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Sustainable recursive social welfare functions*

Geir B. Asheim[†] Tapan Mitra[‡] Bertil Tungodden[§]

July 10, 2006

Abstract

Koopmans's (*Econometrica* **28**, 287–309) axiomatization of discounted utilitarianism is based on seemingly compelling conditions, yet this criterion leads to hard-to-justify outcomes. The present analysis considers a class of sustainable recursive social welfare functions within Koopmans's general framework. This class is axiomatized by means of a weak new equity condition ("Hammond Equity for the Future") and general existence is established. Any member of the class satisfies the key axioms of Chichilnisky's (*Social Choice and Welfare* **13**, 231–257) "sustainable preferences". The analysis singles out one of Koopmans's original conditions as particularly questionable from an ethical perspective.

Keywords and Phrases: Intergenerational justice, sustainability, discounted utilitarianism

JEL Classification Numbers: D63, D71, Q01

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1 Introduction

How should we treat future generations? From a normative point of view, what are the present generation's obligations towards the future? What ethical criterion for intergenerational justice should be adopted if one seeks to respect the interests of future generations?

These questions can be approached and answered in at least two ways:

1. Through an axiomatic analysis one can investigate on what fundamental ethical conditions various criteria for intergenerational justice are based, and then proceed to evaluate the normative appeal of these conditions.

2. By considering different kinds of technological environments, one can explore the consequences of various criteria for intergenerational justice, and compare the properties of the intergenerational utility streams that are generated.

It is consistent with Rawls' (1971) *reflective equilibrium* to do both: criteria for intergenerational justice should be judged both by the ethical conditions on which they build and by their consequences in specific environments. In particular, we may question the appropriateness of a criterion for intergenerational justice if it produces unacceptable outcomes in relevant technological environments. This view has been supported by many scholars, including Atkinson (2001, p. 206), Dasgupta and Heal (1979, p. 311), and Koopmans (1967).

When evaluating long-term policies, economists usually suggest to maximize the sum of discounted utilities. On the one hand, such *discounted utilitarianism* has been given a solid axiomatic foundation by Koopmans (1960). On the other hand, this criterion has ethically questionable implications when applied to economic models with resource constraints. This is demonstrated by Dasgupta and Heal (1974) in the so-called Dasgupta-Heal-Solow (DHS) model of capital accumulation and resource depletion (Dasgupta and Heal, 1974, 1979; Solow, 1974), where discounted utilitarianism for any positive discount rate undermines the well-being of generations in far future, even if sustainable streams with non-decreasing well-being are feasible.

In this paper we revisit Koopmans's framework. In Section 2 we consider conditions that are sufficient to numerically represent the social welfare relation by means of a recursive social welfare function. In this framework we introduce a new equity condition ("Hammond Equity for the Future"), capturing the following ethical intuition: A sacrifice by the present generation leading to a uniform gain for all future generations cannot lead to a less desirable stream of well-beings if the present remains better-off than the future even after the sacrifice.

In Section 3 we point out that "Hammond Equity for the Future" is weak, as it is implied by all the standard consequentialist equity conditions suggested in the literature. We show that adding this condition leads to a class of sustainable recursive social welfare functions, where the well-being of the present generation is taken into account if and only if the future is better-off. Furthermore, we establish general existence by means of an algorithmic construction. Finally, we show that any member of the class of sustainable recursive social welfare functions satisfies the key axioms of Chichilnisky's (1996) "sustainable preferences", namely "No Dictatorship of the Present" and "No Dictatorship of the Future". In a companion paper (Asheim, Buchholz and Mitra, 2006) we demonstrate how a sustainable recursive social welfare function can be used to solve the distributional conflicts in the DHS model.

In Section 4 we offer results that identify which of the conditions used by Koopmans (1960) to axiomatize discounted utilitarianism is particularly questionable from an ethical perspective. The condition in question, referred to as "Independent Present" by us and listed as Postulate $3'$a by Koopmans (1960), requires that the evaluation of two streams which differ during only the first two periods *not* depend on what the common continuation stream is. It is only by means of "Independent Present"—which in the words of Heal (2005) is "restrictive" and "surely not innocent"—that Koopmans moves beyond the recursive form to arrive at discounted utilitarianism. We single out "Independent Present" as the culprit by showing that the addition of this condition contradicts both "Hammond Equity for the Future" and the Chichilnisky (1996) conditions. All the proofs are relegated to an appendix.

Koopmans (1960) has often been interpreted as presenting the definitive case for discounted utilitarianism. In Section 5 we discuss how our results contribute to a weakening of this impression, by exploring other avenues within the general setting of his approach. We also investigate the scope for our new equity condition "Hammond Equity for the Future" if we step outside the Koopmans framework by *not* imposing that the social welfare relation is numerically representable.

2 Formal setting and basic result

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}_+$ the set of non-negative integers. Denote by ${}_0\mathbf{x} = (x_0, x_1, \ldots, x_t, \ldots,)$ an infinite stream, where $x_t \in Y$ is a one-dimensional indicator of the well-being of generation t, and $Y \subseteq \mathbb{R}$ is an interval of admissible well-beings which is not a singleton.[1] We will consider the set $\mathbf{X}$ of infinite streams bounded in well-being (see Koopmans, 1986b, p. 89); i.e., $\mathbf{X}$ is given by

$$\mathbf{X} = \{{}_0\mathbf{x} \in \mathbb{R}^{\mathbb{Z}_+} \mid [\inf_t x_t, \, \sup_t x_t] \subseteq Y\} .$$

By setting $Y = [0, 1]$, this includes the important special case where $\mathbf{X} = [0, 1]^{\mathbb{Z}_+}$, while allowing for cases where Y is not compact.

Denote by ${}_0\mathbf{x}_{T-1} = (x_0, x_1, \ldots, x_{T-1})$ and ${}_T\mathbf{x} = (x_T, x_{T+1}, \ldots, x_{T+t}, \ldots,)$ the T-head and T-tail of ${}_0\mathbf{x}$. Write ${}_{\text{con}}z = (z, z, \ldots)$ for the stream of a constant level of well-being equal to $z \in Y$. Throughout this paper we assume that the indicator of well-being is at least ordinally measurable and level comparable; i.e. what Blackorby, Donaldson and Weymark (1984) refer to as "level-plus comparability".

For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, we write ${}_0\mathbf{x} \geq {}_0\mathbf{y}$ if and only if $x_t \geq y_t$ for all $t \in \mathbb{Z}_+$, ${}_0\mathbf{x} > {}_0\mathbf{y}$ if and only if ${}_0\mathbf{x} \geq {}_0\mathbf{y}$ and ${}_0\mathbf{x} \neq {}_0\mathbf{y}$, and ${}_0\mathbf{x} \gg {}_0\mathbf{y}$ if and only if $x_t > y_t$ for all $t \in \mathbb{Z}_+$.

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[1]A more general formulation is, as used by Koopmans (1960), to assume that the well-being of generation t depends on a n-dimensional vector $\mathbf{x}_t$ that takes on values in a connected set $\mathbf{Y}$. However, by representing the well-being of generation t by a scalar x_t, we can focus on intergenerational issues. In doing so, we follow, e.g., Diamond (1965), Svensson (1980), Chichilnisky (1996), Basu and Mitra (2003) and Bossert, Sprumont and Suzumura (2005).

A *social welfare relation* (SWR) is a binary relation $\succsim$ on $\mathbf{X}$, where for all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ entails that ${}_0\mathbf{x}$ is deemed socially at least as good as ${}_0\mathbf{y}$. Denote by $\sim$ and $\succ$ the symmetric and asymmetric parts of $\succsim$; i.e., ${}_0\mathbf{x} \sim {}_0\mathbf{y}$ is equivalent to ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ and ${}_0\mathbf{y} \succsim {}_0\mathbf{x}$ and entails that ${}_0\mathbf{x}$ is deemed socially indifferent to ${}_0\mathbf{y}$, while ${}_0\mathbf{x} \succ {}_0\mathbf{y}$ is equivalent to ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ and $\neg\, {}_0\mathbf{y} \succsim {}_0\mathbf{x}$ and entails that ${}_0\mathbf{x}$ is deemed socially preferable to ${}_0\mathbf{y}$.

All comparisons are made at time 0; hence, the notation ${}_T\mathbf{x} \succsim {}_{T'}\mathbf{y}$ where T, $T' \geq 0$ means ${}_0\mathbf{x}' \succsim {}_0\mathbf{y}'$ where, for all t, $x'_t = x_{T+t}$ and $y'_t = y_{T'+t}$.

A *social welfare function* (SWF) representing $\succsim$ is a mapping $W : \mathbf{X} \to \mathbb{R}$ with the property that for all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. A mapping $W : \mathbf{X} \to \mathbb{R}$ is *monotone* if ${}_0\mathbf{x} \geq {}_0\mathbf{y}$ implies $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$.

In the present section we impose conditions on the SWR sufficient to obtain a numerical representation in terms of an SWF with a recursive structure (see Proposition 2 below), similar to but not identical to the one obtained by Koopmans (1960).

To obtain a numerical representation, we impose two "technical" conditions.

Condition O (*Order*) $\succsim$ is complete and transitive.

Condition RC (*Restricted Continuity*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ satisfies $x_t = z$ for all $t \geq 1$, and ${}_0\mathbf{x}^n \in \mathbf{X}$ for $n \in \mathbb{N}$ satisfy $\lim_{n\to\infty} \sup_t |x_t^n - x_t| = 0$ with, for each $n \in \mathbb{N}$, $\neg\, {}_0\mathbf{x}^n \prec {}_0\mathbf{y}$ (resp. $\neg\, {}_0\mathbf{x}^n \succ {}_0\mathbf{y}$), then $\neg\, {}_0\mathbf{x} \prec {}_0\mathbf{y}$ (resp. $\neg\, {}_0\mathbf{x} \succ {}_0\mathbf{y}$).

Condition **RC** is weaker than ordinary supnorm continuity.

Condition C (*Continuity*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}^n \in \mathbf{X}$ for $n \in \mathbb{N}$ satisfy $\lim_{n\to\infty} \sup_t |x_t^n - x_t| = 0$ with, for each $n \in \mathbb{N}$, $\neg\, {}_0\mathbf{x}^n \prec {}_0\mathbf{y}$ (resp. $\neg\, {}_0\mathbf{x}^n \succ {}_0\mathbf{y}$), then $\neg\, {}_0\mathbf{x} \prec {}_0\mathbf{y}$ (resp. $\neg\, {}_0\mathbf{x} \succ {}_0\mathbf{y}$).

Condition **C** is entailed by Koopmans's (1960) Postulate 1. As the analysis of Section 3 will show, the weaker continuity condition **RC** will enable us to show existence of sustainable recursive social welfare functions.

The central condition in Koopmans's (1960) analysis is the stationarity postulate

(Postulate 4). Combined with Koopmans's Postulate 3b, the stationarity postulate is equivalent to the following independence condition (where we borrow the name that Fleurbaey and Michel, 2003, use for this condition in a slightly stronger form).

Condition IF (*Independent Future*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ with $x_0 = y_0$, ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$ if and only if ${}_1\mathbf{x} \succsim {}_1\mathbf{y}$.

Condition **IF** means that an evaluation concerning only generations from the next period on can be made as if the present time (time 0) was actually at time 1; i.e., as if generations $\{0, 1, \dots\}$ would have taken the place of generations $\{1, 2, \dots\}$.

With the well-being of each generation t expressed by a one-dimensional indicator x_t, it is uncontroversial to ensure through the following condition that a higher value of x_t cannot lead to a socially less preferred stream.

Condition M (*Monotonicity*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x} > {}_0\mathbf{y}$, then $\neg\, {}_0\mathbf{y} \succ {}_0\mathbf{x}$.

Condition **M** is obviously implied by the "Strong Pareto" condition.

Condition SP (*Strong Pareto*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x} > {}_0\mathbf{y}$, then ${}_0\mathbf{x} \succ {}_0\mathbf{y}$.

With condition **M** we need not impose Koopmans's (1960) extreme streams postulate (Postulate 5) and can consider the set of infinite streams bounded in well-being.

As the fifth and final condition of our basic representation result (Proposition 2), we impose the following efficiency condition.

Condition RD (*Restricted Dominance*) For all x, $z \in Y$, if $x < z$, then $(x, {}_{\text{con}}z) \prec {}_{\text{con}}z$.

To evaluate the implications of **RD**, consider the following three conditions.

Condition WS (*Weak Sensitivity*) There exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $(x_0, {}_1\mathbf{z}) \succ (y_0, {}_1\mathbf{z})$.

Condition **SP** implies condition **RD**, which in turn implies condition **WS**. Condition **WS** coincides with Koopmans's (1960) Postulate 2.

Condition DF (*Dictatorship of the Future*) For all ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, there exist $\underline{y}, \bar{y} \in Y$ with $\underline{y} \leq x_t, y_t \leq \bar{y}$ for all $t \in \mathbb{Z}_+$ and $T' \in \mathbb{Z}_+$ such that, for any ${}_0\mathbf{z}, {}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$, $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) \succ ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$ for all $T > T'$.

Condition NDF (*No Dictatorship of the Future*) Condition **DF** does not hold.

Conditions **NDF** generalizes one of Chichilnisky's (1996) two main axioms to our setting where we consider the set of infinite streams bounded in well-being.

Proposition 1 *Assume that the* SWR $\succsim$ *satisfies conditions* ***O*** *and* ***IF***. *Then* ***WS*** *is equivalent to* ***NDF***.

Since **RD** strengthens **WS**, it follows from Proposition 1 that **RD** ensures "No Dictatorship of the Future", provided that the SWR satisfies conditions **O** and **IF**. To appreciate why we cannot replace **RD** with an even stronger efficiency condition, we refer to the analysis of Section 3 and the impossibility result of Proposition 4.

To state Proposition 2, introduce the following notation:

$$\mathcal{U} := \{U : Y \to \mathbb{R} \mid U \text{ is continuous and non-decreasing; } U(Y) \text{ is not a singleton}\}$$
$$\mathcal{U}_I := \{U : Y \to \mathbb{R} \mid U \text{ is continuous and increasing}\}$$
$$\mathcal{V}(U) := \{V : U(Y)^2 \to \mathbb{R} \mid V \text{ satisfies (V.0), (V.1), (V.2), and (V.3)}\},$$

where, for all $U \in \mathcal{U}$, $U(Y) := \{u \in \mathbb{R} \mid \exists x \in Y \text{ s.t. } u = U(x)\}$ denotes the range of U, and the properties of the aggregator function V, (V.0)–(V.3), are as follows:

(V.0) $V(u, w)$ is continuous in (u, w) on $U(Y)^2$.

(V.1) $V(u, w)$ is non-decreasing in u for given w.

(V.2) $V(u, w)$ is increasing in w for given u.

(V.3) $V(u, w) < w$ for $u < w$, and $V(u, w) = w$ for $u = w$.

Proposition 2 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies conditions* ***O, RC, IF, M,*** *and* ***RD***.

(2) There exists a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ *representing* $\succsim$ *and satisfying, for some* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{\text{con}}z) = U(z)$ *for all* $z \in Y$.

For a given representation W (with associated utility function U) of an SWR satisfying conditions **O**, **RC**, **IF**, **M**, and **RD**, we will refer to $U(x_t)$ as the *utility* of generation t and $W({}_0\mathbf{x})$ as the *welfare* derived from the infinite stream ${}_0\mathbf{x}$.

3 Hammond Equity for the Future

Discounted utilitarianism satisfies conditions **O**, **RC**, **IF**, **M**, and **RD**. Hence, these conditions do not by themselves prevent "Dictatorship of the Present", in the terminology of Chichilnisky (1996).

Condition DP (*Dictatorship of the Present*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, there exist $\underline{y}$, $\bar{y} \in Y$ with $\underline{y} \leq x_t, y_t \leq \bar{y}$ for all $t \in \mathbb{Z}_+$ and $T' \in \mathbb{Z}_+$ such that, for any ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$, $({}_0\mathbf{x}_{T-1}, {}_T\mathbf{z}) \succ ({}_0\mathbf{y}_{T-1}, {}_T\mathbf{v})$ for all $T > T'$.

Condition NDP (*No Dictatorship of the Present*) Condition **DP** does not hold.

Condition **NDP** generalizes the other of Chichilnisky's (1996) two main axioms to our setting where we consider the set of infinite streams bounded in well-being.

Hence, to ensure "No Dictatorship of the Present" we must impose an equity condition that rules out SWRs that allow for such dictatorship. We do so by a condition which—combined with **RC**—entails that the interest of the present are taken into account only if the present is worse-off than the future. Consider a stream $(x, {}_{\text{con}}z)$ having the property that well-being is constant from the second period on. For such a stream we may unequivocally say that, if $x < z$, then the present is worse-off than the future. Likewise, if $x > z$, then the present is better-off than the future.

Condition HEF (*Hammond Equity for the Future*) For all x, y, z, $v \in Y$, if $x > y > v > z$, then $\neg(x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v)$.

For streams where well-being is constant from the second period on, condition **HEF** states the following: If the present is better-off than the future and a sacrifice now leads to a uniform gain for all future generations, then such a transfer from the present to the future cannot lead to a less desirable stream, as long as the present remains better-off than the future. To appreciate the weakness of condition **HEF**, consider first the standard "Hammond Equity" condition (Hammond, 1976) and a weak version of Lauwers' (1998) non-substitution condition.

Condition HE (*Hammond Equity*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ and ${}_0\mathbf{y}$ satisfy that there exists a pair τ', τ'' such that $x_{\tau'} > y_{\tau'} > y_{\tau''} > x_{\tau''}$ and $x_t = y_t$ for all $t \neq \tau'$, τ'', then $\neg\, {}_0\mathbf{x} \succ {}_0\mathbf{y}$.

Condition WNS (*Weak Non-Substitution*) For all x, y, z, $v \in Y$, if $v > z$, then $\neg(x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v)$.

By assuming, in addition, that well-beings are at least cardinally measurable and fully comparable, we may also consider weak versions of the Lorenz Domination and Pigou-Dalton principles. Such equity conditions have been used in the setting of infinite streams by, e.g., Birchenhall and Grout (1979), Asheim (1991), Fleurbaey and Michel (2001), and Hari, Shinotsuka, Suzumura and Xu (2005).

Condition WLD (*Weak Lorenz Domination*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ and ${}_0\mathbf{y}$ satisfy that there exists $T > 1$ such that ${}_0\mathbf{y}_{T-1}$ Lorenz dominates ${}_0\mathbf{x}_{T-1}$ and ${}_T\mathbf{x} = {}_T\mathbf{y}$, then $\neg\, {}_0\mathbf{x} \succ {}_0\mathbf{y}$.

Condition WPD (*Weak Pigou-Dalton*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$, if ${}_0\mathbf{x}$ and ${}_0\mathbf{y}$ satisfy that there exist a positive number ϵ and a pair τ', τ'' such that $x_{\tau'} - \epsilon = y_{\tau'} \geq y_{\tau''} = x_{\tau''} + \epsilon$ and $x_t = y_t$ for all $t \neq \tau'$, τ'', then $\neg\, {}_0\mathbf{x} \succ {}_0\mathbf{y}$.

While it is clear that condition **HEF** is implied by **WNS**—as **HEF** in contrast to **WNS** does not preclude that a finite improvement for the first generation can compensate for a uniform loss for all future generations, provided that the present is worse-off than the future—it is perhaps less obvious that, under **O** and **M**, **HEF** is not stronger than *each* of **HE**, **WPD**, and **WLD**.

Proposition 3 *Assume that the* SWR $\succsim$ *satisfies conditions **O** and **M**. Then each of **HE**, **WPD**, and **WLD** implies **HEF**.*

Note that condition **HEF** involves a comparison between a sacrifice by a single generation and a uniform gain for each member of an infinite set of generations that are worse-off. Hence, contrary to the standard "Hammond Equity" condition, if well-beings are made (at least) cardinally measurable and fully comparable, then the transfer from the better-off present to the worse-off future specified in condition **HEF** increases the sum of well-beings obtained by summing the well-beings of a sufficiently large number T of generations. This entails that condition **HEF** is implied by both the Pigou-Dalton principle of transfers and the Lorenz Domination principle, independently of what specific cardinal scale of well-beings is imposed (provided that conditions **O** and **M** are satisfied). Hence, "Hammond Equity for the Future" can be endorsed from both an egalitarian and utilitarian point of view. In particular, condition **HEF** is much weaker and more compelling than the standard "Hammond Equity" condition.

However, in line with the Diamond-Yaari impossibility result (Diamond, 1965) on the inconsistency of equity and efficiency conditions under continuity,[2] the equity condition **HEF** is in conflict with the following weak efficiency condition under **RC**.

Condition RS (*Restricted Sensitivity*) There exists x, $z \in Y$ with $x > z$ such that $(x, {}_{\mathrm{con}}z) \succ {}_{\mathrm{con}}z$.

Condition **SP** implies condition **RS**, which in turn implies condition **WS**.

Proposition 4 *There is no* SWR $\succsim$ *satisfying conditions **RC**, **RS**, and **HEF**.*

[2]The Diamond-Yaari impossibility result states that the equity condition of "Weak Anonymity" (deeming two streams socially indifferent if one is obtained from the other through a finite permutation of well-beings) is inconsistent with the efficiency condition **SP** under **C**. See also Basu and Mitra (2003) and Fleurbaey and Michel (2003).

Impossibility results arising from **HEF** are further explored in Asheim, Mitra and Tungodden (2006). Here we concentrate on SWRs that exist under **HEF**. We note that it follows from Proposition 4 that **RD** is the strongest efficiency condition compatible with **HEF** under **RC**, when comparing streams $(x, {}_{\text{con}}z)$ where well-being is constant from the second period on with constant streams ${}_{\text{con}}z$.

The following result establishes that "Dictatorship of the Present" is indeed ruled out by adding condition **HEF** to conditions **O**, **RC**, **IF**, and **M**.

Proposition 5 *Assume that the* SWR $\succsim$ *satisfies conditions **O**, **RC**, **IF**, and **M**. Then **HEF** implies **NDP**.*

How does the basic representation result of Proposition 2 change by imposing also condition **HEF** on a SWR $\succsim$ satisfying conditions **O**, **RC**, **IF**, **M**, and **RD**? To investigate this question, introduce the following notation:

$$\mathcal{V}_S(U) := \{V : U(Y)^2 \to \mathbb{R} \mid V \text{ satisfies (V.0), (V.1), (V.2), and (V.3}')\},$$

where, (V.3′) is given as follows:

(V.3′) $V(u, w) < w$ for $u < w$, and $V(u, w) = w$ for $u \geq w$.

Note that, for each $U \in \mathcal{U}$, $\mathcal{V}_S(U) \subseteq \mathcal{V}(U)$.

Proposition 6 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies conditions **O**, **RC**, **IF**, **M**, **RD**, and **HEF**.*

(2) There exists a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ *representing* $\succsim$ *and satisfying, for some* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}_S(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{\text{con}}z) = U(z)$ *for all* $z \in Y$.

We refer to a mapping satisfying property (2) of Proposition 6 as a *sustainable recursive* SWF. Proposition 6 does not pose the question whether there exists a sustainable recursive SWF for any $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. This question of existence is resolved through the following proposition, which also characterizes the asymptotic properties of such welfare functions.

Proposition 7 *For all* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}_S(U)$*, there exists a monotone mapping* $W : \mathbf{X} \to \mathbb{R}$ *satisfying* $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{con}z) = U(z)$ *for all* $z \in Y$*. Any such mapping* W *satisfies, for each* ${}_0\mathbf{x} \in \mathbf{X}$*,*

$$\lim_{T\to\infty} W({}_T\mathbf{x}) = \liminf_{t\to\infty} U(x_t)\,.$$

By combining Propositions 6 and 7 we obtain our first main result.

Theorem 1 *There exists a class of* SWRs $\succsim$ *satisfying conditions* ***O****,* ***RC****,* ***IF****,* ***M****,* ***RD****, and* ***HEF****.*

The proof of the existence part of Proposition 7 is based on an algorithmic construction. For any ${}_0\mathbf{x} \in \mathbf{X}$ and each $T \in \mathbb{Z}_+$, consider the following finite sequence:

$$\left.\begin{array}{l} w(T,T) = \liminf_{t\to\infty} U(x_t) \\ w(T-1,T) = V(U(x_{T-1}), w(T,T)) \\ \dots \\ w(0,T) = V(U(x_0), w(1,T)) \end{array}\right\} \tag{1}$$

Define the mapping $W_\sigma : \mathbf{X} \to \mathbb{R}$ by

$$W_\sigma({}_0\mathbf{x}) := \lim_{T\to\infty} w(0,T)\,. \tag{W}$$

In the proof of Proposition 7 we show that W_σ is a sustainable recursive SWF.

It is an open question whether W_σ is the *unique* sustainable recursive SWF given $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. As reported in the following proposition, we can show uniqueness if the aggregator function satisfies a condition introduced by Koopmans, Diamond, and Williamson (1964, p. 88): $V \in \mathcal{V}(U)$ satisfies the property of *weak time perspective* if there exists a continuous increasing transformation $g : \mathbb{R} \to \mathbb{R}$ such that $g(w) - g(V(u,w))$ is a non-decreasing function of w for given u.

Proposition 8 *Let* $U \in \mathcal{U}_I$ *and* $V \in \mathcal{V}_S(U)$*. If* V *satisfies the property of weak time perspective, then there exists a unique monotone mapping* $W : \mathbf{X} \to \mathbb{R}$ *satisfying* $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $W({}_{con}z) = U(z)$ *for all* $z \in Y$*. This mapping,* W_σ*, is defined by (W).*

The property of weak time perspective does not follow from the conditions we have imposed, but it is satisfied in special cases; e.g., with V given by

$$V(u,w) \ = \begin{cases} (1-\delta)u + \delta w & \text{if } \ u < w \\ w & \text{if } \ u \geq w \,, \end{cases} \tag{2}$$

where $\delta \in (0,1)$.[3] We can also show that the set of supnorm continuous sustainable recursive SWFs contains at most W_σ. However, even though W_σ is continuous in the weak sense implied by condition **RC**, it need not be supnorm continuous.

Once we drop one of the conditions **RC**, **IF**, and **RD**, and combine the remaining two conditions with **O**, **M**, and **HEF**, new possibilities open up. It is clear that:

- The mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := \liminf_{t\to\infty} U(x_t)$ for some $U \in \mathcal{U}_I$ represents an SWR satisfying **O**, **RC**, **IF**, **M**, and **HEF**, but not **RD**.
- The maximin SWR satisfies **O**, **RC**, **M**, **RD**, and **HEF**, but not **IF**.
- Leximin and undiscounted utilitarian SWRs for infinite streams satisfy **O**, **IF**, **M**, **RD**, and **HEF**, but not **RC** (cf. Proposition 12).

It follows from Propositions 1, 5, and 6 that any sustainable recursive SWF represents an SWR satisfying **NDF** and **NDP**. Chichilnisky (1996) defines "sustainable preferences" by imposing **NDF** and **NDP** as well as conditions **O**, **C**, and **SP**. When showing existence, she considers SWRs violating condition **IF**, and it is open question whether "sustainable preferences" can be combined with **IF**. Hence, through showing general existence for our sustainable recursive SWF, we demonstrate that **NDF** and **NDP** can be combined with **IF** and numerical representability—thus be imposed within the Koopmans framework—provided that efficiency and continuity conditions are appropriately weakened.

[3]Note that an SWR $\succsim$ represented by a sustainable recursive SWF with aggregator function given by (2) satisfies the following restricted form of the **IP** condition introduced in the next section: For all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$ such that $(x_0, x_1, {}_2\mathbf{z})$, $(y_0, y_1, {}_2\mathbf{z})$, $(x_0, x_1, {}_2\mathbf{v})$, $(y_0, y_1, {}_2\mathbf{v})$ are non-decreasing, $(x_0, x_1, {}_2\mathbf{z}) \succsim (y_0, y_1, {}_2\mathbf{z})$ if and only if $(x_0, x_1, {}_2\mathbf{v}) \succsim (y_0, y_1, {}_2\mathbf{v})$.

4 Independent Present

The following condition is invoked as Postulate $3'$a in Koopmans's (1960) characterization of discounted utilitarianism.

Condition IP (*Independent Present*) For all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, $(x_0, x_1, {}_2\mathbf{z}) \succsim (y_0, y_1, {}_2\mathbf{z})$ if and only if $(x_0, x_1, {}_2\mathbf{v}) \succsim (y_0, y_1, {}_2\mathbf{v})$.

In words, condition **IP** requires that the evaluation of two streams which differ during only the first two periods *not* depend on what the common continuation stream is. We suggest in this section that this condition may not be compelling, both through an intuitive argument, and through formal results.

We claim that it might be consistent with ethical intuition to accept that the stream $(1, 4, 5, 5, 5, \dots)$ is socially better than $(2, 2, 5, 5, 5, \dots)$, while not accepting that $(1, 4, 2, 2, 2, \dots)$ is socially better than $(2, 2, 2, 2, 2, \dots)$. It is not obvious that we should treat the conflict between the worst-off and the second worst-off generation presented by the first comparison in the same manner as we treat the conflict between the worst-off and the best-off generation put forward by the second comparison.

Turn now to the formal results. Koopmans (1960) characterizes discounted utilitarianism by means of conditions **IF**, **WS**, and **IP**. However, it turns out that conditions **IF**, **WS**, and **IP** contradict **HEF** under **RC** and **M**. Furthermore, this conclusion is tight, in the sense that an SWR exists if any one of these conditions is dropped. We report this as our second main result.

Theorem 2 *There is no* SWR $\succsim$ *satisfying conditions* ***RC, IF, M, WS, HEF,*** *and* ***IP****. If one of the conditions* ***RC, IF, M, WS, HEF,*** *and* ***IP*** *is dropped, then there exists an* SWR $\succsim$ *satisfying the remaining five conditions as well as condition* ***O****.*

In the following proposition, we reproduce Koopmans's (1960) characterization of discounted utilitarianism within this paper's formal setting.

Proposition 9 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies conditions* ***O**, **RC**, **IF**, **M**, **WS**, and **IP**.*

(2) There exists a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ *representing* $\succsim$ *and satisfying, for some* $U \in \mathcal{U}$ *and* $\delta \in (0,1)$, $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$.

*Strengthening **WS** to **RD** in statement (1) is equivalent to replacing* $\mathcal{U}$ *by* $\mathcal{U}_I$ *in statement (2).*

Furthermore, we note that the discounted utilitarian SWF exists and is unique.

Proposition 10 *For all* $U \in \mathcal{U}$ *and* $\delta \in (0,1)$, *there exists a unique monotone mapping* $W : \mathbf{X} \to \mathbb{R}$ *satisfying* $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ *for all* ${}_0\mathbf{x} \in \mathbf{X}$. *This mapping,* W_δ, *is defined by, for each* ${}_0\mathbf{x} \in \mathbf{X}$,

$$W_\delta({}_0\mathbf{x}) = (1-\delta)\sum\nolimits_{t=0}^{\infty} \delta^t U(x_t)\,.$$

Propositions 9 and 10 have the following implication.

Proposition 11 *There is no* SWR $\succsim$ *satisfying conditions* ***O**, **RC**, **IF**, **M**, **IP**, **NDP**, and **NDF**.*

To summarize, it follows from Theorem 2 and Propositions 1 and 11 that, within a Koopmans framework where **O**, **RC**, **IF**, **M**, and **WS** are imposed, condition **IP** contradicts both **HEF** and **NDP**. Hence, in such a framework, **IP** is in conflict with consequentialist equity conditions that respect the interests of future generations.

5 Concluding remarks

Koopmans (1960) has often been interpreted as presenting the definitive case for discounted utilitarianism. In Sections 2 and 3 we have sought to weaken this impression by exploring other avenues within the general setting of his approach. In particular, by not imposing condition **IP**, used by Koopmans (1960) to characterize discounted utilitarianism, we were able to combine our new equity condition **HEF** with the essential features of the Koopmans framework: (a) numerical representability, (b) condition **IF** which includes Koopmans's stationarity postulate, and (c) sensitivity for the interests of the present generation. This leads to a non-empty class of sustainable recursive social welfare functions. We have argued that condition **HEF** is weak, as it is implied by all the standard consequentialist equity conditions suggested in the literature, yet strong enough to ensure that the Chichilnisky (1996) conditions are satisfied. In a companion paper (Asheim, Buchholz and Mitra, 2006) we demonstrate how a sustainable recursive social welfare function can be used to solve in an appealing way the interesting distributional conflicts that arise in the DHS model of capital accumulation and resource depletion. In particular, it leads to growth and development at first when capital is productive, while protecting the generations in the distant future from the grave consequences of discounting when the vanishing resource stock undermines capital productivity.

In this final section we note that even wider possibilities open up if we are willing to give up numerical representability by not imposing **RC**. In particular, we are then able to combine the equity condition **HEF** and the independence condition **IP** with our basic conditions **O** and **IF**, while strengthening our efficiency conditions **M** and **RD** to condition **SP**.

Proposition 12 *There exists an* SWR $\succsim$ *satisfying conditions* ***O, IF, SP, HEF,*** *and* ***IP***.

The proof of this proposition employs the leximin and undiscounted utilitarian SWRs for infinite streams that have been axiomatized in recent contributions (see Asheim and Tungodden, 2004; Basu and Mitra, 2005; Bossert, Sprumont and Suzumura, 2005).

We end by making the observation that continuity is not simply a "technical" condition without ethical content. In a setting where **RC** (or a stronger continuity condition like **C**) is combined with **RS** (or a stronger efficiency condition like **SP**), it follows from Proposition 4 that condition **HEF** is not satisfied. Hence, on this basis one may claim that, in combination with a sufficiently strong efficiency condition, continuity rules out SWFs that protect the interests of future generations by implying that the equity condition **HEF** does not hold. In the main analysis of this paper we have avoided the trade-off between continuity and numerical representability on the one hand, and the ability to impose the equity condition **HEF** on the other hand, by weakening the efficiency condition in an appropriate way.

Appendix: Proofs

Proof of Proposition 1. *Part I: **WS** implies **NDF**.* Assume that the SWR $\succsim$ satisfies conditions **O** and **WS**. By **WS**, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ with ${}_1\mathbf{x} = {}_1\mathbf{y}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$. Let ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$ be given by ${}_0\mathbf{z} = {}_0\mathbf{v} = {}_0\mathbf{x}$. We have that, for any $\underline{y}$, $\bar{y} \in Y$ satisfying $\underline{y} \le x_t, y_t \le \bar{y}$ for all $t \in \mathbb{Z}_+$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$. Still, for all $T > 0$, $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) = {}_0\mathbf{x} = ({}_0\mathbf{x}_{T-1}, {}_T\mathbf{y}) = ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$, implying by **O** that $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) \sim ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$. This contradicts **DF**.

*Part II: **NDF** implies **WS**.* Assume that the SWR $\succsim$ satisfies conditions **O**, **IF** and **NDF**. Suppose that **WS** does not hold, e.g., for all ${}_0\mathbf{x}'$, ${}_0\mathbf{y}' \in \mathbf{X}$ with ${}_1\mathbf{x}' = {}_1\mathbf{y}'$, we have that ${}_0\mathbf{x}' \sim {}_0\mathbf{y}'$. By **NDF**, there exists ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ such that ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, since **DF** holds trivially otherwise. Let ${}_0\mathbf{z}$, ${}_0\mathbf{v}$ be arbitrary streams in $\mathbf{X}$. We have that ${}_{T-1}\mathbf{x} \sim (z_{T-1}, {}_T\mathbf{x})$ for all $T > 0$ since **WS** does not hold. By **IF** and the above argument,

$$ {}_{T-2}\mathbf{x} = (x_{T-2}, {}_{T-1}\mathbf{x}) \sim (x_{T-2}, z_{T-1}, {}_T\mathbf{x}) \sim ({}_{T-2}\mathbf{z}_{T-1}, {}_T\mathbf{x}) . $$

By invoking **O** and applying **IF** and the above argument repeatedly, it follows that ${}_0\mathbf{x} \sim ({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x})$ for all $T > 0$. Likewise, ${}_0\mathbf{y} \sim ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$ for all $T > 0$. By **O**, $({}_0\mathbf{z}_{T-1}, {}_T\mathbf{x}) \succ ({}_0\mathbf{v}_{T-1}, {}_T\mathbf{y})$ for all $T > 0$. This contradicts **NDF**. ■

The following lemma is useful for proving Proposition 2 and subsequent results.

Lemma 1 *Assume that the* SWR $\succsim$ *satisfies conditions **O**, **RC**, **M**. Then, for all* ${}_0\mathbf{x} \in \mathbf{X}$, *there exists* $z \in Y$ *such that* ${}_{\text{con}}z \sim {}_0\mathbf{x}$. *If condition **RD** is added, then* z *is unique.*

Proof. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, and **M**. By **O**, **M**, and the definition of $\mathbf{X}$, there exists $z \in Y$ such that $\inf\{v \in Y \mid {}_{\text{con}}v \succsim {}_0\mathbf{x}\} \le z \le \sup\{v \in Y \mid {}_{\text{con}}v \precsim {}_0\mathbf{x}\}$. By **O** and **RC**, ${}_{\text{con}}z \sim {}_0\mathbf{x}$.

If condition **RD** is added, then by **O**, **M**, and **RD** we have that

$$ {}_{\text{con}}v = (v, {}_{\text{con}}v) \precsim (v, {}_{\text{con}}z) \prec {}_{\text{con}}z \quad \text{if } v < z , \tag{3} $$

so that $\inf\{v \in Y \mid {}_{\text{con}}v \succsim {}_0\mathbf{x}\} = \sup\{v \in Y \mid {}_{\text{con}}v \precsim {}_0\mathbf{x}\}$ and z is unique. ■

Proof of Proposition 2. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, and **RD**. In view of Lemma 1, determine $W : \mathbf{X} \to Y$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W({}_0\mathbf{x}) = z$ where ${}_{\text{con}}z \sim {}_0\mathbf{x}$. By **O** and (3), $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. By **M**, W is monotone.

Let $U \in \mathcal{U}_I$ be given by $U(x) = x$ for all $x \in Y$, implying that $U(Y) = Y$. Hence, by construction of W, $W({}_{\text{con}}z) = z = U(z)$ for all $z \in Y$. It follows from **IF** that, for given $x_0 \in Y$, there exists an increasing transformation $V(U(x_0), \cdot) : Y \to Y$ such that, for all ${}_1\mathbf{x} \in \mathbf{X}$, $W(x_0, {}_1\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$. This determines $V : Y \times Y \to Y$, where $V(u, w)$ is increasing in w for given u, establishing that V satisfies (V.2). By **M**, $V(u, w)$ is non-decreasing in u for given w, establishing that V satisfies (V.1). Since $\neg(x, {}_{\text{con}}z) \prec {}_{\text{con}}v$ (resp. $\neg(x, {}_{\text{con}}z) \succ {}_{\text{con}}v$) if and only if

$$V(x, z) = V(U(x), W({}_{\text{con}}z)) = W(x, {}_{\text{con}}z) \geq v \quad \text{(resp. } \leq v\text{)},$$

RC implies that V satisfies (V.0). Finally, since

$$\begin{aligned} V(z, z) &= V(U(z), W({}_{\text{con}}z)) = W({}_{\text{con}}z) = z \\ V(x, z) &= V(U(x), W({}_{\text{con}}z)) = W(x, {}_{\text{con}}z) < W({}_{\text{con}}z) = z \text{ if } x < z\,, \end{aligned}$$

by invoking **RD**, it follows that V satisfies (V.3). Hence, $V \in \mathcal{V}(U)$.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}_I$ and $V \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. Since the SWR $\succsim$ is represented by the SWF W, it follows that $\succsim$ satisfies **O**. Moreover, $\succsim$ satisfies **M** since W is monotone, $\succsim$ satisfies **IF** since V satisfies (V.2), and $\succsim$ satisfies **RD** since $U \in \mathcal{U}_I$ and V satisfies (V.3). The following argument shows that $\succsim$ satisfies **RC**.

Let ${}_0\mathbf{x}, {}_0\mathbf{y} \in \mathbf{X}$, and let $x_t = z$ for all $t \geq 1$. Let ${}_0\mathbf{x}^n \in \mathbf{X}$ for $n \in \mathbb{N}$, with the property that $\lim_{n\to\infty} \sup_t |x_t^n - x_t| = 0$ and, for each $n \in \mathbb{N}$, $\neg\, {}_0\mathbf{x}^n \prec {}_0\mathbf{y}$. We have to show that $\neg\, {}_0\mathbf{x} \prec {}_0\mathbf{y}$, or equivalently, $W({}_0\mathbf{x}) \geq W({}_0\mathbf{y})$. Define $\epsilon(n)$ for $n \in \mathbb{N}$ by, for each $n \in \mathbb{N}$, $\epsilon(n) := \sup_t |x_t^n - x_t|$, so that $\lim_{n\to\infty} \epsilon(n) = 0$. For each $n \in \mathbb{N}$,

$$\begin{aligned} V(U(x_0 + \epsilon(n)), U(z + \epsilon(n))) &= V(U(x_0 + \epsilon(n)), W({}_{\text{con}}(z + \epsilon(n)))) \\ &= W(x_0 + \epsilon(n), {}_{\text{con}}(z + \epsilon(n))) \geq W({}_0\mathbf{x}^n) \geq W({}_0\mathbf{y}) \end{aligned}$$

since W is monotone and represents $\succsim$, and $\neg\, {}_0\mathbf{x}^n \prec {}_0\mathbf{y}$. This implies that

$$W({}_0\mathbf{x}) = V(U(x_0), W({}_{\text{con}}z)) = V(U(x_0), U(z)) \geq W({}_0\mathbf{y})$$

since U and V are continuous and $\lim_{n\to\infty} \epsilon(n) = 0$. The same kind of argument can be used to show that $\neg\, {}_0\mathbf{x} \succ {}_0\mathbf{y}$ if, for each $n \in \mathbb{N}$, $\neg\, {}_0\mathbf{x}^n \succ {}_0\mathbf{y}$. ■

Proof of Proposition 3. Assume $x > y > v > z$. We must show under **O** and **M** that each of **HE**, **WLD**, and **WPD** implies $\neg(x, {}_{\text{con}}z) \succ (y, {}_{\text{con}}v)$.

Since $x > y > v > z$, there exist an integer T and utilities $x', z' \in [0, 1]$ satisfying $y > x' \geq v > z' > z$ and $x - x' = T(z' - z)$.

By **O** (completeness) and **HE**, $(x', z', {}_{\text{con}}z) \succsim (x, {}_{\text{con}}z)$, and by **M**, $(y, {}_{\text{con}}v) \succsim (x', z', {}_{\text{con}}z)$. By **O** (transitivity), $(y, {}_{\text{con}}v) \succsim (x, {}_{\text{con}}z)$.

Consider next **WLD** and **WPD**. Let ${}_0\mathbf{x}^0 = (x, {}_{\text{con}}z)$, and define ${}_0\mathbf{x}^n$ for $n \in \{1, \dots, T\}$ inductively as follows:

$$\begin{aligned} x_t^n &= x_t^{n-1} - (z' - z) && \text{for } t = 0 \\ x_t^n &= z' && \text{for } t = n \\ x_t^n &= x_t^{n-1} && \text{for } t \neq 0, n\,. \end{aligned}$$

By **O** (completeness) and **WLD**, ${}_0\mathbf{x}^T \succsim {}_0\mathbf{x}^0$, and by **M**, $(y, {}_{\text{con}}v) \succsim {}_0\mathbf{x}^T$. By **O** (transitivity), $(y, {}_{\text{con}}v) \succsim (x, {}_{\text{con}}z)$ since ${}_0\mathbf{x}^0 = (x, {}_{\text{con}}z)$.

By **O** (completeness) and **WPD**, ${}_0\mathbf{x}^n \succsim {}_0\mathbf{x}^{n-1}$ for $n \in \{1, \dots, T\}$, and by **M**, $(y, {}_{\text{con}}v) \succsim {}_0\mathbf{x}^T$. By **O** (transitivity), $(y, {}_{\text{con}}v) \succsim (x, {}_{\text{con}}z)$ since ${}_0\mathbf{x}^0 = (x, {}_{\text{con}}z)$. ■

Proof of Proposition 4. Suppose there exists an SWR $\succsim$ satisfying conditions **RC**, **RS**, and **HEF**.

Step 1: By **RS**, there exists $x, z \in Y$ with $x > z$ such that $(x, {}_{\text{con}}z) \succ {}_{\text{con}}z$. Define $a = x - z$. We claim that there is $b \in (0, a)$ such that

$$(x, {}_{\text{con}}z) \succ (z + b, {}_{\text{con}}z)\,.$$

If not, $\neg\,(x, {}_{\text{con}}z) \succ (z + b, {}_{\text{con}}z)$ for every $b \in (0, a)$. By letting $b \to 0$ and using **RC**, $\neg\,(x, {}_{\text{con}}z) \succ {}_{\text{con}}z$. This contradicts $(x, {}_{\text{con}}z) \succ {}_{\text{con}}z$ and establishes our claim.

Step 2: For every $c \in (0, b)$, noting that $x > z+b > z+c > z$, **HEF** implies $\neg(x, {}_{\text{con}}z) \succ (z + b, {}_{\text{con}}(z + c))$. By letting $c \to 0$ and using **RC**, we get

$$\neg(x, {}_{\text{con}}z) \succ (z + b, {}_{\text{con}}z)\,.$$

This contradicts the claim proved in Step 1, and establishes the proposition. ∎

Proof of Proposition 5. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **M**, **IF**, and **HEF**. Let ${}_0\mathbf{x}$, ${}_0\mathbf{y} \in \mathbf{X}$ satisfy ${}_0\mathbf{x} \succ {}_0\mathbf{y}$, and let $\underline{y}$, $\bar{y} \in Y$ satisfy $\underline{y} \leq x_t, y_t \leq \bar{y}$ for all $t \in \mathbb{Z}_+$. For any $T \in \mathbb{Z}_+$ with $x_{T-1} > \underline{y}$, Proposition 4 implies that $(x_{T-1}, {}_{\text{con}}\underline{y}) \succ {}_{\text{con}}\underline{y}$ would contradict **RC** and **HEF**. Hence, since $x_{T-1} \geq \underline{y}$, it follows from **O** and **M** that $(x_{T-1}, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}$ for all $T > 0$. By **IF** and the above argument,

$$({}_{T-2}\mathbf{x}_{T-1}, {}_{\text{con}}\underline{y}) = (x_{T-2}, x_{T-1}, {}_{\text{con}}\underline{y}) \sim (x_{T-2}, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y} .$$

By invoking **O** and applying **IF** and the above argument repeatedly, $({}_0\mathbf{x}_{T-1}, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}$ for all $T > 0$. Likewise, $({}_0\mathbf{y}_{T-1}, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y}$ for all $T > 0$.

Let ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in [\underline{y}, \bar{y}]^{\mathbb{Z}_+}$ be given by ${}_0\mathbf{z} = {}_0\mathbf{v} = {}_{\text{con}}\underline{y}$. Since $({}_0\mathbf{x}_{T-1}, {}_{\text{con}}\underline{y}) \sim {}_{\text{con}}\underline{y} \sim ({}_0\mathbf{y}_{T-1}, {}_{\text{con}}\underline{y})$ for all $T > 0$, we have by **O** that $({}_0\mathbf{x}_{T-1}, {}_T\mathbf{z}) \sim ({}_0\mathbf{y}_{T-1}, {}_T\mathbf{v})$ for all $T > 0$. This contradicts **DP**. ∎

The following result is useful for the proof of Proposition 6.

Lemma 2 *Assume that the* SWR $\succsim$ *satisfies conditions* ***O****,* ***RC****,* ***IF****,* ***M****,* ***RD****, and* ***HEF****. Then, for all* ${}_0\mathbf{x} \in \mathbf{X}$ *and* $T \in \mathbb{Z}_+$*,* ${}_T\mathbf{x} \precsim {}_{T+1}\mathbf{x}$*.*

Proof. Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **RD**, and **HEF**. By the interpretation of ${}_T\mathbf{x}$, it is sufficient to show that we will arrive at a contradiction if ${}_0\mathbf{x} \succ {}_1\mathbf{x}$. Therefore, suppose ${}_0\mathbf{x} \succ {}_1\mathbf{x}$. By Lemma 1, there exist z^0, $z^1 \in Y$ such that ${}_{\text{con}}z^0 \sim {}_0\mathbf{x}$ and ${}_{\text{con}}z^1 \sim {}_1\mathbf{x}$, where, by **O**, (3), and ${}_0\mathbf{x} \succ {}_1\mathbf{x}$, it follows that $z^0 > z^1$. Furthermore, since ${}_1\mathbf{x} \sim {}_{\text{con}}z^1$, it follows by **IF** that $(x_0, {}_1\mathbf{x}) \sim (x_0, {}_{\text{con}}z^1)$. Hence, ${}_0\mathbf{x} \sim (x_0, {}_{\text{con}}z^1)$.

If $x_0 \leq z^0$, then,

$$\begin{aligned} {}_0\mathbf{x} &\sim (x_0, {}_{\text{con}}z^1) \prec (x_0, {}_{\text{con}}z^0) && \text{by (3) and condition } \mathbf{IF} \text{ since } z^1 < z^0 \\ &\precsim (z^0, {}_{\text{con}}z^0) = {}_{\text{con}}z^0 \sim {}_0\mathbf{x} && \text{by conditions } \mathbf{O} \text{ and } \mathbf{M} \text{ since } x_0 \leq z^0. \end{aligned}$$

This contradicts condition **O**, ruling out this case.

If $x_0 > z^0$, then, by selecting some $v \in (z^1, z^0)$,

$$\begin{aligned} {}_0\mathbf{x} &\sim (x_0, {}_{\text{con}}z^1) \precsim (z^0, {}_{\text{con}}v) && \text{by conditions } \mathbf{O} \text{ and } \mathbf{HEF} \text{ since } x_0 > z^0 > v > z^1 \\ &\prec (z^0, {}_{\text{con}}z^0) \sim {}_0\mathbf{x} && \text{by (3) and condition } \mathbf{IF} \text{ since } v < z^0. \end{aligned}$$

This contradicts condition **O**, ruling out also this case. ∎

Proof of Proposition 6. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **RD**, and **HEF**. By Proposition 2, the SWR $\succsim$ is represented by a monotone SWF $W : \mathbf{X} \to \mathbb{R}$ satisfying, for some $U \in \mathcal{U}_I$ and $V \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. It remains to be shown that $V(u, w) = w$ for $u > w$, implying that V satisfies (V.3$'$) and, thus, $V \in \mathcal{V}_S(U)$.

Therefore, since $V(u, w)$ is non-decreasing in u for given $w \in U(Y)$ and $V(u, w) = w$ for $u = w$, suppose that $V(u, w) > w$ for some u, $w \in U(Y)$ with $u > w$. Since $U \in \mathcal{U}_I$, the properties of W imply that there exist x, $z \in Y$ with $x > z$ such that

$$\begin{aligned} W(x,\, {}_{\text{con}}z) &= V(U(x), W({}_{\text{con}}z)) = V(U(x), U(z)) \\ &= V(u, w) > w = U(z) = W({}_{\text{con}}z)\,. \end{aligned}$$

Since the SWR $\succsim$ is represented by the SWF W, it follows that $(x,\, {}_{\text{con}}z) \succ {}_{\text{con}}z$. This contradicts Lemma 2.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}_I$ and $V_S \in \mathcal{V}(U)$, $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. By Proposition 2, it remains to be shown that the SWR $\succsim$, represented by the SWF W, satisfies **HEF**. The following argument shows that $\succsim$ satisfies **HEF**.

Let x, y, z, $v \in Y$ satisfy $x > y > v > z$. We have to show that $\neg(x,\, {}_{\text{con}}z) \succ (y,\, {}_{\text{con}}v)$, or equivalently, $W(x,\, {}_{\text{con}}z) \leq W(y,\, {}_{\text{con}}v)$. By the properties of W,

$$\begin{aligned} W(x,\, {}_{\text{con}}z) &= V(U(x), W({}_{\text{con}}z)) = V(U(x), U(z)) = U(z) \\ &< U(v) = V(U(y), U(v)) = V(U(y), W({}_{\text{con}}v)) = W(y,\, {}_{\text{con}}v)\,, \end{aligned}$$

since $x > y > v > z$, $U \in \mathcal{U}_I$, and $V \in \mathcal{V}_S(U)$ ■

Proof of Proposition 7. Fix $U \in \mathcal{U}_I$ and $V \in \mathcal{V}_S(U)$. The proof has two parts.

Part I: $\lim_{T\to\infty} W({}_T\mathbf{x}) = \liminf_{t\to\infty} U(x_t)$. Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfies $W({}_0\mathbf{x}) = V(U(x_0), W({}_1\mathbf{x}))$ for all ${}_0\mathbf{x} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$. Hence, by Proposition 6, the SWF W represents a SWR $\succsim$ satisfying **O**, **RC**, **M**, **RD**, **IF**, and **HEF**. By Lemma 1, for all ${}_0\mathbf{x} \in \mathbf{X}$, there exists $z \in Y$ such that ${}_{\text{con}}z \sim {}_0\mathbf{x}$. By Lemma 2, $W({}_t\mathbf{x})$ is non-decreasing in t.

Step 1: $\lim_{t\to\infty} W({}_t\mathbf{x})$ *exists.* Suppose $W({}_\tau\mathbf{x}) > \limsup_{t\to\infty} U(x_t)$ for some $\tau \in \mathbb{Z}_+$. By the premise and the fact that $U \in \mathcal{U}_I$, there exists $z \in Y$ satisfying

$$W({}_\tau\mathbf{x}) \;\geq\; U(z) \;>\; \limsup_{t\to\infty} U(x_t)$$

and $T \geq \tau$ such that $z > v := \sup_{t \geq T} x_t$. By **RD**, **O**, and **M**, ${}_{\text{con}}z \succ (v, {}_{\text{con}}z) \succsim {}_T\mathbf{x}$, and hence, by **O**, ${}_{\text{con}}z \succ {}_T\mathbf{x}$. However, since $W({}_t\mathbf{x})$ is non-decreasing in t, $W({}_T\mathbf{x}) \geq W({}_\tau\mathbf{x}) \geq U(z)$. This contradicts that W is an SWF. Hence, $W({}_t\mathbf{x})$ is bounded above by $\limsup_{t\to\infty} U(x_t)$, and the result follows since $W({}_t\mathbf{x})$ is non-decreasing in t.

Step 2: $\lim_{t\to\infty} W({}_t\mathbf{x}) \geq \liminf_{t\to\infty} U(x_t)$. Suppose $\lim_{t\to\infty} W({}_t\mathbf{x}) < \liminf_{t\to\infty} U(x_t)$. By the premise and the fact that $U \in \mathcal{U}_I$, there exists $z \in Y$ satisfying

$$\lim_{t\to\infty} W({}_t\mathbf{x}) \leq U(z) < \liminf_{t\to\infty} U(x_t)$$

and $T \geq 0$ such that $z < v := \inf_{t \geq T} x_t$. By **O**, **M**, and **RD**, ${}_{\text{con}}z \precsim (z, {}_{\text{con}}v) \prec {}_{\text{con}}v \precsim {}_T\mathbf{x}$, and hence, by **O**, ${}_{\text{con}}z \prec {}_T\mathbf{x}$. However, since $W({}_t\mathbf{x})$ is non-decreasing in t, $W({}_T\mathbf{x}) \leq \lim_{t\to\infty} W({}_t\mathbf{x}) \leq U(z)$. This contradicts that W is an SWF.

Step 3: $\lim_{t\to\infty} W({}_t\mathbf{x}) \leq \liminf_{t\to\infty} U(x_t)$. Suppose $\lim_{t\to\infty} W({}_t\mathbf{x}) > \liminf_{t\to\infty} U(x_t)$. By Lemma 1, there exists, for all $t \in \mathbb{Z}_+$, $z^t \in Y$ such that ${}_{\text{con}}z^t \sim {}_t\mathbf{x}$. Since $U \in \mathcal{U}_I$, $z \in Y$ defined by $z := \lim_{t\to\infty} z^t$ satisfies $U(z) = \lim_{t\to\infty} W({}_t\mathbf{x})$. By the premise and the fact that $U \in \mathcal{U}_I$, there exists $x \in Y$ satisfying

$$\liminf_{t\to\infty} U(x_t) < U(x) < U(z)$$

and a subsequence $(x_{t_\tau}, z^{t_\tau})_{\tau \in \mathbb{Z}_+}$ such that, for all $\tau \in \mathbb{Z}_+$, $x_{t_\tau} \leq x < z^{t_\tau}$. Then

$${}_{\text{con}}z^{t_\tau} \sim {}_{t_\tau}\mathbf{x} = (x_{t_\tau}, {}_{t_\tau+1}\mathbf{x}) \precsim (x_{t_{\tau+1}-1}, {}_{t_{\tau+1}}\mathbf{x}) \sim (x_{t_{\tau+1}-1}, {}_{\text{con}}z^{t_{\tau+1}}) \precsim (x, {}_{\text{con}}z)\,,$$

since z^t is non-decreasing in t. By **O**, **RC**, and the definition of z, ${}_{\text{con}}z \precsim (x, {}_{\text{con}}z)$. Since $x < z$, this contradicts **RD**.

Part II: Existence. Let ${}_0\mathbf{x} \in \mathbf{X}$, implying that there exist $\underline{y}, \bar{y} \in Y$ such that, for all $t \in \mathbb{Z}_+$, $\underline{y} \leq x_t \leq \bar{y}$. For each $T \in \mathbb{Z}_+$, consider $\{w(t,T)\}_{t=0}^T$ determined by (1).

Step 1: $w(t,T)$ is non-increasing in T for given $t \leq T$. Given $T \in \mathbb{Z}_+$,

$$w(T, T+1) = V(U(x_T), w(T+1, T+1)) \leq w(T+1, T+1) = \liminf_{t\to\infty} U(x_t) = w(T,T)$$

by (1) and (V.3′). Thus, applying (V.2), we have

$$w(T-1, T+1) = V(U(x_{T-1}), w(T, T+1)) \leq V(U(x_{T-1}), w(T,T)) = w(T-1, T)\,.$$

Using (V.2) repeatedly, we then obtain

$$w(t, T+1) \leq w(t,T) \ \text{ for all } t \in \{0, ..., T-1\}\,,$$

which establishes that $w(t,T)$ is non-increasing in T for given $t \leq T$.

Step 2: $w(t,T)$ is bounded below by $U(\underline{y})$. By (1), (V.1), (V.2), and (V.3′), $w(T,T) = \liminf_{t\to\infty} U(x_t) \geq U(\underline{y})$, and for all $t \in \{0, ..., T-1\}$,

$$w(t+1,T) \geq U(\underline{y}) \text{ implies } w(t,T) = V(U(x_t), w(t+1,T)) \geq V(U(\underline{y}), U(\underline{y})) = U(\underline{y})\,.$$

Hence, it follows by induction that $w(t,T)$ is bounded below by $U(\underline{y})$.

Step 3: Definition and properties of W_σ. By steps 1 and 2, $\lim_{T\to\infty} w(t,T)$ exists for all $t \in \mathbb{Z}_+$. Define the mapping $W_\sigma : \mathbf{X} \to \mathbb{R}$ by (W). We have that W_σ is monotone by (1), (V.1), and (V.2). As $w(0,T) = V(U(x_0),\, w(1,T))$ and V satisfies (V.0), we have that $W_\sigma({}_0\mathbf{x}) = V(U(x_0), W_\sigma({}_1\mathbf{x}))$. Finally, if ${}_0\mathbf{x} = {}_{\text{con}}z$ for some $z \in Y$, then it follows from (1) and (V.3′) that $w(t,T) = U(z)$ for all $T \in \mathbb{Z}_+$ and $t \in \{0, ..., T\}$, implying that $W_\sigma({}_0\mathbf{x}) = U(z)$. ■

Proof of Proposition 8. Suppose there exists a monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfying $W({}_0\mathbf{y}) = V(U(y_0), W({}_1\mathbf{y}))$ for all ${}_0\mathbf{y} \in \mathbf{X}$ and $W({}_{\text{con}}z) = U(z)$ for all $z \in Y$ such that $W({}_0\mathbf{x}) \neq W_\sigma({}_0\mathbf{x})$. Since V satisfies the property of weak time perspective, there is a continuous increasing transformation $g : \mathbb{R} \to \mathbb{R}$ such that $|g(W({}_0\mathbf{x})) - g(W_\sigma({}_0\mathbf{x}))| = \epsilon > 0$, and furthermore, $|g(W({}_t\mathbf{x})) - g(W_\sigma({}_t\mathbf{x}))| = |g(V(U(x_t), W({}_{t+1}\mathbf{x}))) - g(V(U(x_t), W_\sigma({}_{t+1}\mathbf{x})))| \leq |g(W({}_{t+1}\mathbf{x})) - g(W_\sigma({}_{t+1}\mathbf{x}))|$ for all $t \in \mathbb{Z}_+$. It now follows, by induction, that

$$|g(W({}_T\mathbf{x})) - g(W_\sigma({}_T\mathbf{x}))| \geq \epsilon > 0$$

for all $T \in \mathbb{Z}_+$. However this contradicts that, for all $T \in \mathbb{Z}_+$,

$$\lim_{T\to\infty} W({}_T\mathbf{x}) = \liminf_{t\to\infty} U(x_t) = \lim_{T\to\infty} W_\sigma(\mathbf{x})$$

by Proposition 7, since g is a continuous increasing transformation. ■

For the proofs of the results of Section 4, the following notation is useful, where ${}_0\mathbf{z} = (z_0,\, {}_1\mathbf{z}) = (z_0,\, z_1,\, {}_2\mathbf{z}) \in \mathbf{X}$ is a fixed but arbitrary reference stream:

$x_0 \succsim_0^{\mathbf{z}} y_0$	means	$(x_0,\, {}_1\mathbf{z}) \succsim (y_0,\, {}_1\mathbf{z})$
${}_1\mathbf{x}\ {}_1\!\succsim^{\mathbf{z}}\ {}_1\mathbf{y}$	means	$(z_0,\, {}_1\mathbf{x}) \succsim (z_0,\, {}_1\mathbf{y})$
$(x_0,\, x_1)\ {}_0\!\succsim_1^{\mathbf{z}} (y_0,\, y_1)$	means	$(x_0,\, x_1,\, {}_2\mathbf{z}) \succsim (y_0,\, y_1,\, {}_2\mathbf{z})$
${}_2\mathbf{x}\ {}_2\!\succsim^{\mathbf{z}}\ {}_2\mathbf{y}$	means	$(z_0,\, z_1,\, {}_2\mathbf{x}) \succsim (z_0,\, z_1,\, {}_2\mathbf{y})$
$x_1 \succsim_1^{\mathbf{z}} y_1$	means	$(z_0,\, x_1,\, {}_2\mathbf{z}) \succsim (z_0,\, y_1,\, {}_2\mathbf{z})\,.$

Say that $\succsim_0^{\mathbf{z}}$ is *sensitive* if there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_0 \succ_0^{\mathbf{z}} y_0$, and likewise for ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$. Say that $\succsim_0^{\mathbf{z}}$ is *independent of* ${}_0\mathbf{z}$ if, for all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, $x_0 \succsim_0^{\mathbf{z}} y_0$ if and only if $x_0 \succsim_0^{\mathbf{v}} y_0$, and likewise for ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$.

In this notation and terminology, condition **IF** implies that ${}_1\succsim^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$, condition **WS** states that $\succsim_0^{\mathbf{z}}$ is sensitive, while condition **IP** states that ${}_0\succsim_1^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$. The following result indicates that imposing condition **IP** is consequential.

Lemma 3 *Assume that the* SWR $\succsim$ *satisfies conditions **IF** and **IP**. Then* $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, *and* $\succsim_1^{\mathbf{z}}$ *are independent of* ${}_0\mathbf{z}$.

Proof. Assume that the SWR $\succsim$ satisfies conditions **IF** and **IP**. By repeated application of **IF**, ${}_1\succsim^{\mathbf{z}}$ and ${}_2\succsim^{\mathbf{z}}$ are independent of ${}_0\mathbf{z}$, while **IP** states that ${}_0\succsim_1^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$. By **IF**, $(x_1, {}_2\mathbf{z}) \succsim (y_1, {}_2\mathbf{z})$ is equivalent to $(z_0, x_1, {}_2\mathbf{z}) \succsim (z_0, y_1, {}_2\mathbf{z})$, which, by **IP**, is equivalent to $(z_0, x_1, {}_2\mathbf{v}) \succsim (z_0, y_1, {}_2\mathbf{v})$, which in turn, by **IF**, is equivalent to $(x_1, {}_2\mathbf{v}) \succsim (y_1, {}_2\mathbf{v})$, which finally, by **IF**, is equivalent to $(v_0, x_1, {}_2\mathbf{v}) \succsim (v_0, y_1, {}_2\mathbf{v})$, where ${}_0\mathbf{v} \in \mathbf{X}$ is some arbitrary stream. Hence, $\succsim_0^{z}$ and $\succsim_1^{z}$ are independent of ${}_0\mathbf{z}$. ■

Proof of Theorem 2. *Part I:* This part is proved in three steps.

Step 1: By Lemma 3, **IF** and **IP** imply that $\succsim_0^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$.

Step 2: By condition **WS**, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_0 \succ_0^{\mathbf{z}} y_0$. This rules out that $x_0 = y_0$, and by **M**, $x_0 < y_0$ would lead to a contradiction. Hence, $x_0 > y_0$. Since $\succsim_0^{\mathbf{z}}$ is independent of ${}_0\mathbf{z}$, this implies **RS**.

Step 3: By Proposition 4, there is no SWR $\succsim$ satisfying **RC**, **RS**, and **HEF**.

Part II: To establish this part, consider dropping a single condition.

*Dropping **IP**.* Existence follows from Theorem 1 since **RD** implies **WS**.

*Dropping **HEF**.* Existence follows from Propositions 9 and 10.

*Dropping **WS**.* All the remaining conditions are satisfied by the SWF $\succsim$ being represented by the mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := \liminf_{t\to\infty} x_t$.

*Dropping **M**.* All the remaining conditions are satisfied by the SWF $\succsim$ being represented by the mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := -x_0 + \liminf_{t\to\infty} x_t$.

*Dropping **IF**.* All the remaining conditions are satisfied by the SWF $\succsim$ being represented by the mapping $W : \mathbf{X} \to \mathbb{R}$ defined by $W({}_0\mathbf{x}) := \min\{x_0, x_1\}$.

*Dropping **RC**.* Existence follows from Proposition 12 since **SP** implies **M** and **WS**. ■

Proof of Proposition 9. *Part I: (1) implies (2).* Assume that the SWR $\succsim$ satisfies conditions **O**, **RC**, **IF**, **M**, **WS**, and **IP**.

By **WS**, $\succsim_0^{\mathbf{z}}$ is sensitive. By **IF**, $(x_1, {}_2\mathbf{z}) \succ (y_1, {}_2\mathbf{z})$ implies $(z_0, x_1, {}_2\mathbf{z}) \succ (z_0, y_1, {}_2\mathbf{z})$. Since $\succsim_0^{\mathbf{z}}$ is sensitive, there exist ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$ such that $x_1 \succ_1^{\mathbf{z}} y_1$, meaning that $\succsim_1^{\mathbf{z}}$ is sensitive. This implies that ${}_1\succsim^{\mathbf{z}}$ is sensitive and, by applying **IF**, that ${}_2\succsim^{\mathbf{z}}$ is sensitive. By Lemma 3, $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$ are independent of ${}_0\mathbf{z}$.

By **O** and **M**, there exists a continuous function $\tilde{U} : Y \to \mathbb{R}$ satisfying $\tilde{U}(z) \geq \tilde{U}(v)$ if and only if ${}_{\text{con}}z \succsim {}_{\text{con}}v$. In view of Lemma 1, determine $\tilde{W} : \mathbf{X} \to \mathbb{R}$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $\tilde{W}({}_0\mathbf{x}) = \tilde{U}(y)$ where ${}_{\text{con}}y \sim {}_0\mathbf{x}$. By **O**, $\tilde{W}({}_0\mathbf{x}) \geq \tilde{W}({}_0\mathbf{y})$ if and only if ${}_0\mathbf{x} \succsim {}_0\mathbf{y}$. By construction of $\tilde{W}$, $\tilde{W}({}_{\text{con}}z) = \tilde{U}(z)$ for all $z \in Y$. By **IF**, for given $x_0 \in Y$, there exists an increasing transformation $\tilde{V}(\tilde{U}(x_0), \cdot) : \mathbb{R} \to \mathbb{R}$ such that, for all ${}_1\mathbf{x} \in \mathbf{X}$, $\tilde{W}(x_0, {}_1\mathbf{x}) = \tilde{V}(\tilde{U}(x_0), \tilde{W}({}_1\mathbf{x}))$. This determines $\tilde{V} : \tilde{U}(Y)^2 \to \mathbb{R}$. Since $\neg(x, {}_{\text{con}}z) \prec {}_{\text{con}}v$ (resp. $\neg(x, {}_{\text{con}}z) \succ {}_{\text{con}}v$) if and only if

$$\tilde{V}(\tilde{U}(x), \tilde{U}(z)) = \tilde{V}(\tilde{U}(x), \tilde{W}({}_{\text{con}}z)) = \tilde{W}(x, {}_{\text{con}}z) \geq \tilde{U}(v) \quad (\text{resp.} \leq \tilde{U}(v)),$$

RC implies that $\tilde{V}$ is continuous in (u, w) on $\tilde{U}(Y)^2$.

Hence, on the set of streams in $\mathbf{X}$ of the form $(x_0, x_1, {}_{\text{con}}v)$, $\succsim$ is represented by $\tilde{W}(x_0, x_1, {}_{\text{con}}v) = \tilde{V}(x_0, \tilde{W}(x_1, {}_{\text{con}}v)) = \tilde{V}(x_0, \tilde{V}(x_1, \tilde{U}(v)))$, which is continuous in (x_0, x_1, v) on Y^3. Since $\succsim_0^{\mathbf{z}}$, $\succsim_1^{\mathbf{z}}$, and ${}_2\succsim^{\mathbf{z}}$ are sensitive (in the case of ${}_2\succsim^{\mathbf{z}}$ also within the set of constant streams, by **O**, **M**, and **WS**), and $\succsim_0^{\mathbf{z}}$, ${}_1\succsim^{\mathbf{z}}$, ${}_0\succsim_1^{\mathbf{z}}$, ${}_2\succsim^{\mathbf{z}}$, and $\succsim_1^{\mathbf{z}}$ are all independent of ${}_0\mathbf{z}$, it now follows from standard results for additively separable representations (Debreu, 1960; Gorman, 1968; Koopmans, 1986a) that there exist continuous functions $U_0 : Y \to \mathbb{R}$, $U_1 : Y \to \mathbb{R}$, and $U : Y \to \mathbb{R}$, such that $W_0 : \{{}_0\mathbf{x} \in \mathbf{X} \mid x_t = v \text{ for all } t \geq 2\} \to \mathbb{R}$ defined by

$$W_0(x_0, x_1, {}_{\text{con}}v) = U_0(x_0) + U_1(x_1) + U(v) \tag{4}$$

is an SWF. By repeated applications of **IF**, it follows from Lemma 1 that W_0 can be extended to all ${}_0\mathbf{x} \in \mathbf{X}$:

$$W_0({}_0\mathbf{x}) = U_0(x_0) + U_1(x_1) + U(W^*({}_2\mathbf{x})),$$

where $W^* : \mathbf{X} \to Y$ maps any ${}_0\mathbf{y} \in \mathbf{X}$ into *some* $z \in Y$ satisfying ${}_{\text{con}}z \sim {}_0\mathbf{y}$. It follows from **IF** that $W_1 : \mathbf{X} \to \mathbb{R}$ defined by

$$W_1({}_0\mathbf{x}) = U_1(x_0) + U(W^*({}_1\mathbf{x})),$$

is also an SWF. The additively separable structure between time 0 and times $1, 2, \ldots$ means that, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W_1({}_0\mathbf{x}) = \delta W_0({}_0\mathbf{x}) + \epsilon$, $U_1(x_0) = \delta U_0(x_0) + \epsilon$, and

$$U(W^*({}_1\mathbf{x})) = \delta\big(U_1(x_1) + U(W^*({}_2\mathbf{x}))\big) + \epsilon\,. \tag{5}$$

Furthermore, by inserting ${}_{\text{con}}z$ in (5) and keeping in mind that $U(W^*({}_{\text{con}}z)) = U(z)$, we obtain $U(z) = \delta\big(U_1(z) + U(z)\big) + \epsilon$, or equivalently,

$$U_1(z) = \tfrac{1-\delta}{\delta}U(z) - \tfrac{\epsilon}{\delta} \tag{6}$$

for all $z \in Y$. By defining $W : \mathbf{X} \to \mathbb{R}$ by, for all ${}_0\mathbf{x} \in \mathbf{X}$, $W({}_0\mathbf{x}) = U(W^*({}_0\mathbf{x}))$, it follows from (5) and (6) that the SWF W satisfies $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ for all ${}_0\mathbf{x} \in \mathbf{X}$, where $\delta \in (0,1)$ since both $\succsim_0^{\mathbf{z}}$ and ${}_1\succsim^{\mathbf{z}}$ are sensitive. By **M**, W is monotone and U is non-decreasing. By **WS**, $U(Y)$ is not a singleton; hence, $U \in \mathcal{U}$.

If **WS** is strengthened to **RD**, then it follows from (3), (4), and repeated applications of **IF** that $U(Y)$ is increasing; hence, $U \in \mathcal{U}_I$.

Part II: (2) implies (1). Assume that the monotone mapping $W : \mathbf{X} \to \mathbb{R}$ is an SWF and satisfies, for some $U \in \mathcal{U}$ and $\delta \in (0,1)$, $W({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W({}_1\mathbf{x})$ for all ${}_0\mathbf{x} \in \mathbf{X}$. Note that, for each $U \in \mathcal{U}$ and each $\delta \in (0,1)$, $V : U(Y)^2 \to \mathbb{R}$ defined by $V(u,w) = (1-\delta)u + \delta w$ is an element of $\mathcal{V}(U)$; hence,

$$\{V : U(Y)^2 \to \mathbb{R} \mid V(u,w) = (1-\delta)u + \delta w \text{ for some } \delta \in (0,1)\} \subseteq \mathcal{V}(U)\,.$$

Also, $W({}_{\text{con}}z) = (1-\delta)U(z) + \delta W({}_{\text{con}}z)$ implies $W({}_{\text{con}}z) = U(z)$. Hence, by Proposition 2, if $U \in \mathcal{U}_I$, it remains to be shown that the SWR $\succsim$, represented by the SWF W, satisfies **IP**. The following argument shows that $\succsim$ satisfies **IP**.

Let ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z}$, ${}_0\mathbf{v} \in \mathbf{X}$, and let $(x_0, x_1)\ {}_0\succsim_1^{\mathbf{z}} (y_0, y_1)$, or equivalently, $W(x_0, x_1, {}_2\mathbf{z}) \geq W(y_0, y_1, {}_2\mathbf{z})$. We have to show that $(x_0, x_1)\ {}_0\succsim_1^{\mathbf{v}} (y_0, y_1)$, or equivalently, $W(x_0, x_1, {}_2\mathbf{v}) \geq W(y_0, y_1, {}_2\mathbf{v})$. By the properties of W,

$$\begin{aligned} W(x_0, x_1, {}_2\mathbf{z}) - W(y_0, y_1, {}_2\mathbf{z}) &= (1-\delta)\big[\big(U(x_0) - U(y_0)\big) + \delta\big(U(x_1) - U(y_1)\big)\big] \\ &= W(x_0, x_1, {}_2\mathbf{v}) - W(y_0, y_1, {}_2\mathbf{v})\,, \end{aligned}$$

since $W({}_0\mathbf{x}') = (1-\delta)\big(U(x_0') + \delta U(x_1')\big) + \delta^2 W({}_2\mathbf{x}')$ for all ${}_0\mathbf{x}' \in \mathbf{X}$.

If $U \in \mathcal{U}\backslash\mathcal{U}_I$, then above analysis goes through, except that it does not follow that the SWR $\succsim$ satisfies **RD**. Instead, the property that $U(Y)$ is not a singleton implies that SWR $\succsim$ satisfies **WS**. ■

Proof of Proposition 10. Fix $U \in \mathcal{U}$ and $\delta \in (0,1)$, and let ${}_0\mathbf{x} \in \mathbf{X}$, implying that there exist $\underline{y}$, $\bar{y} \in Y$ such that, for all $t \in \mathbb{Z}_+$, $\underline{y} \leq x_t \leq \bar{y}$.

Part I: Existence. For each $T \in \mathbb{Z}_+$, consider the following finite sequence:

$$
\begin{aligned}
&w(T,T) = U(\bar{y}) \\
&w(T-1,T) = (1-\delta)U(x_{T-1}) + \delta w(T,T) = (1-\delta)U(x_{T-1}) + \delta U(\bar{y}) \\
&\dots \\
&w(0,T) = (1-\delta)U(x_0) + \delta w(1,T) = (1-\delta)\sum\nolimits_{t=0}^{T-1}\delta^t U(x_t) + \delta^T U(\bar{y})
\end{aligned}
$$

Since $w(t,T)$ is non-increasing in T for given $t \le T$ and bounded below by $U(\underline{y})$, $\lim_{T\to\infty} w(t,T)$ exists for all $t \in \mathbb{Z}_+$. Define the monotone mapping $W_\delta : \mathbf{X} \to \mathbb{R}$ by

$$W_\delta({}_0\mathbf{x}) := \lim\nolimits_{T\to\infty} w(0,T) = (1-\delta)\sum\nolimits_{t=0}^{\infty}\delta^t U(x_t)\,.$$

As $w(0,T) = (1-\delta)U(x_0) + \delta w(1,T)$, we have that $W_\delta({}_0\mathbf{x}) = (1-\delta)U(x_0) + \delta W_\delta({}_1\mathbf{x})$.

Part II: Uniqueness. Suppose there exists a monotone mapping $W : \mathbf{X} \to \mathbb{R}$ satisfying $W({}_0\mathbf{y}) = (1-\delta)U(y_0) + \delta W({}_1\mathbf{y}))$ for all ${}_0\mathbf{y} \in \mathbf{X}$ such that $W({}_0\mathbf{x}) \neq W_\delta({}_0\mathbf{x})$. Since $W({}_t\mathbf{x}) - W_\delta({}_t\mathbf{x}) = \delta\big(W({}_{t+1}\mathbf{x}) - W_\delta({}_{t+1}\mathbf{x})\big)$ for all $t \in \mathbb{Z}_+$,

$$|W({}_T\mathbf{x}) - W_\delta({}_T\mathbf{x})| = \tfrac{1}{\delta^T}|W({}_0\mathbf{x}) - W_\delta({}_0\mathbf{x})| > U(\bar{y}) - U(\underline{y})$$

for some $T \in \mathbb{Z}_+$. However this contradicts that, for all $T \in \mathbb{Z}_+$,

$$U(\underline{y}) = W({}_{\text{con}}\underline{y}) \le W({}_T\mathbf{x}) \le W({}_{\text{con}}\bar{y}) = U(\bar{y})$$

(and likewise for $W_\delta({}_T\mathbf{x})$) by the facts that W is monotone and $W({}_{\text{con}}z) = (1-\delta)U(z) + \delta W({}_{\text{con}}z)$ implies $W({}_{\text{con}}z) = U(z)$. ■

Proof of Proposition 11. Assume that the SWR $\succsim$ satisfying conditions **O**, **RC**, **IF**, **M**, **IP**, and **NDF**. By Proposition 1, **O**, **IF**, and **NDF** imply **WS**. Hence, by Propositions 9 and 10, the SWR $\succsim$ is represented by $W_\delta : \mathbf{X} \to \mathbb{R}$ defined by, for each ${}_0\mathbf{x} \in \mathbf{X}$,

$$W_\delta({}_0\mathbf{x}) = (1-\delta)\sum\nolimits_{t=0}^{\infty}\delta^t U(x_t)\,,$$

for some $U \in \mathcal{U}$ and $\delta \in (0,1)$. This implies **DP**, thus contradicting **NDP**. ■

Proof of Proposition 12. Asheim and Tungodden (2004), Basu and Mitra (2005), and Bossert, Sprumont and Suzumura (2005) define different kinds of incomplete leximin and undiscounted utilitarian SWRs, each of which is given an axiomatic characterization. Denote by $\succsim$ one such incomplete SWR. It can be verified that $\succsim$ is reflexive, transitive and satisfies **IF**, **SP**, **HEF** (with $(x, {}_{\text{con}}z) \precsim (y, {}_{\text{con}}v)$ if $x > y > v > z$), and **IP**. Completeness (and thereby condition **O**) can be satisfied by invoking Arrow's (1951) version of Szpilrajn's (1930) extension theorem (see also Svensson, 1980).

Since $\succsim$ satisfies conditions **SP** and **HEF** (with $(x, {}_{\text{con}}z) \precsim (y, {}_{\text{con}}v)$ if $x > y > v > z$), so will any completion. Since, for all ${}_0\mathbf{x}$, ${}_0\mathbf{y}$, ${}_0\mathbf{z} \in \mathbf{X}$, $(x_0, x_1)\ {}_0\!\succsim_1^{\mathbf{z}}\ (y_0, y_1)$ or $(x_0, x_1)\ {}_0\!\precsim_1^{\mathbf{z}}\ (y_0, y_1)$, and $\succsim$ satisfies **IP**, so will any completion. However, special care must be taken to ensure that the completion satisfies **IF**.

Consider $\mathbf{X}_0^2 = \{({}_0\mathbf{x}, {}_0\mathbf{y}) \in \mathbf{X}^2 \mid x_0 \neq y_0\}$, and invoke Arrow's (1951) version of Szpilrajn's (1930) extension theorem to complete $\succsim$ on this subset of $\mathbf{X}^2$. For any $({}_0\mathbf{x}, {}_0\mathbf{y}) \in \mathbf{X}$ with ${}_0\mathbf{x} \neq {}_0\mathbf{y}$, let ${}_0\mathbf{x}$ be at least as good as ${}_0\mathbf{y}$ if and only if ${}_T\mathbf{x}$ is at least as good as ${}_T\mathbf{y}$ according to the completion of $\succsim$ on $\mathbf{X}_0^2$, where $T := \min\{t \mid x_t \neq y_t\}$. Since $\succsim$ satisfies **IF**, this construction constitutes a complete SWR satisfying **IF**. ■

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