL REVISITED:
VISUALISING THE DEA PRODUCTION FRONTIER*

by

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Abstract: The contributions of the paper are threefold: i) compare with mathematical rigour the Charnes, Cooper, and Rhodes DEA model and the Farrell model exhibiting constant returns to scale, ii) reinterpret the contribution of Farrell and Fieldhouse that extended the analysis to variables returns to scale and establish the connection with the approach in Banker, Charnes and Cooper, iii) provide graphical visualisation of properties of the frontier function. Both papers by Farrell emphasised the importance of graphical visualisation of non-parametric frontier functions, but, to our knowledge, this is seldom followed up in the literature. We use a graphical package (EffiVision) with a numerical representation of the frontier functions, representing the contemporary development of visualisation. By making suitable cuts through the DEA frontier in multidimensional space, various graphical representations of features of economic interest can be done. Development of ray average cost function and scale elasticity are novel illustrations.

Key words: Farrell efficiency measure, DEA, variable returns to scale, visualisation, grouping method

JEL classification: C61, D24

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1. Introduction

It is now the 50th anniversary of the seminal paper by Farrell (1957), which established definitions of efficiency measures and introduced methods of estimating efficiency scores in the case of a non-parametric frontier exhibiting constant returns to scale (CRS). This paper is by now well known and diffused within a wide range of research environments (Førsund and Sarafoglou, (2002), 2005). The number of citations in social science journals (ISI Web of Knowledge) is over 1400 (July 2007). However, the following-up paper, Farrell and Fieldhouse (1962) is not so well known, having 95 citations in social science journals.

In the 1962 paper linear programming (LP) was used to find the frontier and estimate efficiency scores, as suggested by Hoffman (1957) in the discussion of Farrell's 1957 paper, but applied only to the single output case. Attempts to formulate the CRS problem for the multiple output case was described theoretically in the 1957 paper for the case of solving a system of linear equations, and for the LP case in the 1962 paper. However, we will show that the procedure set up in the 1957 paper was not quite correct, and that the approach described in the 1962 paper, as a generalisation to multiple outputs, was somewhat awkwardly called a CRS model, but was in fact a model of variable returns to scale. The formal LP problem with multiple outputs identical to the Charnes *et al*. (1978) CRS model was indicated in Boles (1967) and set out in Boles (1971), including a Fortran computer program.

Farrell (1957) was concerned about making the results easy understandable by use of various graphical representations. He made use of (rather simple) histograms of the distribution of efficiency scores and exhibited figures of isoquants for various two-dimensional combinations in his attempt also to say something "about the shape of the production function itself" (Farrell and Fieldhouse (1962), p. 256).

Although CRS was assumed in Farrell (1957) for the empirical application, he also discussed how to deal with increasing and diminishing returns to scale (Section 2.4, pp. 258-259). He suggested that the case of diseconomies of scale could be dealt with by what he called the

Grouping method. In the single output case, observations were sorted according to output size and put into similar size groups, and then the model with a CRS specification was applied separately for each group. Farrell and Fieldhouse (1962) followed up this idea empirically using agricultural data for England deemed suitable for the approach.

As for the theoretical discussion of variable returns to scale (VRS), attention was restricted to cases where increasing returns to scale is followed, either by constant returns or by decreasing returns, but never decreasing returns followed by increasing returns. This assumption implies that the average cost function is U-shaped although the production function does not have to be concave. Ray average cost functions were drawn for observations, using a series of isoquants of the efficient VRS production function in the single output case as reference for efficiency score calculations, keeping the observed input ratios. Such efficiency scores are proportional to average cost since the input prices are assumed to be equal for every unit. Notice that we do not have to know the input prices. The efficiency scores were then plotted against the output level along the ray in question.

In Section 2 we give a thorough review of Farrell's estimation methods for the non-parametric case, showing the connection between the approach in the single output case and the Charnes *et al*. (1978) (CCR) approach, and we point out the deficiency of the solution method proposed in the 1957 paper for the multi-output case. In Section 3 the treatment of variable returns to scale (VRS) is discussed. The difference between the VRS approach in Farrell and Fieldhouse (1962) and the approach in Banker *et al*. (1984), which is a classical paper with 981 citations in the ISI social citation database (July 2007), is demonstrated. In Section 4 the data in Farrell and Fieldhouse (1962) is used to exhibit the ray average cost function graphically, employing the software package EffiVision (Krivonozhko *et al*., 2004). The package allows for various cuts to be made through the frontier function. Extending the Farrell approach of providing graphical insights into properties of the frontier function, the scale elasticity properties are exhibited by cutting the frontier with a two-dimensional plane, and calculating right-hand and left-hand scale elasticities for corners of ray frontier functions, following Førsund *et al*. (2007). Scale elasticity curves are displayed, exposing the special nature of the development of the scale elasticity along a factor ray in the case of a DEA frontier. Conclusions and ideas for further research are offered in Section 5.

2. The Farrell legacy

Efficiency concepts and frontier estimation method

In the case of constant returns to scale (CRS) and a single output, Farrell (1957) illustrated his definition of three efficiency measures; technical efficiency, price (allocative) efficiency and overall (cost) efficiency, by using a unit isoquant portrayed in the input coefficient space. The original illustration is shown in Diagram 1 in Figure 1. This illustration is widely reproduced, probably because it gives an immediate intuition about the nature of the efficiency measures. Single output and constant returns to scale (CRS) are assumed, thus the unit isoquant in input coefficient space contains all information about the frontier function. The three efficiency measures for an inefficient observation P are

i) *Technical efficiency*, defined as inputs needed at best practice to produce observed outputs relative to observed input quantities, keeping observed input ratios; OQ/OP.

ii) *Price efficiency* (also called allocative efficiency), defined as minimum costs of producing observed output at observed factor prices relative to costs assuming technical efficiency and observed input ratios; OR/OQ.

iii) *Overall efficiency*, defined as costs of producing observed output, assuming both technical efficiency and price efficiency, relative to observed costs; OR/OP = (OQ/OP)(OR/OQ).

Farrell showed how the unit isoquant could be estimated when specifying a CRS non-parametric piecewise linear frontier function by enveloping the best practice units with a convex isoquant (Diagram 2 in Figure 1). Hoffman (1957) suggested in the discussion of Farrell's paper that the newly developed linear programming (LP) could be used to solve Farrell's model. This technique was adopted in Farrell and Fieldhouse (1962).[1]

[1] Fieldhouse was a PhD student (not Farrell's) at Cambridge University and was the person at that time that could run a LP program on the EDSAC II computer (private communication from Martin Fieldhouse).

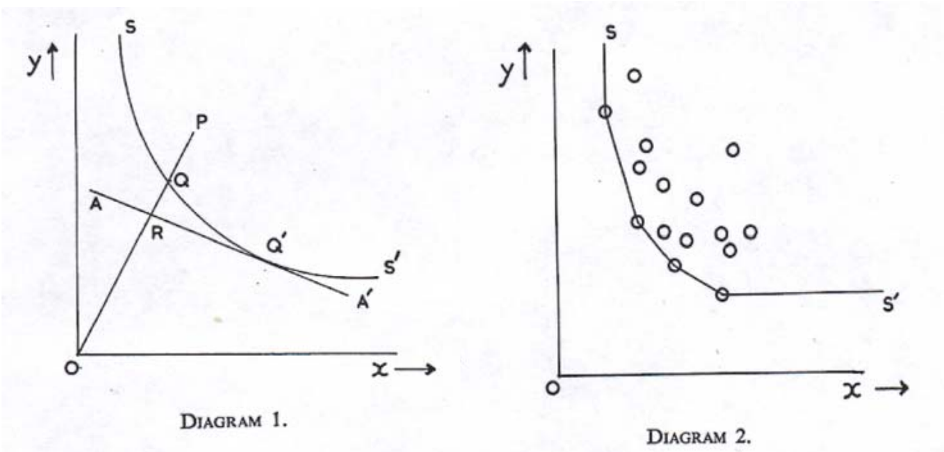


Figure1. Farrell efficiency measures and frontier estimation

The basic Farrell efficiency model

How the model used by Farrell compares with the now perceived DEA model of Charnes *et al.* (1978) may not be so well understood. We will therefore examine his model in detail and expose the connection with contemporary DEA models.

Farrell (1957) and Farrell and Fieldhouse (1962) considered the following basic model:

$$I_1 = \max \{\lambda_1 + ... + \lambda_{n+m}\}$$
subject to
$$\sum_{j=1}^{n+m} P_j \lambda_j = P_k \qquad (1)$$
$$\lambda_j \geq 0 \, , j = 1,...,n+m$$

where $P_j \in E^m (j = 1,...,n)$ are observations of actual production units (the nature of the observations will become clear below), and

$$P_{n+i} = (0,...,\infty,...,0) \, , i = 1,...,m \qquad (2)$$

are artificial observations where infinity appears in the ith position and there are zeros elsewhere. P_k belongs to the feasible set of (1), and the technical efficiency of P_k is the inverse of the optimal value, $1/I_1$.

All models in Farrell (1957) and Farrell and Fieldhouse (1962) were reduced to model (1). We will establish how the model (1) is associated with present-day DEA models. For this purpose, we consider the following optimisation model:

$$I_2 = \min\ \theta$$
subject to
$$\sum_{j=1}^{n+m} P_j \lambda_j = \theta P_k \qquad (3)$$
$$\sum_{j=1}^{n+m} \lambda_j = 1$$
$$\lambda_j \geq 0,\ j = 1,\ldots,n+m$$

The model (3) is closer to the contemporary DEA form; however, it is not yet a DEA model.

Theorem 1. Suppose the origin does not belong to the convex hull of all P_j , $j = 1,\ldots,n + m$. Then problems (1) and (3) are equivalent; this means that the optimum of the value functions are inversely related, $I_2 = 1/I_1$, and optimal solutions sets of both problems coincide.

Proof. Suppose that we obtain an optimal solution of problem (1). Let J_B be the index set of optimal basic variables, and let $\lambda_j^*, j \in J_B$, be the optimal variables. Optimisation theory (Dantzig, 1998) ensures that in this optimal solution vector not more than m of the λ_j^* will be strictly positive and the rest will be zero. Suppose also that

$$\sum_{j \in J_B} \lambda_j^* \neq 1 \qquad (4)$$

Let us consider the hyperplane H going through vectors $P_j, j \in J_B$, which form the optimal basis. Such basis always exists and contains precisely m vectors. Farrell introduced artificial points in problem (1) to ensure the existence of feasible bases. Observe that hyperplane H does not contain the origin, since otherwise vectors $P_j, j \in J_B$, would be linearly dependent. Then, according to convex analysis (Nikaido, 1968), line OP_k contains point $P_k' \in H$ that is represented in the form

$$P_k = \eta_1 P_k' = \eta_1 \sum_{j \in J_B} \lambda_j' P_j ,$$
$$\sum_{j \in J_B} \lambda_j' P_j = P_k' , \sum_{j \in J_B} \lambda_j' = 1 \tag{5}$$

From (1) and (5) it follows that

$$I_1 = \sum_{j \in J_B} \lambda_j^* = \eta_1 \sum_{j \in J_B} \lambda_j' = \eta_1 \tag{6}$$

So, we associate the value function (1) with the distance along line OP_k' from point P_k to point P_k' that belongs to the convex combination of vectors $P_j, j \in J_B$.

It can be shown that

a) if $\eta_1 < 1$, then point P_k and the origin are on the same side of hyperplane *H*,

b) if $\eta_1 > 1$, then hyperplane *H* separates the origin and the point P_k,

c) if $\eta_1 = 1$, then points P_k' and P_k coincide.

Hence, the value function of problem (1) reaches at least the value η_1. However, suppose for the moment that the vectors $P_j, j \in J_B$ do not belong to some facet Γ. Then the value function can be improved. Indeed, consider the segment OP_k'. Point P_k' is an interior point of the feasible set (1). So, according to convex analysis (Nikaido, 1968), there exists a unique point P_k'' on the segment which belongs to some facet of the polyhedron (1). Repeating arguments of the previous case, we can write

$$P_k = \eta_1 P_k' = \eta_1 \eta_2 P_k'', \ \eta_2 > 1$$
$$\sum_{j = J_1} \lambda_j'' P_j = P_k'', \ \sum_{j = J_1} \lambda_j'' = 1 \tag{7}$$

where $P_j, j \in J_1$ are vertices of the feasible set (1) which determines this facet. Hence, from (5) and (7), it follows that the value function is equal to

$$I_1 = \eta_1 \eta_2 \sum_{j=J_1} \lambda_j'' = \eta_1 \eta_2 > \eta_1 \tag{8}$$

Relation (8) contradicts the assumption that vectors $P_j, j \in J_B$ do not belong to some facet Γ that separates the origin and point P_k. Hence, vectors $P_j, j \in J_B$ belong to some facet of the polyhedron and we can write

$$P_k' = \frac{1}{\eta_1^*} P_k = \theta^* P_k, \ \theta^* < 1, \tag{9}$$

where η_1^*, θ^* are maximal and minimal values of η_1 and θ, respectively, that allow point P_k' to belong to the feasible set of (1).

If we assume equality in relation (4), this means that point P_k is equal to P_k' and we could start our consideration immediately from point P_k''.

Thus, we have shown that maximisation of the value function of problem (1) means minimisation of the length of the segment OP_k' defined by (9) while point P_k' still belongs to the feasible set of problem (1). According to the theory of convex analysis (Nikaido, 1968) such a point P_k' is unique.

Conversely, suppose that λ^* is an optimal solution of problem (3). Minimisation of the length of segment OP_k', where $P_k' = \theta P_k$ while θP_k still belongs to the feasible set of problem (3), means according to (9) maximisation of η_1. Next, from (5) and (6) it follows that the optimal solution λ^* of problem (1) is also obtained. This completes the proof. □

Notice that, though the point P_k' is unique, solution vectors in the variables $\lambda_j^*, j \in J_B$, may not be unique, they may constitute a whole set of optimal solutions.

Comparing the basic Farrell model and the CCR DEA model

Model (3) looks like a DEA model; however, it is not still a DEA model. Next, we show that model (3) is equivalent to the Charnes *et al*. (1978) (CCR) model with one output.

In the single output CRS case, Farrell and Fieldhouse (1962) considered the input coefficient vectors $X_j{}' = (X_j / y_j)$, where input observations $X_j \in E^m, j = 1,...,n,$ and y_j is a scalar output (P_j in (1) is here termed $X_j{}'$ to facilitate comparison with CCR). So, we can write this model in Farrell's basic form (1)

$$\begin{aligned} & I_3 = \max \sum_{j=1}^{n+m} \lambda_j' \\ & \text{subject to} \\ & \sum_{j=1}^{n+m} X_j' \lambda_j' = X_k' \\ & \lambda_j' \geq 0, j = 1,...,n+m, \end{aligned} \tag{10}$$

where the "weight" variables λ_j in (1) are written λ_j' to correspond with variables X_j'. According to Theorem 1, rewrite problem (10) in an equivalent form

$$\begin{aligned} & I_4 = \min\ \theta \\ & \text{subject to} \\ & \sum_{j=1}^{n+m} X_j{}' \lambda_j{}' = \theta X_k{}' \\ & \sum_{j=1}^{n+m} \lambda_j{}' = 1 \\ & \lambda_j' \geq 0, j = 1,...,n+m \end{aligned} \tag{11}$$

Now, consider the CCR model with the same input observations $X_j, j = 1,...,n$ and one scalar output y_j :

$I_5 = \min\,\theta$

subject to

$$\sum_{j=1}^{n} X_j \lambda_j \le \theta X_k \tag{12}$$

$$\sum_{j=1}^{n} y_j \lambda_j \ge y_k,$$

$$\lambda_j \ge 0,\, j = 1,\ldots,n$$

In order to prove the equivalence of problem (11) and (12), we may transform problem (12). First, divide the relation associated with outputs by y_k. This allows us to get unity on the right-hand side of the second inequality in problem (12). Second, divide every column of (12) associated with λ_j- variables by y_j / y_k. As a result we obtain the sum of λ_j on the left-hand side of the second inequality. Third, divide every row of the first inequality of (12) associated with inputs by y_k. Then the input coefficients (X_j/y_j) appear on the left-hand side and (X_k/y_k) on the right-hand side. Furthermore, write the two first inequalities as equalities by introducing slack variables. As a result, we obtain

$I_5 = \min\,\theta$

subject to

$$\sum_{j=1}^{n} X_j' \lambda_j' + S = \theta X_k' \tag{13}$$

$$\sum_{j=1}^{n} \lambda_j' - s_o = 1,$$

$$\lambda_j' \ge 0,\, j = 1,\ldots,n,\;\; s_i \ge 0,\, i = 1,\ldots,m,$$

where $S = (s_1\ ,\ldots,s_m\ )$ is a vector of slack variables, s_o is a slack variable associated with the output inequality, and

$$X_j' = X_j / y_j\,,\; j = 1,\ldots,n,\;\; X_k' = X_k / y_k\,, \qquad \lambda_j' = \lambda_j y_j\,,\; j = 1,\ldots,n \tag{14}$$

It is worth emphasising that all those algebraic transformations do not change the equivalence of problems (12) and (13). This means that the value functions of both problems are equal, variables λ_j and $\lambda_j'\,, j = 1,\ldots,n$ are transformed in accordance with (14), and variable θ does not change. Observe that in the optimal solution of (13) the slack variable $s_o^* = 0$. This follows from the proof of Theorem 1. (A value greater than one of the sum of λ_j' cannot be part of an optimal solution.)

Taking into account all that is written above about the equivalence of problems (12) and (13) we can formulate the following theorem:

Theorem 2. Problem (12) is equivalent to problem (13), this means that the value functions of both problems are equal, the variable θ does not change, and variables λ_j and λ_j', $j = 1,...,n$ are related in accordance with (14).

Now, we can compare model (11) to the CCR model (12) using model (13). The only difference is that artificial vertices (2) with corresponding variables λ'_{n+i}, $i = 1,...,m$ are introduced in model (11), and slack variables $s_i \ (i = 1,...,m)$, s_o are inserted in model (13). The feasible set of (11) is a polyhedron with *m* vertices at infinity. This enabled Farrell (1957) to avoid difficulties associated with explaining weakly Pareto-efficient points.

The feasible set in problem (13) is a polyhedral set, so in problem (13) weakly Pareto-efficient units may appear (Krivonozhko *et al*., 2005). Moreover, problem (13) has to be solved in two stages in order to separate efficient and weakly efficient points. However, the form of model (12) is more universal, and this enabled researches to develop a large family of DEA models.

These differences of models (11) and (13) are more of a theoretical nature than practical ones. From a computational point of view both models are almost equivalent. Farrell (1957) had to introduce artificial units

$$\tilde{P}_{n+i} = (0,...,M,...,0), \ i = 1,...,m \tag{15}$$

with large *M* instead of units (2) at infinity in order to solve his model practically. This situation is very similar to introducing an infinitesimal (non-Archimedean) constant, ε, in DEA models (Cooper *et al*., 2000).

We checked our hypothesis by repeating the computations in Farrell and Fieldhouse (1962) on their data, which had four inputs and a single output, both using their own model (1) and the CCR model (12). All efficiency scores coincided completely.

On generalising the basic Farrell model to multiple outputs

Farrell (1957) wanted to generalise model (1) to the case of multi outputs. A model of the following type was considered:

$$\max \sum_{j=1}^{n+m+r} \lambda_j$$

subject to

$$\sum_{j=1}^{n+m+r} X_j \lambda_j = X_k \tag{16}$$

$$\sum_{j=1}^{n+m+r} Y_j \lambda_j = (\sum_{j=1}^{n+m+r} \lambda_j) Y_k$$

$$\lambda_j \geq 0, j = 1,\ldots,n+m+r$$

where $X_j \in E^m$, $j = 1,\ldots,n$ are input vectors, $Y_j \in E^r$, $j = 1,\ldots,n$ are output vectors of observations of actual production units and (X_i, Y_i), $i = n+1,\ldots,n+m+r$ are artificial units at infinity introduced as in (1) and (2). The complete optimisation problem was not written in Farrell, relation (5) on p. 257 holds only for some facet.

However, model (16) cannot be considered as a generalisation of (1). Consider the following two-dimensional example with two units,

$$P_1 = (X_1, Y_1) = (4,1),\ P_2 = (X_2, Y_2) = (3,3) \ .$$

Then problem (16) reduces to:

$$\max\{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\}$$

subject to

$$4\lambda_1 + 3\lambda_2 + M\lambda_3 = 4$$

$$\lambda_1 + 3\lambda_2 + M\lambda_4 = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$$

$$\lambda_j \geq 0,\ j = 1,\ldots,4$$

where M is a very large number, P_3 = (M, 0) and P_4 = (0, M) are artificial units, and unit (X_1, Y_1) is under investigation (k = 1). The solution of this problem is $\lambda_1 = 1,\ \lambda_2, \lambda_3, \lambda_4 = 0$.

Hence, according to Farrell (1957), the efficiency score is $\theta = 1/\sum_{j=1}^{4} \lambda_j = 1$. However, point P_1 is not efficient as this observation is dominated in both the input and the output directions by P_2. Point θP_1 does not reach the frontier.

We think that Farrell understood that model (16) was not quite correct, since he did not consider this model in his second paper.

3. The treatment of variable returns to scale

The grouping method

Farrell (1957) wanted to generalise the CRS assumption and discussed the case of economies and diseconomies of scale. He suggested a method of grouping the data according to size of output. This *grouping method* was applied in Farrell and Fieldhouse (1962). The grouping method was based on calculating constant returns to scale frontiers within a limited number of size groups. To our knowledge this method has not been followed up in the literature. It certainly relies on adequate data sets.

Applying the grouping method to farm survey data for England and Wales 1952 – 1953 with a single output, gross sales (value), and four inputs, land (adjusted acres), fodder (purchase), labour (man-year equivalents) and capital (value), Farrell and Fieldhouse (1962) divided all observations (already aggregated from 2363 farms to 208 groups based on labour and land) into ten groups according to output, and then estimated the efficient production function and efficiency scores for units within each group separately. According to Farrell and Fieldhouse model (10) was used for the computations. Data used for calculations are given in Table 1 (pp. 254-255) of that article, and efficiency scores for every unit in every group are shown in their Table 2 (pp. 260-261). We used the same data and solved separately ten CCR models of the type (12). Then we compared efficiency scores obtained for every unit. All efficiency scores coincided completely.

In our opinion, Farrell and Fieldhouse (1962) tried to create a variable returns to scale model. For this very reason ten constant returns to scale models were solved, i.e. for every group its own production frontier were determined. We illustrate our assumption by Figure 2. Three groups of observations are shown, sorted according to the value of the output along the vertical axis, and the estimated CRS frontier F_i (i =1,2,3) associated with each group. The broken line indicates the use of this grouping method to infer scale properties.

The overall method

The more general method introduced in Farrell and Fieldhouse (1962) termed the *Overall method* was first set up as linear programming problems based on the unit isoquant concept since only a single output was specified. Farrell and Fieldhouse at first looked at observations $P_j = (X_j, y_j)$ where input vectors are $X_j \in E^m, j = 1,...,n$ and y_j is an output scalar quantity. Then these observations were transformed to

$$P_j' = (X_j / y_j, y_j) = (X_j', y_j), \;\; j = 1,...,n \tag{17}$$

in order to determine a production function with a convex production possibility set.

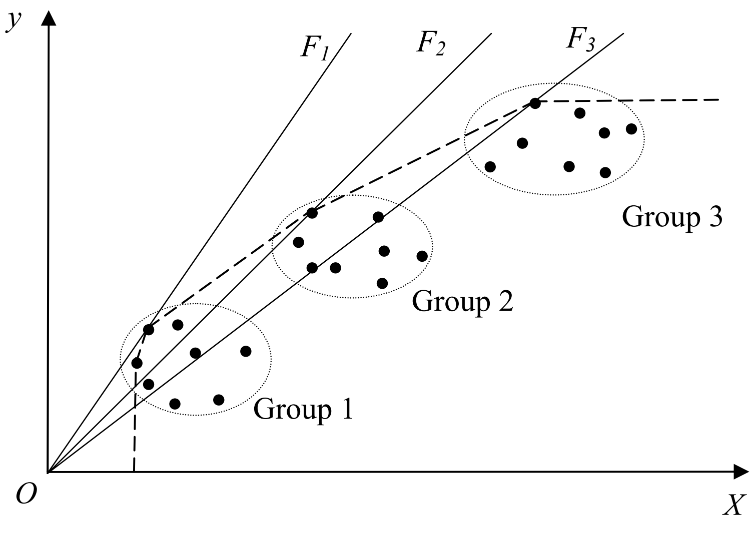


Figure 2. The grouping method:
CRS production functions for different size groups

The production possibility set for this model is a convex combination of all production units (17) (see Farrell and Fieldhouse (1962), Figure 3, p. 259). For every unit P_k', a projection P_1 on the frontier is searched along line $O'P_k'$ that is parallel to the axis OX or, in other words, the minimal possible value of input per unit of output is found at a given output y_k. The efficiency score for unit P_k' is determined as $\theta = O'P_1 / O'P_k'$.

Again, this model was reduced to the basic form (1). It can be seen from the proof of Proposition 1 that the search of minimal point is accomplished on the line going through the origin of the coordinates. It is important for the basic model (1). For this reason the axis OX' was moved to the position $O'P_k'$. This corresponds to the following transformation of observations (17)

$$P_j'' = (X_j', y_j - y_k),\ j = 1,...,n \tag{18}$$

Such type of transformation should be done for every unit P_k' under investigation. An illustration is provided in Figure 3 (Farrell and Fieldhouse (1962), p. 259), adding the calculation of the efficiency score for unit P_k' to the original figure and changing the notation for inputs and output.

Notice that Farrell and Fieldhouse (1962) measured input per unit output along the vertical axis and that the horizontal axis corresponds to output. In our paper we stick to the standard designations of axes of input- and output variables used in the DEA framework. Farrell and Fieldhouse (1962) gave a very detail description of this model. From this description, it follows

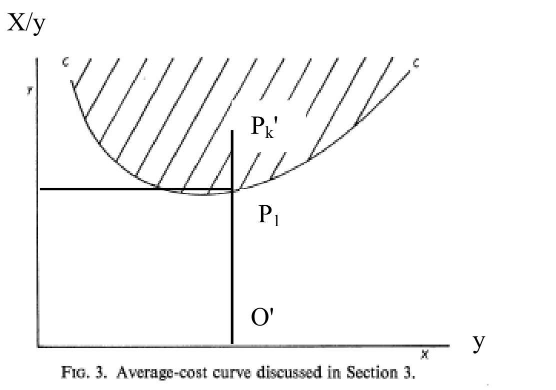


Figure 3. Ray-average cost curve.

immediately that this model can be written in the following present-day formulation

$$\min \theta$$

subject to

$$\sum_{j=1}^{n} X_j{}'\lambda_j = \theta X_k{}' \tag{19}$$

$$\sum_{j=1}^{n} y_j\lambda_j = y_k,$$

$$\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0,\ j = 1,...,n$$

Again, we checked our assumption by repeating computations which were done in Farrell and Fieldhouse using the overall method on data given in Table 1 (pp. 254-255). However, we used model (19) in our computations. In the overall method Farrell and Fieldhouse did not divide units into groups, but included all observations in one model. Our computations of efficiency scores according to model (19) coincide completely with the results obtained in Table 2 (pp. 260-261), using the basic model (1) with observation P_j defined in (17).

Generalising to multiple outputs

Farrell and Fieldhouse indicated how the overall method can be modified to calculate efficiency scores in the case of multiple outputs. In this case every input vector was not divided by some output, since output is also a vector. According to the description in Farrell and Fieldhouse we can rewrite this case in the contemporary formulation

$$\min \theta$$

subject to

$$\sum_{j=1}^{n} X_j\lambda_j = \theta X_k \tag{20}$$

$$\sum_{j=1}^{n} Y_j\lambda_j = Y_k$$

$$\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0,\ j = 1,...,n$$

where $X_j \in E^m, j = 1,...,n$ are vectors of inputs, and $Y_j \in E^r, j = 1,...,n$ are vectors of outputs. Notice that observations in (20) are taken in their original form without any transformation.

The problems (19) and (20) are similar to the model in Banker *et al*. (1984) (BCC). The difference is that in these problems only convex combinations are taken, while in the BCC model the inequalities used instead of equalities ensures that a free disposal hull of real observations is considered.

Figure 4 clarifies the difference. The efficiency score for the Farrell and Fieldhouse model is determined as $\theta_1 = O'P_1 / O'P_k$ with reference to the frontier of the polyhedron, part of which is indicated by solid lines. For the BCC model, however, it is calculated as $\theta_2 = O'P_2 / O'P_k$ with reference to the frontier of the polyhedral set. The vertical broken curve at the start and the horizontal broken curve at the end, outline, together with the solid lines in-between, the production possibility set of the BCC model. Observe also that in problems (19) and (20) artificial or slack variables are not used, since projections of actual production units do not go beyond the convex combinations of actual observations.

It should be added that the three postulates introduced by BCC; convexity, free disposability, and tightness of envelopment, leading to the shape of the estimated production possibility set as

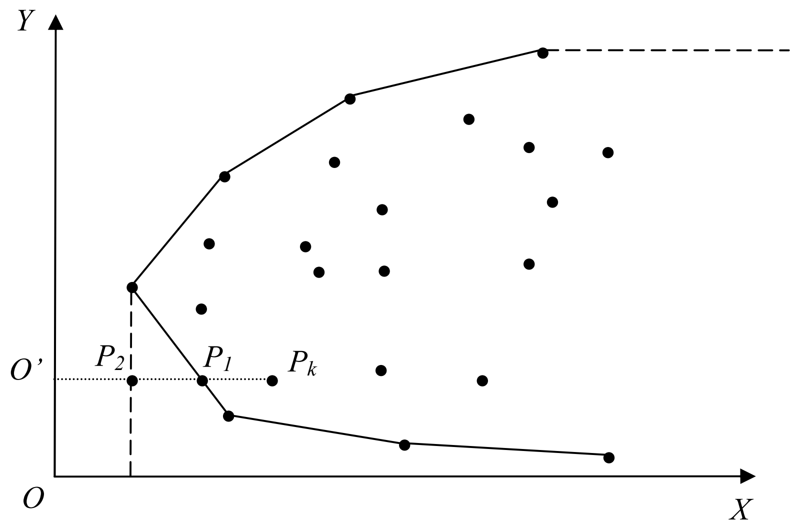


Figure 4. Production possibility sets.
The Farrell and Fieldhouse model and the BCC model

shown in Figure 4, are the most reasonable assumptions for a production possibility set. These conditions are universally accepted by researchers in the field. Farrell and Fieldhouse did not address the question of implied properties of the production set, being preoccupied with a convex production possibility set as shown in Figure 3.

The overall method has, as far as we know, only been followed up in Seitz (1970), (1971). The formal LP model used by Seitz has, as far as we can understand, been developed by Boles (see Boles, 1971). However, reading the papers by Seitz and Boles it is difficult to see that the overall method as described in Farrell and Fieldhouse (1962), and interpreted above, has been applied. Too few explanations are offered in these papers for us to try to comment upon the deviation from Farrell and Fieldhouse.

4. Graphical representations of frontier functions

The EffiVision software

In their Figure 2, Farrell and Fieldhouse (1962) tried to visualise the multidimensional space by using graphics in the two-dimensional plane. An arbitrary ray in the space of inputs (or an output corresponding to points on that ray) was taken as the horizontal axis. The efficiency score (derived variable) was considered as an independent variable (vertical axis).

From the present-day point of view it is much more expedient to visualise multidimensional production possibility sets in original variables (inputs and outputs). Then important derived economic quantities (scale elasticity, marginal rates and so on) can be calculated as by-products. However, it is necessary to determine strictly what isoquants in a general multi-inputs/multi-outputs model are. Let us concentrate on this in more detail.

Following Shephard (1970), we define the input possibility set *L(Y)* and the output possibility set *P(X)* as follows: $L(Y) = \{X \mid (X,Y) \in T\}$, $P(X) = \{Y \mid (X,Y) \in T\}$, where T is a production possibility set. Shephard also introduced two subsets of these. The input isoqant corresponding to Y is written as

$$Isoq\,L(Y) = \left\{ X \middle| X \in L(Y),\ \lambda X \notin L(Y),\ if\,\lambda \in [0,1) \right\} \quad (21)$$

The output isoquant corresponding to X is defined by

$$Isoq\,P(X) = \left\{ Y \middle| Y \in P(X),\ \theta Y \notin P(X),\ if\,\theta \in (1,+\infty) \right\} \quad (22)$$

Now we proceed to visualise the production possibility set in the multidimensional space. Define a two-dimensional plane in space E^{m+r} as

$$Pl(X_k, Y_k, d_1, d_2) = (X_k, Y_k) + \alpha d_1 + \beta d_2,\ (X_k, Y_k) \in T \quad (23)$$

where α and β are any real numbers, directions $d_1, d_2 \in E^{m+r}$, and d_1 is not parallel to d_2. The plane goes through the point (X_k, Y_k) in E^{m+r} and is spanned by vectors d_1 and d_2.

Next, define the intersection of the boundary of T with this plane as

$$Sec(X_k, Y_k, d_1, d_2) = \left\{ (X,Y) \middle| (X,Y) \in Pl(X_k, Y_k, d_1, d_2) \cap\ WEff_P T, d_1, d_2 \in E^{m+r} \right\} \quad (24)$$

where $WEff_P T$ is the set of weakly Pareto-efficient points. Krivonozhko *et al*. (2005) have proved that the following relation hold; $WEff_P T = Bound\,T$, where the latter is the boundary of T.

Krivonozhko *et al*. (2004) developed parametric optimisation methods to construct intersections of the type (24). By choosing different directions d_1 and d_2 we construct various sections going through the point (X_k, Y_k) and cutting the frontier. The curves obtained generalised the well-known functions in economics: production function, isoquant, isocost, isoprofit, etc.

Substituting $d_1 = (X_k, 0) \in E^{m+r}$ and $d_2 = (0, Y_k) \in E^{m+r}$ into (24), we obtain the curve $Sec_1(X_k, Y_k)$ that shows the dependence on maximum output, while input is changing along direction $\alpha X_k, \alpha > 0$, as in a one-input/one-output production model. Therefore we called this curve the *ray production function* associated with unit (X_k, Y_k) (Førsund *et al*., 2007).

In a similar way, substituting $d_1 = (e_p, 0) \in E^{m+r}$ and $d_2 = (e_s, 0) \in E^{m+r}$, where e_p and e_s are m identity vectors with 1 in position p and s, respectively, we obtain the curve $Sec_2(X_k, Y_k, p, s)$ that determines points on the frontier in a two-dimensional space of inputs that produce output Y_k. We call this curve the input isoquant associated with input variables x_p and x_s. Again, taking $d_1 = (0, e_p) \in E^{m+r}$, $d_2 = (0, e_s) \in E^{m+r}$, where e_p and e_s are m identity vectors with 1 in position p and s, respectively, we obtain the curve $Sec_3(X_k, Y_k, p, s)$ that shows a part of the frontier in a two-dimensional space of outputs variables y_p and y_s under given input vector X_k.

As we mentioned above, in mathematical economics a more general notion of input and output isoquants (21) – (22) were introduced. Krivonozhko *et al.* (2004) have proved that $Sec_2(X_k, Y_k, p, s) \subset Isoq\, L(Y_k)$ and $Sec_3(X_k, Y_k, p, s) \subset Isoq\, P(X_k)$ under certain conditions.

Parametric optimisation methods, which allow us to construct curves of the type (24), have been implemented in the software package EffiVision. Moreover, many important derived economic quantities (scale elasticity, marginal rates of substitution and so on) can be calculated as by-products using those economic functions.

Isoquants

Graphical representations of isoquants are basic tools in textbooks to promote the understanding of substitution properties. Several isoquants are plotted in Farrell (1957). In the case of more than two inputs, partial isoquants for pairs of inputs were shown for some given values of the remaining inputs. The method used to construct the isoquants was not explained.

Using EffiVision, we can draw isoquants that are derived from a BCC VRS estimate of the production possibility set. In Figure 5, these are estimated from the Farrell and Fieldhouse data, and show the substitution possibilities of labour and land. The isoquants are drawn separately for four groups, based on the output size quartiles, and the values of output and the other two inputs (fodder and capital) are kept constant at their average values in each group. The substitution regions in the figure are characterised by almost linear isoquant segments implying an unlimited

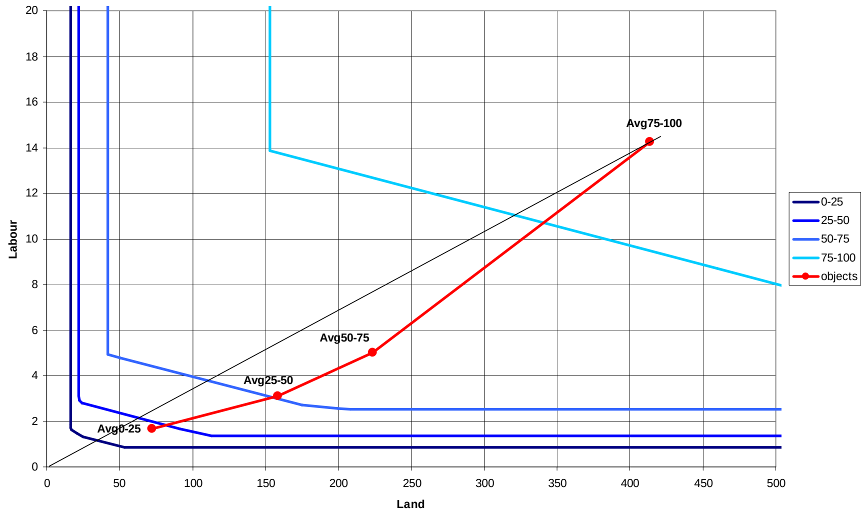
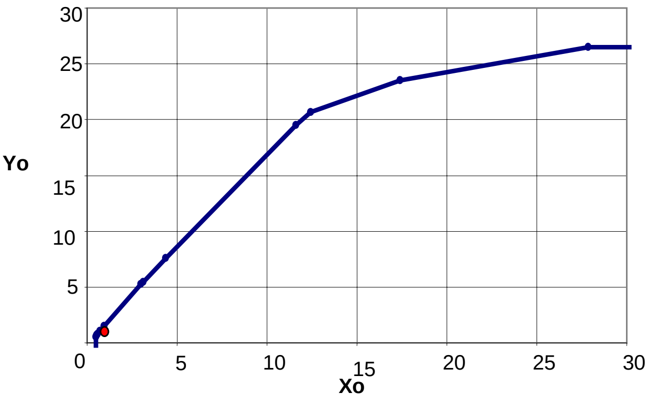


Figure 5. Isoquants for labour and land for four output levels and average levels of the other inputs fodder and capital

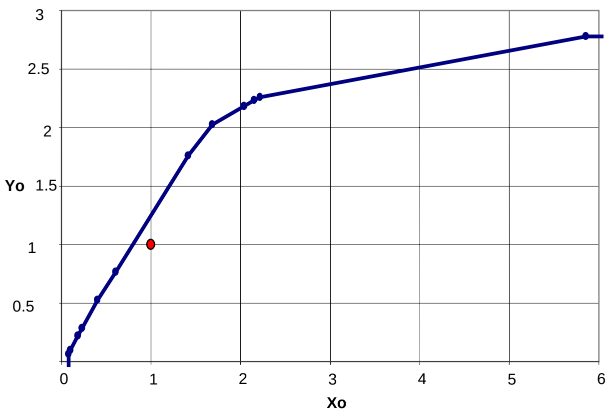
elasticity of substitution. The substitution region becomes relatively wider with increasing output size. In the figure also the average observed labour and land values for each group are plotted. Somewhat unexpectedly the largest farms are most labour intensive relative to land, and then come the smallest group, as shown by the ray from the origin to the average point of units in the fourth size quartile. The reason may be that more labour intensive dairy farming dominates large farms.

Ray frontier functions

The shape of the frontier function can be portrayed by making two-dimensional cuts through the multidimensional frontier. A choice of direction for the cut has to be made. We have chosen to cut the frontier function through the average point for each of the groups representing quartiles of the output value. The results for two groups; the smallest and the largest, are set out in panels (a), (b) of Figure 6. The values shown on the axes are all normalised by setting the average values equal to one for each panel. The ray frontier function has 12 to 13 corner points. The first segment is by construction vertical, and the last segment horizontal. The average farm point is fairly close to the frontier for all four groups and belongs to the first part. The shape of the ray frontiers are dominated by almost constant slopes of the first part and the last part. The segments constituting a third part are rather short. The shapes of the ray frontiers are quite similar



Panel (a)



Panel (b)

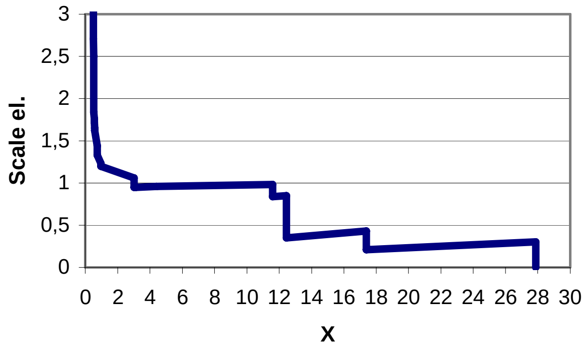
Figure 6. Cut through the average point of the first and fourth quartile of output size

independent of size class with a tendency for the smallest size class to have fewer and shorter segments with slopes in between the two dominating slopes of segments.

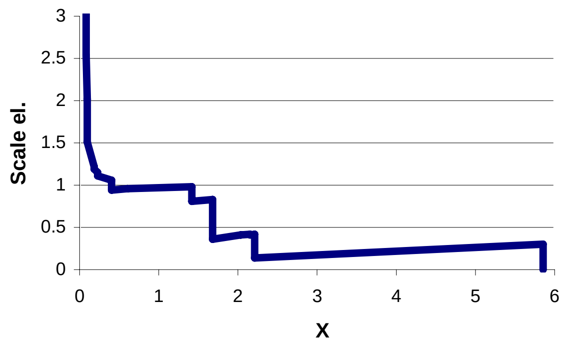
Scale properties

An instructive way of portraying scale properties of a frontier function is to calculate the scale elasticities along a ray frontier at each corner point and trace the development of the scale elasticity along the cut. EffiVision allows us to do this by means of numerical algorithms (Førsund *et al*., 2007). The change of values from above one to below one is of special interest since optimal scale along the ray frontier function is then identified.

The development of the scale elasticity corresponding to the ray frontier functions in panels (a) and (b) in Figure 6 is shown in panels (a) and (b) in Figure 7. The scale elasticity value is set out on the vertical axis and the ray value of inputs (relative to the means) is set out along the horizontal axis. The scale elasticity values start at infinity for the starting vertical segments shown in the panels, and then fall rapidly for the first part of the ray frontier functions. The crossing of the line for the value of 1 identifies the optimal scale size in terms of the ray input value. Since each panel is calibrated on the average values of the units within the group, abscissa values are not directly comparable across groups. Optimal scale is obtained rather early along all ray frontier cuts.



Panel (a)



Panel (b)

Figure 7. Scale elasticity development for the first and fourth quartile of output size

The relative length of the segment with a scale elasticity value close to one can be compared for the panels. We see that the smallest size group has the longest segment with a value close to one. For the largest size group the last segment with the lowest scale elasticity values is the dominating one regarding relative length.

A special property of the scale elasticity variation is that the values are decreasing monotonically; moving outwards along segments for values greater than one. For values smaller than one the value is *increasing* along each segment, but falling at the start of each new segment to values lower than the end value at the previous segment. (The increase is hyperbolic, but a linear interpolation, being very close, is used in the panels.) Thus, the Regular Ultra Passum (RUP) law of Frisch (1965) is not obeyed (as pointed out in Førsund and Hjalmarsson (2004) and Førsund *et al*., 2007).

The saw-tooth shape of the scale elasticity curves for values for the region of diseconomies of scale is clearly portrayed. The RUP law was introduced by Frisch (1965) as a sufficient condition for U-shaped average- and marginal cost curves (Førsund and Hjalmarsson, 2004). The shape of cost functions for DEA frontiers remains to be investigated theoretically and empirically.

The ray average-cost function

Farrell and Fieldhouse (1962) sought a graphical representation for investigating economies of scale. They observed that in the case of a single output, and assuming the same input price for all units, the technical efficiency score is proportional to average cost on the frontier along a ray to the observation. Using a space with input coefficients along the vertical axis and output along the horizontal axis, the boundary from below of the production possibility set for one output and one input represents the development of the input coefficient when output changes, i.e. the curve will also represent average costs (see Figure 3, Farrell and Fieldhouse (1962), p. 259). A *ray average-cost function* can therefore be depicted as a relationship between the efficiency scores and output.

However, using model (19) it is more convenient to hold on to the input coefficients and generate the average cost function by using the EffiVision graphical software to cut through the boundary of the five-dimensional frontier in question by specifying a direction. Figure 8 illu-

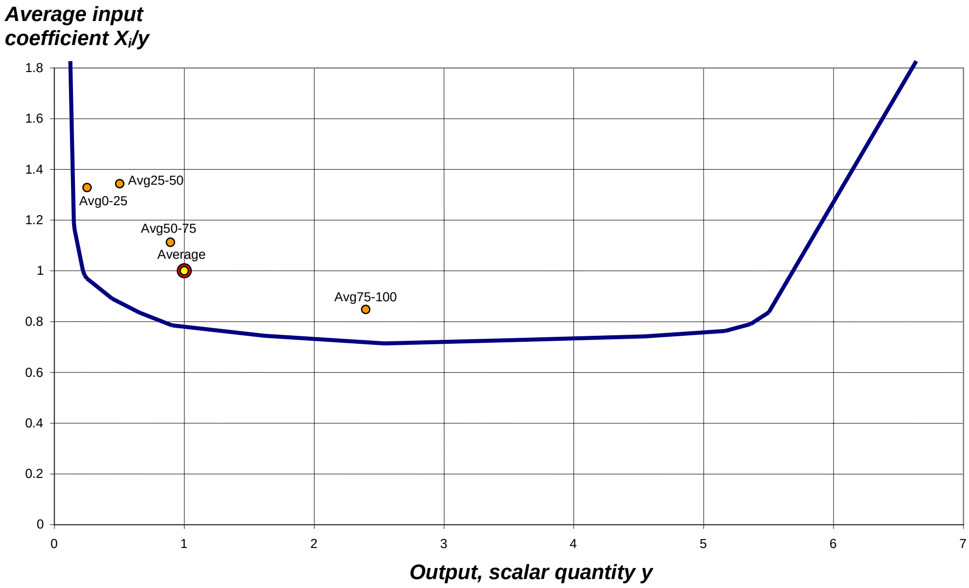


Figure 8. Ray average cost function in five-dimensional space

strates the ray average-cost function constructed by cutting the frontier with a plane going through the origin and spanned by two vectors; the average input vector and the average output vector. The plane goes through the average sample point normalised to (1,1) (projections of average points of four output-size groups are also indicated in the figure). Economies of scale are rapidly exhausted and then there is a long flat part before diseconomies of scale sets in.

Efficiency scores

Farrell (1957) found it instructive to look at the frequency distribution of efficiencies and provided histograms with efficiency score groups along the horizontal axis and relative frequency along the vertical axis (p.p. 270-271). Inspired by input coefficient diagrams in Salter (1960) we follow the exposition of efficiency distributions in Førsund and Hjalmarsson (1979), using a cumulated size variable along the horizontal axis and efficiency scores along the vertical axis, sorted in ascending order. Figure 9 shows the cumulative efficiency distribution for the Farrell and Fieldhouse data using the BCC VRS model, specifying input orientation to facilitate comparisons with the scores published in Farrell and Fieldhouse, Table 2, calculated using the polyhedron set as the production possibility set as explained in the discussion of Figure 4. Each histogram represents one of the 208 units, and the width of the histogram is proportional to the

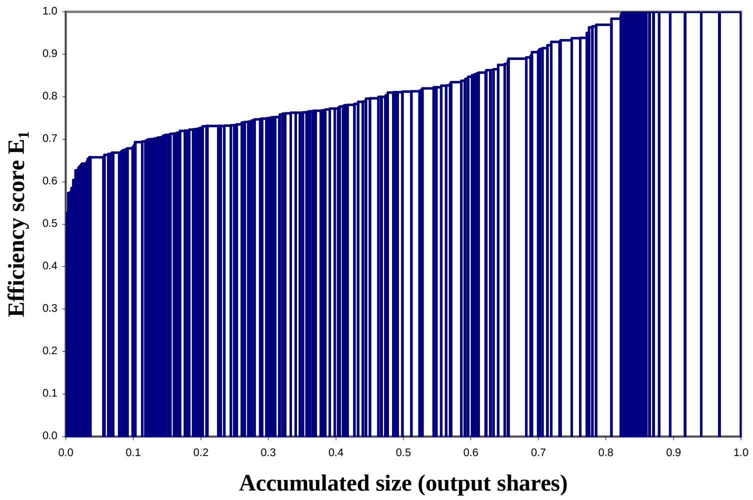


Figure 9. Cumulative efficiency distribution of input-oriented efficiency scores. BCC VRS model for the Farrell and Fieldhouse (1962) agriculture data.

relative size of the unit measured by the single output. The range of the efficiency distribution is from 0.53 to 1.00. There is a small left-hand tail of the most inefficient units in the range of 0.53 – 0.65 for the efficiency score composed of very small units. From there the distribution is quite even up to the efficient tail. We see that small units are overrepresented in the first half of the distribution, while large units are overrepresented among the efficient units, this part having 17 % of the total output. But we see that there are a substantial number of very small units that are also efficient. Such units will determine the left-hand part of the boundary of the production set as illustrated in Figure 4.

5. Conclusion

Farrell was the originator of both definitions of efficiency measures and of non-parametric estimation methods. However, we feel that Farrell (1957) is cited more than actually read and understood. By examining his models for estimating non-parametric piecewise linear frontiers we have rigorously shown how his basic model for a single output relates to Charnes *et al*. (1978), and how he failed to generalise the CRS model to multiple outputs. A key result of this paper is to demonstrate the connection between the attempt in Farrell and Fieldhouse (1962) to model economies and diseconomies of scale, and the approach of Banker *et al*. (1984) (BCC). It turned out that the crucial difference is whether to envelop the data by a polyhedron, as in the Farrell and Fieldhouse case, or by a free disposal hull as in the case of BCC.

Farrell solved several difficult problems for his time. First, he introduced notion of efficiency measure for the case of multi inputs for production units. Second, he explained this notion to a wide range of economists with the help of one isoquant for two inputs and one output model. His brilliant Diagram 1 (Farrell (1957), p. 254) has been re-published in many textbooks and scientific papers. Third, Farrell did all this correctly from a mathematical point of view.

Both Farrell (1957) and Farrell and Fieldhouse (1962) were quite preoccupied with making the analyses understood using graphical representations. However, Farrell did not succeed in visualising production frontiers in the general case of multi-input, multi-output models. In our opinion, optimisation theory and methods had not yet been developed properly at the time. He could not exploit appropriately these techniques to generalise his approach. In this paper we have shown the contemporary methods of visualisation of multi-input multi-output production frontiers. A novel contribution is that we have presented graphically the development of scale elasticity along ray frontiers. The revealed development shows the need for further research into the shape of cost functions for piecewise linear frontiers.

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