# MEMORANDUM

**No 20/2004**

# Contract Renewal

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ISSN: 0801-1117

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University of Oslo

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| | |
|---|---|
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| No 11 | Rolf Golombek and Michael Hoel<br>Climate Agreements and Technology Policy. 38 pp. |
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# CONTRACT RENEWAL

HELGE HOLDEN, LARS HOLDEN, AND STEINAR HOLDEN

Abstract. Consider a contract between two players, describing the payment an agent obtains from the principal, in exchange for a good or service supplied. At each point in time, either player may unilaterally demand a renegotiation of the contract, involving renegotiation costs for both players. Players' payoffs from trade under the contract, as well as from a renegotiated contract, are stochastic, following the exponential of a Lévy process. It is argued that the optimal strategy for each player is to require a renegotiation when the contract payment relative to the outcome of a renegotiation passes a certain threshold, depending on the stochastic processes, the discount rate, and the renegotiation costs. There is strategic substitutability in the choice of thresholds, so that if one player becomes more aggressive by choosing a threshold closer to unity, the other player becomes more passive. If players may invest in order to reduce the renegotiation costs, there will be over-investment compared to the welfare maximizing levels.

## 1. Introduction

In most economies, a large part of the transactions take place within long-term relationships. Most workers stay in their job for many years. Some tenants rent the same dwelling for decades. Firms may trade with the same supplier for long periods. Usually, such long-term relationships are within a framework of long-term contracts, reducing the risk that either of the parties may be left without a trading partner on short notice.

However, even if the parties may gain from long-term relationships, economic circumstances, both internal and external, may change so as to make one or both parties unsatisfied with the contract. While contracts in principle might be written in such a detailed and foresighted manner that this should never happen, in practice contracts cannot cover all the complexities that may arise. Thus, in practice, contracts are incomplete and the terms need to be adjusted over time.

We consider the choice faced by the parties to a contract on whether and when to require a renegotiation of the contract. Clearly, if more favorable contract terms are feasible, requiring a renegotiation is attractive. However, contract renegotiation is not costless. Furthermore, obtaining a more favorable contract now may lead the opponent to demand a renegotiation in the future, involving both additional renegotiation costs and a less favorable contract.

*Date*: August 30, 2004.

2000 *Mathematics Subject Classification.* Primary: 91A10, 91A15; Secondary: 91B40.

*Key words and phrases.* Contract renewal, renegotiation, stochastic models, differential games.

JEL Classification: C72, C73, C78, E31.

We dedicate this paper to our parents.

In our framework, there are two players, a principal (P) and an agent (A). The agent performs a fixed service, or delivers a specific good, for a payment specified in a contract. Trade takes place in continuous time. The payoff to the parties from trading under the contract is given by a stochastic process.

At each point in time, either of the players may unilaterally demand a renegotiation of the contract, involving renegotiation costs for both players. One possible interpretation of this is that there is an explicit renegotiation clause in the contract. Another possible interpretation is that a player may enforce a renegotiation by credibly threatening to disrupt trade, even without an explicit renegotiation clause. This interpretation allows two alternatives. MacLeod and Malcomson [16] and Hart and Moore [11] assume that in case no trade takes place, a court cannot verify why (i.e., which party chose not to trade). Thus, if one party violates the contract, the other party cannot verify this for a court, and is thus unable to sue for damages. Alternatively, it may well be verifiable which party chooses not to trade, but the courts will not enforce any penalty provisions. This is the motivation of Grout [10]; the Trade Union Immunity Laws in the United Kingdom prevent an employer from suing a trade union to recover any losses during an industrial dispute, even if the dispute violates a previous agreement. This assumption is in contrast to the "specific performance contracts" analyzed in Aghion, Dewatripont, and Rey [1], where courts can enforce a level of trade specified in the contract.

A demand for renegotiation may be caused by changes in the "inside" or "outside option". First, players' payoff from trade under the contract may change, so that one of the players benefit from a renegotiation. We capture this effect by assuming, realistically, that the contract is set in nominal terms, so that the real value of the contract payment depends on the stochastic aggregate price level. Second, outside alternatives may change, which we capture by assuming that the outcome of a renegotiation is given by an exogenous stochastic process, known to both parties at the time when a renegotiation is demanded. (Thus, we do not go into details of the renegotiation process; see MacLeod and Malcomson [16] and Holden [12] for analysis of renegotiation of contracts of trades in continuous time.)

In principle, strategies may depend on anything that has happened in the history of the game, and thus be immensely complicated. To keep the analysis tractable, we follow the tradition of the differential games literature (see Isaacs [14] and Dockner, Jørgensen, Van Long, and Sorger [8]) of restricting attention to Markov strategies, i.e., strategies where actions are allowed to depend on past history through the current value of the state variables only.

The model we consider is simple; two players, trading with each other under a contract, the payoff from trade being stochastic, and the only choice variable is when to demand a renegotiation. Yet the decision problem facing the players is very complex. When deciding whether to require a renegotiation, a player must weigh the gain from a possible improvement in contract terms against the costs of a renegotiation. However, the player must also take into consideration that a renegotiation now, making the contract terms more favorable to himself, will also make the contract less favorable to the opponent. This may cause the opponent to require a renegotiation at an earlier point in time than he otherwise would have done, involving both renegotiation costs and less favorable contract terms for the first player.

We show that the optimal strategies of the players are given by critical thresholds for the ratio of the real value of the contract payment relative to the real value of a renegotiated contract. Thus, the agent will demand a renegotiation whenever this ratio is below the agent's threshold, irrespective of whether this is caused by high inflation eroding the real value of the contract payment, or by an increase in the real payment that may be achieved by a renegotiation. Conversely, the principal will require a renegotiation whenever the ratio of the real contract payment to the real renegotiation payment is above the principal's threshold.

The thresholds depend, among other things, on the costs of renegotiation. As expected, we show that the higher the costs of renegotiation for a player, the more passive is the player, i.e., the farther is the critical threshold from unity.

Our key result is that there is strategic substitutability in the choice of threshold values, so that if one player becomes more aggressive (i.e. setting a threshold closer to unity), the opponent will become more modest (i.e. setting a threshold farther from unity). The intuition for this result is that if a player becomes more aggressive, the expected time until this player requires a renegotiation is reduced. Thus, the expected duration of a change in contract terms induced by the opponent is reduced, which makes it less attractive for the opponent to require a renegotiation.

One implication of strategic substitutability is that asymmetries between the players may be exacerbated. For example, if requiring a renegotiation becomes less costly for one of the players, making this player more aggressive, the strategic effect will make the opponent more passive. As the opponent becomes more passive, the first player becomes even more aggressive, exacerbating the direct effect of reduced renegotiation costs. Numerical examples suggest that the strategic effect in some cases may be substantial.

The paper is related to a considerable literature which studies the optimal choice of nominal prices (or wages) under a stochastic evolution of money or aggregate prices (see, e.g., Sheshinsky and Weiss [19], Danziger [7], Caplin and Spulber [6], and Caplin and Leahy [5]). As in our paper, adjustment of the nominal price is costly. As in our model, optimal behavior is typically characterized by threshold strategies, often termed (S,s) strategies, where the prevailing price is changed if it is sufficiently far from the optimal new price, so that the gain from adjustment covers the adjustment costs. However, in this literature, the price is set unilaterally by the firm, avoiding the complexities arising from the fact that a renegotiation requested by one player now may cause the opponent to demand a renegotiation at an earlier stage than he otherwise would have done, inflicting additional costs on both players.

The model we consider has close similarities to the model studied in Andersen and Christensen [3]. As in our model, the model of Andersen and Christensen involves trade in continuous time between two players according to a given contract, where each player at each point in time may require a renegotiation of the contract. The outcome of the renegotiation is assumed to be given by a geometric Brownian motion. However, an important limitation of Andersen and Christensen [3] is that it only allows for one contract renegotiation, implying that if one player has required a renegotiation, this option is no longer open to the other player. Thus, Andersen and Christensen find strategic complementarity in players' contract renegotiation decisions, i.e., that the more reluctant one player is to demand a renegotiation, the more reluctant the opponent will be, in contrast to our finding of strategic substitutability.

In Andersen and Christensen [2] the model is extended to a large but finite number of contract renewals, and the model is solved by use of backward induction from the last possible contract renewal. However, it is not stated whether the strategic complementarity also holds in this case.

The remainder of the paper is organized as follows. The basic model is described in Section 2. Section 3 analyzes the case when both players use critical thresholds. In Section 4, we assume that the stochastic environment is continuous, and we prove the existence of a Nash equilibrium. In Section 5, we allow for discontinuities in the stochastic environment, and show that in this case equilibrium may involve randomization. We show that in some specific cases, Nash equilibrium requires that one of the players uses a mixed strategy, in the sense that the player randomizes between two threshold values. Technically, randomization may follow when the best reply function of one player (i.e., the optimal threshold as a function of the threshold of the opponent) is a discontinuous function. In the example we consider, there is a possibility that the real renegotiation payment may take a large fall, which may lead the principal to require a renegoation. In this case, the agent is faced with the choice of whether to set a "low" threshold, potentially allowing for a "low" real contract payment, but with the advantage that when the real contract payment is "low", a fall in the real renegotiation payment will not induce a renegotiation. Alternatively, the agent may set a higher threshold, preventing a low contract payment, but implying that a fall in the real renegotiation payment will induce an immediate and costly renegotiation. For some parameter values, the agent will randomly choose one of the two strategies. In Section 6, we extend the basic model by allowing for a stage prior to the basic model, where players may invest in reducing the renegotiation cost, and we consider the efficiency of this investment decision. Section 7 discusses the case where only one player is allowed to demand a renegotiation of the contract. An approximate formula for the equilibrium is given in Section A. Section 8 summarizes some of the main results. Proofs are provided in Section 9.

## 2. The model

Consider a contract between two players, according to which one player, the agent, supplies a good or undertakes a service for the other player, the principal, receiving in exchange a payment from the principal. Trade takes place continuously, and the contract specifies a nominal payment. The nominal contract payment set at $t_i$ can be viewed as the product of two components, $Z(t_i,\omega)Q(t_i,\omega)$, where $Z(t_i,\omega)$ is the real value of the payment set at time $t_i$, and $Q(t_i,\omega)$ is the aggregate price level at $t_i$. We will refer to $Z(t,\omega)$ as the real renegotiation payment, reflecting that $Z$ is that real payment that will be agreed upon in a renegotiation. The parameter $\omega$ denotes that $Z(t,\omega)$ and $Q(t,\omega)$ are stochastic; specifically they are exogenous stochastic processes, see below. At each time $t$, the contemporaneous values $Z(t,\omega)$ and $Q(t,\omega)$ are known to the players, but the future values $Z(s,\omega)$ and $Q(s,\omega)$ for $s > t$ are unknown.

Let the times of contract renegotiations be denoted $t_1, t_2, \dots$. The real value of the contract payment at time $t$, where $t_{i+1} > t > t_i$, is found by deflating the nominal contract payment by the aggregate price level at time $t$; we shall refer to this as the real contract payment $R(t,\omega) = Z(t_i,\omega)Q(t_i,\omega)/Q(t,\omega)$. Players' flow payoffs are constant elasticity functions of the real contract payment, so that $R^{\eta_\nu}$ is the flow payoff of the agent ($\nu = a$) and the principal ($\nu = p$). Clearly, $\eta_a > 0$ and $\eta_p < 0$ so that the agent prefers a high real payment, and the principal prefers a low real payment[1].

At any point $t$ in time, either player may require a renegotiation of the payment specified in the contract, paying a fee that is proportional to the new real payment $Z(t,\omega)$. Specifically, the renegotiation fee is $\tau_\nu Z^{\eta_\nu}(t,\omega)$, where $\tau_\nu$ is assumed to be strictly positive, deterministic and independent of which player is initiating the renegotiation[2]. See, however, observation II in Section 4.

One may argue that when the outcome of the renegotiation is known to the players in advance, the renegotiation costs, which reflect time and uncertainty associated with reaching a new agreement, should be negligible. However, it is straightforward, but cumbersome, to extend the model so that players at time $t$ only know the expected outcome of a renegotiation at time $t$, and where the actual renegotiation outcome at time $t$ is stochastic. The expected outcome may either be a function of the previous renegotiation outcome, i.e., for $t > t_i$ the expected outcome in real terms is described by $E\{Z(t,\omega) \mid Z(t_i,\omega)\} = \exp(c(t-t_i))Z(t_i,\omega)$ for a constant $c$, or the expected outcome may be a stochastic process similarly to $Z$ and $Q$ in the presented model. Under both alternatives, the qualitative results would be unaffected.

The overall objective function of the players is the discounted sum of flow payoffs

$$U_\nu(t_1,\dots,\omega) = \int_0^\infty R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds - \tau_\nu \sum_{j=0}^{\infty} Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1}) \tag{1}$$

where the discount rate $\beta > 0$. To avoid unimportant additional constants, we normalize by setting $R(0,\omega) = Z(0,\omega) = 1$, and $t_0 = 0$. The players choose the times for renegotiation in order to maximize their objective functions.

[1]The payoff functions are motivated by a union-firm setting, with a constant elasticity production function with labor as the only input, a constant elasticity of demand, the union's payoff an isoelastic function of the real wage, and the payoff of the firm an isoelastic function of the real profits. Note that as $\eta_a > 0$ and $\eta_p < 0$, the model is not symmetric. However, by use of the same method, it can be shown that the analysis and results would be qualitatively the same in a symmetric model where the flow payoff of the principal is $-R^{\eta_p}$, and $\eta_p = \eta_a > 0$.

[2]Proportional renegotiation fees, adjusted for the constant elasticity $\eta_\nu$, yield tractable solutions. In a labor contract, renegotiation costs may reflect time spent on bargaining, and the real contract payment (i.e., the real wage) seems an appropriate measure of the costs of time.

To ensure a high degree of generality, we assume that the real renegotiation payment and the aggregate price are given by the exponential of a Lévy process. Thus we assume that $Z(s,\omega) = \exp(F(s,\omega))$ and $Q(s,\omega) = \exp(G(s,\omega))$ where $F, G$ are Lévy processes. Lévy processes include geometric Brownian motion, jump processes that follow a Poisson distribution and many other stochastic processes that are, e.g., asymmetric or have heavier tails. For the benefit of the reader we recall the definition of a Lévy process (see, e.g., Sato [18]).

**Definition 2.1** (Lévy process)**.** *A stochastic process $X_t$ is a Lévy process provided the following conditions hold:*
*(i) For any $n$ and any $0 \le t_0 < \cdots < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.*
*(ii) $X_0 = 0$ almost surely.*
*(iii) The distribution of $X_{s+t} - X_s$ is independent of $s$.*
*(iv) The process is stochastically continuous, i.e., $\lim_{t\downarrow 0} \operatorname{Prob}(|X_t| > \epsilon) = 0$ for all $\epsilon > 0$.*
*(v) The process is right-continuous with left limits.*

For Lévy processes we have the Lévy–Khintchine formula for the characteristic function of $X_t$ (see, e.g., Sato [18])

$$E\{\exp(i\lambda X_t)\} = \exp\Big(t\big(i\alpha\lambda - \tfrac{1}{2}\lambda^2 a^2 + \int_{-\infty}^{\infty} (e^{i\lambda x} - 1 - i\lambda x \chi_{\{|x|\le 1\}}(x)) d\sigma(x)\big)\Big), \tag{2}$$

where $d\sigma$ is a $\sigma$-finite measure, denoted the Lévy-measure, with $\sigma(\{0\}) = 0$ and $\int_{-\infty}^{\infty} \min(|x|^2, 1) d\sigma(x) < \infty$. The process is uniquely defined by the quantities $(\alpha, a, d\sigma)$. The measure $\sigma$ describes the size and intensity of the jumps in the process. The process is Gaussian if and only if $\sigma = 0$, and in that case, $\alpha$ denotes the drift and $a$ the volatility. If $\sigma$ satisfies

$$\int_1^{\infty} e^{\eta x} d\sigma(x) < \infty,$$

we may conclude that

$$E\{\exp(\eta X_t)\} = \exp\Big(t\big(\alpha\eta + \tfrac{1}{2}\eta^2 a^2 + \int_{-\infty}^{\infty} (e^{\eta x} - 1 - \eta x \chi_{\{|x|\le 1\}}(x)) d\sigma(x)\big)\Big)$$

holds and is finite.

To ensure that the objective functions are finite, it is necessary to bound $Z$ and $Q$ relative to the discount rate $\beta$. This requires two additional assumptions. First, we assume that the volatility of the non-gaussian part is bounded, by assuming that the Lévy-measures satisfy

$$\int_1^{\infty} e^{\eta_h x} d\sigma_h(x) < \infty, \quad h = z, q$$

for some $\eta_h$. We may then define the drift in the processes by

$$\mu_{\nu,h} = \alpha_h \eta_\nu + \tfrac{1}{2}\eta_\nu^2 a_h^2 + \int_{-\infty}^{\infty} (e^{\eta_\nu x} - 1 - \eta_\nu x \chi_{\{|x|\le 1\}}(x)) d\sigma_h(x), \quad \eta_\nu \le \eta_h,$$

for $\nu = a, p$ and $h = z, q$. We have that $E\{Z^{\eta_\nu}(t,\omega)\} = \exp(t\mu_{\nu,z})$ and $E\{Q^{\eta_\nu}(t,\omega)\} = \exp(t\mu_{\nu,q})$. For example, $\mu_{a,z}$ is the expected rate of increase in the real renegotiation payment, adjusted for the relative rate of risk aversion of the agent, $\eta_a$.

The second assumption is that the drift parameters $\mu_{\nu,h}$ must be bounded by the discount rate $\beta$.

**Definition 2.2** (Property $\mathcal{F}$)**.** *We say that the stochastic price model has property $\mathcal{F}$ if the real renegotiation payment $Z(s,\omega) = \exp(F(s,\omega))$ and $Q(s,\omega) = \exp(G(s,\omega))$ where $F$ and $G$ are Lévy processes given by $(\alpha_z, a_z, d\sigma_z)$ and $(\alpha_q, a_q, d\sigma_q)$, respectively. Assume that there exists $\eta_h$ such that*

$$\int_1^\infty e^{\eta_h x} d\sigma_h(x) < \infty, \quad h = z, q.$$

*and consider $\eta_\nu \leq \eta_h$ for $\nu = a, p$ and $h = z, q$. Furthermore, we assume that*

$$\mu_{\nu,h} < \beta, \quad \nu = a, p,\ h = z, q.$$

Note that by assuming that payoff functions exhibit constant elasticity in the real contract payment $R$, and that the stochastic processes are given by the exponential of Lévy processes, we ensure that the situation is the same after each renegotiation, subject to a constant $Z(t,\omega)$. This property is crucial for the analysis, as it implies that the same strategies are optimal after each renegotiation.

The strategy of a player is defined as a description of the criteria applied when the player will require a renegotiation of the contract. In principle, strategies may depend on anything that has happened in the history of the game. However, we will follow the tradition of the differential games literature and restrict attention to Markov strategies where the players' choice of action only depend the state of the game. Thus, players may condition their play on the real contract payment $R$, the real renegotiation payment $Z$, or any combination of these variables. We do not allow players to condition their play on the opponent's play, except for any effect via the state variables $R$ and $Z$. For example, we do not consider strategies where players punish a rapid renegotiation by the opponent by another renegotiation, inflicting further renegotiation costs on both players.

The theorem below states that if one of the players uses a Markov strategy, there exists no strategy for the other player that gives higher expected values of the objective function than having a critical threshold for the ratio $R/Z$. Other variables like $R$ or $Z$ separately, calendar time or the time duration since the previous renegotiation, need not be used in the strategy.

**Theorem 2.3.** *Assume the stochastic price model satisfies property $\mathcal{F}$. Assume that one player uses a Markov strategy. Then there exists no strategy for the other player that gives a higher expected objective function than what is possible to obtain having a critical threshold for the ratio $R/Z$, i.e., require renegotiation whenever the real contract payment relative to real renegotiation payment $R/Z$ passes a specified value.*

Let $r_p$ and $r_a$ denote the critical thresholds. Clearly, the agent will demand a renegotiation if the real contract payment relative to real renegotiation payment of the contract is too low, so that $r_a < 1$, while the principal will demand a renegotiation if the real contract payment relative to real renegotiation payment is too high, so that $r_p > 1$. However, it does not follow that there exists a pair $(r_a, r_p)$ where $r_a$ is the optimal response to $r_p$ and $r_p$ is the optimal response $r_a$; in Section 5, a counterexample is given.

## 3. The model when both players have critical thresholds

In the previous section it was proved that when one player uses a Markov strategy, then the other player may obtain the maximum of his objective function by having a critical threshold. Thus, in this and the following sections we assume that both players have a critical threshold. Then we may give formulas for the expected objective functions and their derivatives. Figure 1 shows a realization of the process $R$ when both players use threshold strategies.

Define the expected values of the objective functions

$$u_\nu(r_a, r_p) = E\{U_\nu(r_a, r_p, \omega)\} \tag{3}$$

where $U_\nu(r_a, r_p, \omega)$, with a slight abuse of notation, is defined from (1) when the players have critical thresholds $r_a$ and $r_p$. Let $T(r_a, r_p, \omega)$ be the time of the first renegotiation given the thresholds $r_a$ and $r_p$, i.e., the first time after $t = 0$ that the contract payment relative to renegotiation payment is either equal or below $r_a$ or equal or above $r_p$, viz.,

$$T(r_a, r_p, \omega) = \inf\{t > 0 \mid R(t, \omega)/Z(t, \omega) \notin (r_a, r_p)\}.$$

Given the thresholds, define the expected contribution to the objective function of player $\nu$ from the start at $t = 0$ to the first contract renegotiation,

$$f_\nu(r_a, r_p) = E\{\int_0^{T(r_a, r_p, \omega)} R^{\eta_\nu}(s, \omega) \exp(-\beta s) ds\}.$$

The expected flow payoff just after the first renegotiation, discounted down to time $t = 0$, is defined by

$$h_\nu(r_a, r_p) = E\{Z^{\eta_\nu}(T(r_a, r_p, \omega), \omega) \exp(-\beta T(r_a, r_p, \omega))\}.$$

Note that in the special case where the real renegotiation payment $Z$ is a constant, $h_\nu$ is a pure discount factor. Note also that the second inequality in Definition 2.2 ensures that $h_\nu < 1$.

Then we may formulate the following theorem.

**Theorem 3.1.** *Assume the real renegotiation payment $Z$ and the aggregate price $Q$ satisfy property $\mathcal{F}$, and that the contract is renegotiated as soon as the contract payment relative to the renegotiation payment $R/Z$ exits the interval $(r_a, r_p)$. Then the following properties hold:*
*(i) The expected values of the objective functions immediately after a renegotiation satisfy*

$$u_\nu(r_a, r_p) = \frac{f_\nu(r_a, r_p) - \tau_\nu h_\nu(r_a, r_p)}{1 - h_\nu(r_a, r_p)}, \quad \nu = a, p \tag{4}$$

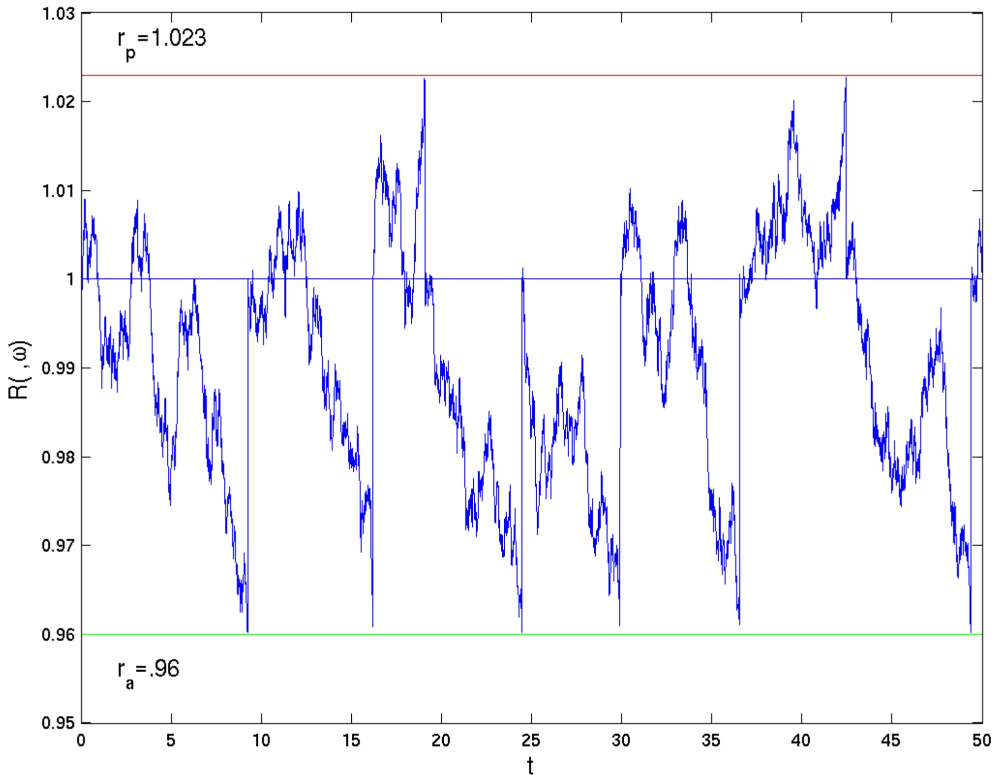


FIGURE 1. The figure shows one realization of the process $R(\cdot,\omega)$ in the case with $Z = 1$ and $Q$ is geometric (or exponential) Brownian motion $Q = \exp((\alpha_q - a_q^2/2)t + a_q B_t)$ with drift $\alpha_q = .002$ and volatility $a_q = .01$. Here $B_t$ denotes standard Brownian motion. When the unit of time is interpreted as one month, this corresponds to 2.4% annual inflation. The process is sampled at 5000 points.

*and are defined for* $0 \leq r_a < 1 < r_p \leq \infty$.
*(ii) The derivatives satisfy*

(5) $$\frac{\partial u_\nu}{\partial r_\mu} = \frac{\frac{\partial f_\nu}{\partial r_\mu} + (u_\nu - \tau_\nu)\frac{\partial h_\nu}{\partial r_\mu}}{1 - h_\nu}, \quad \nu = a, p, \quad \mu = a, p.$$

We make the following observations:

(I) The expressions on the right hand side of equation (4) can be computed numerically, and possibly also analytically, for specific stochastic processes.

(II) To facilitate the interpretation of equation (4), one may compare with the deterministic case where the contribution $f_\nu$ is deterministic between renegotiations, and where the real renegotiation payment $Z$ is a constant, implying that $h_\nu$ is a pure discount factor. Recalling the formula for the sum of an infinite geometric series $(a + ka + k^2a + \cdots = a/(1-k)$

when $|k| < 1$), we see that equation (4) is on the same form, where $a = f - \tau_\nu h_\nu$ is the payoff associated with the time interval between two renegotiations, including the discounted renegotiation costs, and $k = h_\nu$ is the discount factor.

(III) Equation (5) captures the opposing effects of increasing the thresholds: Increasing the threshold for, say, the agent, $r_a$, the expected time until the next renegotiation is reduced. This will reduce the expected payoff until the next renegotiation, i.e., $\frac{\partial f_\nu}{\partial r_a} < 0$. Furthermore, reducing the expected time until the next renegotiation raises the discount factor $\frac{\partial h_\nu}{\partial r_a} > 0$, reflecting that the renegotiation cost $\tau_\nu$ is incurred earlier, but also that the value of the objective function after a renegotiation $u_\nu$ is received earlier. The denominator reflects that in expected terms, all intervals between renegotiations are identical, so the effect of one interval is multiplied up.

## 4. Assuming the real renegotiation payment $Z$ and the aggregate price $Q$ are continuous

In order to prove the existence of a Nash equilibrium (see, e.g., Gibbons [9]) in a game where players set critical thresholds, it is necessary with additional definitions and assumptions. Define the expected value of the objective function of each of the players, given optimal response of this player, as

$$\begin{aligned} u_{m,a}(r_p) &= \sup_{r_a} u_a(r_a, r_p), \\ u_{m,p}(r_a) &= \sup_{r_p} u_p(r_a, r_p), \end{aligned}$$

and the optimal thresholds $m_a(r_p)$ and $m_p(r_a)$ as follows

$$\begin{aligned} m_a(r_p) &= \inf\{r_a \in [0,1) \mid u_a(r_a, r_p) = u_{m,a}(r_p)\}, \\ m_p(r_a) &= \sup\{r_p \in (1,\infty] \mid u_p(r_a, r_p) = u_{m,p}(r_a)\}. \end{aligned}$$

In general, the optimal threshold may not be unique, and the definitions above in this case pick the most lenient value, i.e., the value farthest from unity. However, in Theorem 4.1 below, we show that when $Z$ and $Q$ are continuous, then the optimal threshold is indeed unique. If it is optimal for a player never to require contract renegotiation, then $m_a(r_p) = 0$ (the agent) or $m_p(r_a) = \infty$ (the principal).

When $Z$ and $Q$ are continuous, $f_\nu$, $h_\nu$, and hence $u_\nu$ are all continuous by Theorem 3.1. Then $u_{m,\nu}$ are continuous and $m_\nu$ are well-defined, piecewise continuous and $u_a(m_a(r_p), r_p) = u_{m,a}(r_p)$ and $u_p(r_a, m_p(r_a)) = u_{m,p}(r_a)$. Furthermore, the Lévy measure vanishes, and $Z$ and $Q$ are geometric Brownian motions. We may then state the following theorem regarding uniqueness of the optimal value and the existence of an equilibrium point.

**Theorem 4.1.** *Assume the real renegotiation payment $Z$ and the aggregate price $Q$ are geometric Brownian motions, satisfying property $\mathcal{F}$. Then the following properties hold:*
*(i) The expected objective function for the agent, $u_a(r_a, r_p)$, is increasing in $r_p$, i.e., $\frac{\partial u_a}{\partial r_p} > 0$, while the expected objective function for the principal, $u_p(r_a, r_p)$, is decreasing in $r_a$, i.e.,*

$\frac{\partial u_p}{\partial r_a} > 0$.
*(ii) Given $r_p$, there exists a unique value $0 \leq r_a = m_a(r_p) < 1$ that maximizes $u_a(r_a, r_p)$. Correspondingly, given $r_a$, there exists a unique value $\infty \geq r_p = m_p(r_a) > 1$ that maximizes $u_p(r_a, r_p)$.*
*(iii) If $m_a(r_p) > 0$, then the function $m_a(r_p)$ is strictly increasing, and if $m_p(r_a) < \infty$, then the function $m_p(r_a)$ is strictly increasing.*
*(iv) If $m_a(r_p) > 0$, then $m_a(r_p)$ is strictly decreasing in $\tau_a$. Correspondingly, if $m_p(r_a) < \infty$, then $m_p(r_a)$ is strictly increasing in $\tau_p$.*
*(v) There is at least one Nash equilibrium point $(r_a^e, r_p^e)$ such that*

$$r_a^e = \text{argmax}_{r<1}\{u_a(r, r_p^e)\},$$
$$r_p^e = \text{argmax}_{r>1}\{u_p(r_a^e, r)\}.$$

Theorem 4.1 ensures that both players have unique best response functions in the form of thresholds $r_\nu$. Furthermore, there exists a Nash equilibrium in thresholds. It is possible to prove existence of a Nash equilibrium under weaker assumptions than $Z$ and $Q$ being continuous. The essential criterion is that $m_\nu$ are continuous. But this assumption leads to rather technical assumptions on $Z$ and $Q$.

Although we have not been able to construct cases with multiple Nash equilibria when $Z$ and $R$ are continuous, we have been unable to prove uniqueness of the Nash equilibrium in the general case. Thus, for each set of stochastic processes, it is necessary to verify that there is only one equilibrium point. Andersen and Christensen [2] prove that the equilibrium is unique in their model for a logGaussian price.

As expected, Theorem 4.1, (iv), shows that higher renegotiation costs make a player more reluctant to require a renegotiation, by pushing the threshold value $m_\nu$ further from unity. The interaction effects are, however, more interesting. First, part (i) show that if one player bcomes more aggressive (that is, has a threshold close to unity), this reduces the expected value of the objective function for the opponent. The opponent loses from both more frequent renegotiation costs and on average a less favorable contract payment.

Second, and more important, Theorem 4.1,(iii), identifies strategic substitutability in the choice of thresholds. This follows from the optimal thresholds $m_a(r_p)$ and $m_p(r_a)$ being increasing functions. If, in equilibrium, one player becomes more aggressive by choosing a threshold closer to unity, the other player becomes more passive by choosing a threshold further from unity. In other words, if, say, the renegotiation fee of the principal is reduced, the principal will respond by becoming more active, but this will induce the agent to become more passive. The intuition for this result is as follows. Demanding a renegotiation involves an immediate cost, and then a gain by a more favorable contract payment until the next renegotiation. If the opponent is aggressive, i.e. the threshold of the opponent is close to unity, the expected time until the next renegotiation is short, so that the gain from a more favorable contract will be shortlived. In contrast, if the opponent is more passive, with a threshold farther from unity, the gain from a more favorable contract is likely to last longer, making a renegotiation more attractive.

The strategic substitutability effect is in contrast to Andersen and Christensen [3], who find strategic complementarity in the choice of thresholds. Their result appears to be due to the fact that they consider only one contract renegotiation, implying an incentive for players to preempt the opponent. Thus, if one player is aggressive, the opponent has an incentive to also be aggressive, to increase the likelihood of being the player who obtains the advantage of asking for a renegotiation at a suitable moment. Andersen and Christensen [2] consider the model with a finite, but large number of contract renewals, but it is not stated whether the strategic complementarity holds in that model.

Other observations include:

(I) If the renegotiation payment $ZQ$ is monotone, then only one of the players will be active and the critical threshold of the other player is immaterial. Hence, there is no unique optimal strategy for this player. This is discussed in Section 7.

(II) The model may be generalized to the case where the renegotiation costs $\tau_a$ and $\tau_p$ depend on which player that requires contract renegotiation. In equations (4) and (5), this would require that $\tau_\nu$ is replaced by the expected value of the contract renegotiation fee, which again would be a function of $r_a$ and $r_p$. Theorem 4.1 is also valid in the generalized model, but in equations (19), (20), and (21) below, and the calculation leading to these equations, $\tau_\nu$ must be interpreted as the contract renegotiation fee when the agent requires a contract renegotiation. The model may also be generalized to allow for the renegotiation fees being stochastic, where $\tau_\nu$ is the expected value of the renegotiation fee.

(III) In special cases it is possible to find analytic expressions for some of the variables. Assume the real renegotiation payment relative to the real contract payment is given by a geometric Brownian motion $Z(t,\omega)/R(t,\omega) = Z(t,\omega)Q(t,\omega) = \exp((\alpha - a^2/2)t + aB_t)$ where $Z(0,\omega)Q(0,\omega) = 1$. Then (see Borodin and Salminen [4, p. 233, formula 3.0.1])

$$E\{\exp(-\beta T(r_a, r_p, \omega))\} = \big(r_a^\gamma(r_p^\sigma - r_p^{-\sigma}) - r_p^\gamma(r_a^\sigma - r_a^{-\sigma})\big)\big((r_p/r_a)^\sigma - (r_p/r_a)^{-\sigma}\big)^{-1}$$

with $\gamma = \alpha a^{-2} - 1/2$ and $\sigma = \sqrt{\gamma^2 + 2\beta a^{-2}}$. By differentiating this expression with respect to $\beta$ at $\beta = 0$ we find, where $\tilde\sigma = \sqrt{a^2 + 8\beta}/(2a)$,

$$\begin{aligned} E\{T(r_a, r_p, \omega)\} &= \frac{1}{a^2\gamma(r_a r_p)^{1/2}}\Big(\Big(\frac{r_p}{r_a}\Big)^{\tilde\sigma} - \Big(\frac{r_p}{r_a}\Big)^{-\tilde\sigma}\Big)^{-1} \\ &\quad \times \Big[\ln(r_a)(r_a^{\tilde\sigma+1/2} - r_a^{-\tilde\sigma+1/2}) - \ln(r_p)(r_p^{\tilde\sigma+1/2} - r_p^{-\tilde\sigma+1/2}) \\ &\qquad - \ln(r_p/r_a)\frac{\big(\frac{r_p}{r_a}\big)^{\tilde\sigma} + \big(\frac{r_p}{r_a}\big)^{-\tilde\sigma}}{\big(\frac{r_p}{r_a}\big)^{\tilde\sigma} - \big(\frac{r_p}{r_a}\big)^{-\tilde\sigma}} \\ &\qquad \times \Big((r_a^{\tilde\sigma+1/2} - r_a^{-\tilde\sigma+1/2}) + (r_p^{\tilde\sigma+1/2} - r_p^{-\tilde\sigma+1/2})\Big)\Big] \end{aligned}$$

is the expected time to the first renegotiation.

In Figures 2–4, we present how key variables depend on the critical threshold $r_a$ and $r_p$, treating the thresholds as exogenous. Note that in almost all simulations, we include a positive drift in the aggregate price level $Q$, representing inflation, implying a tendency that the real value of the contract payment, $R$, falls over time, relative to the real renegotiation

Table 1. The Nash equilibrium point for the example illustrated in Figures 2–4.

| $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|
| .960 | 1.023 | 196.7 | 200.8 | .65 | 9.7 |

Table 2. The Nash equilibrium points for various values of $\tau_\nu$. Drift $\alpha_q = .002$ and volatility $a_q = .01$. Other parameters as in Figure 2.

| $\tau_a$ | $\tau_p$ | $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|---|---|
| .05 | .05 | .973 | 1.020 | 197.3 | 199.3 | .64 | 5.3 |
| .07 | .05 | .968 | 1.017 | 196.3 | 200.3 | .58 | 5.9 |
| .05 | .07 | .976 | 1.026 | 198.0 | 197.9 | .74 | 5.7 |
| .07 | .07 | .971 | 1.023 | 197.1 | 199.1 | .69 | 6.7 |
| .13 | .13 | .965 | 1.029 | 196.4 | 198.9 | .75 | 9.9 |
| .13 | .23 | .970 | 1.044 | 197.6 | 195.9 | .87 | 10.9 |
| .23 | .13 | .951 | 1.024 | 193.7 | 201.7 | .64 | 12.1 |
| .23 | .23 | .957 | 1.035 | 195.2 | 199.0 | .78 | 13.7 |

payment, $Z$. Thus, it will usually be the agent who demands a renegotiation, unless the critical threshold of the principal is close to unity.

Figure 5 illustrates the game in setting thresholds. The curves show the best response functions $m_\nu$ for different values of renegotiation fees $\tau_\nu$. The intersections indicate Nash equilibrium for the appropriate renegotiation fees. We observe that higher renegotiation fee for one player leads to less aggressive play by this player, in the form of a threshold farther from unity. The strategic substitutability effect is also apparent: reducing, say, the renegotiation fee of the agent, so that we consider thin curves further to the right, involves higher thresholds $r_a$ for the agent, but in Nash equilibrium (represented by the intersections), also higher thresholds for the principal (i.e., lower values of $1/r_p$, indicating more passive play). The strategic effect varies between the different cases, but in some cases it is rather strong.

If we reduce the renegotiation costs of the agent, $\tau_a$, from .35 to .05, keeping $\tau_p$ constant at .35, $r_a$ increases from .950 to .985, implying that the agent now requires a renegotiation whenever he can increase the real contract payment by 1.5 percent, as opposed to a critical threshold of 5 percent before the change. Then the strategic effect implies that critical threshold of the principal increases, from a threshold at 4 percent reduction in real contract payment to a threshold of 8.2 percent reduction ($1/r_p$ falls from .960 to .918).

Comparison of Tables 2, 4–5 indicates that the threshold of the principal is an increasing function of the drift. This may reflect that when the drift is strong, there is less reason for the principal to demand a renegotiation even if he has been "unlucky" with the random movement, so that the contract payment is high relative to the renegotiation payment. The reason is that when the drift is strong, the disadvantageous period is unlikely to last long, so it is better for the principal to "let the drift do the job" than to incur the costs of a renegotiation. In contrast, when the drift is weak, the principal must use a lower threshold to avoid lengthy periods of a disadvantageous contract payment. This result is

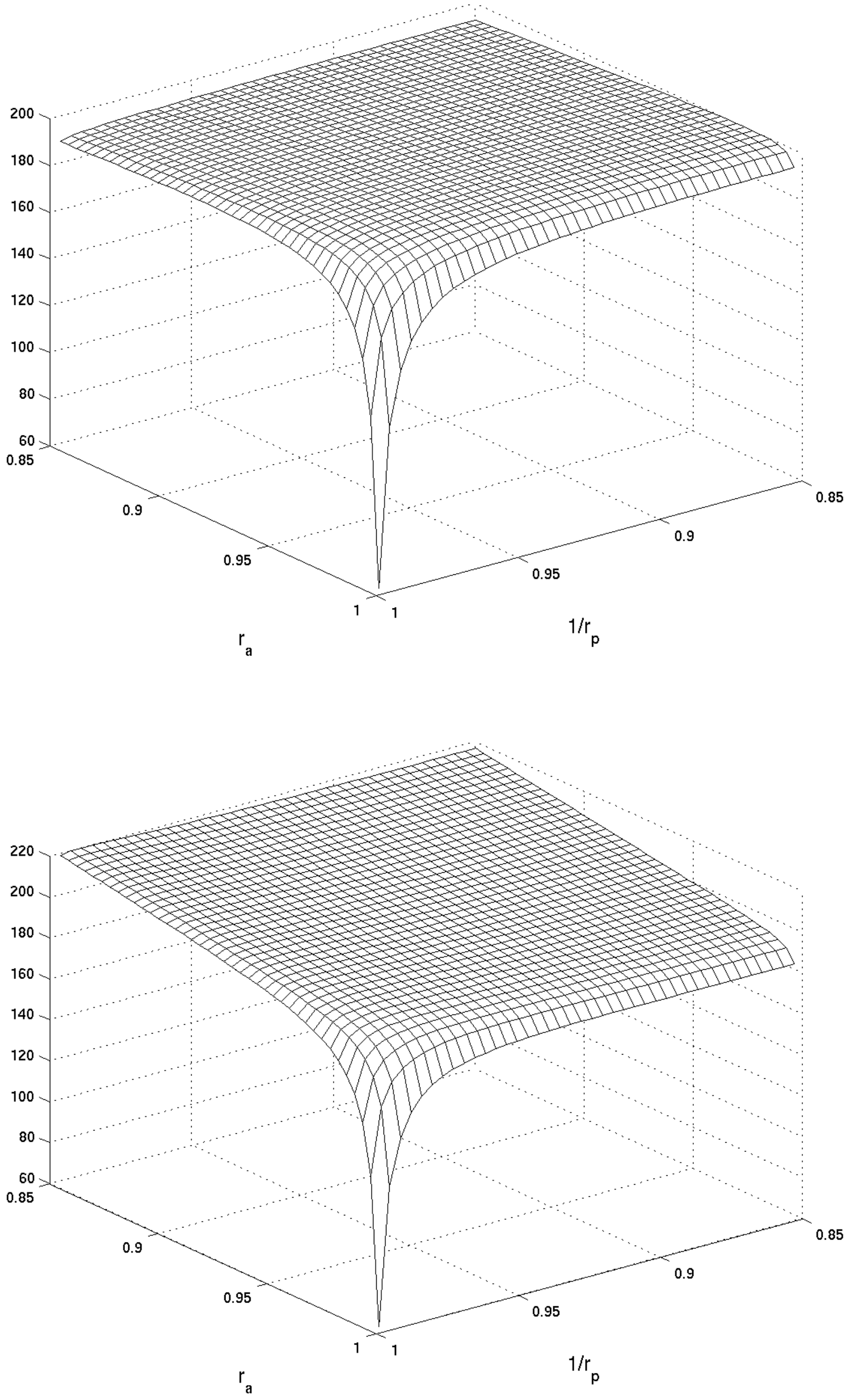


FIGURE 2. The function $u_a$ (top) and $u_p$ (bottom) for $r_a, 1/r_p \in [.85, 1]$ with the same process and parameters as in Figure 1. Furthermore, $\beta = .005$, $\eta_a = 1$, $\eta_p = -1.5$, and $\tau_a = \tau_p = .1$. The plots are computed using $10^5$ realizations, each sampled at $2 \cdot 10^5$ points up to time 200. Note that $u_a$ increases when $r_p$ increases. Because of the drift in the aggregate price level $Q$, there is a tendency that the real value of the contract payment falls over time, inducing the agent to demand a renegotiation. Thus, $r_a$ is more important for both $u_a$ and $u_p$ than $r_p$ is, except when $r_p$ is close to 1.

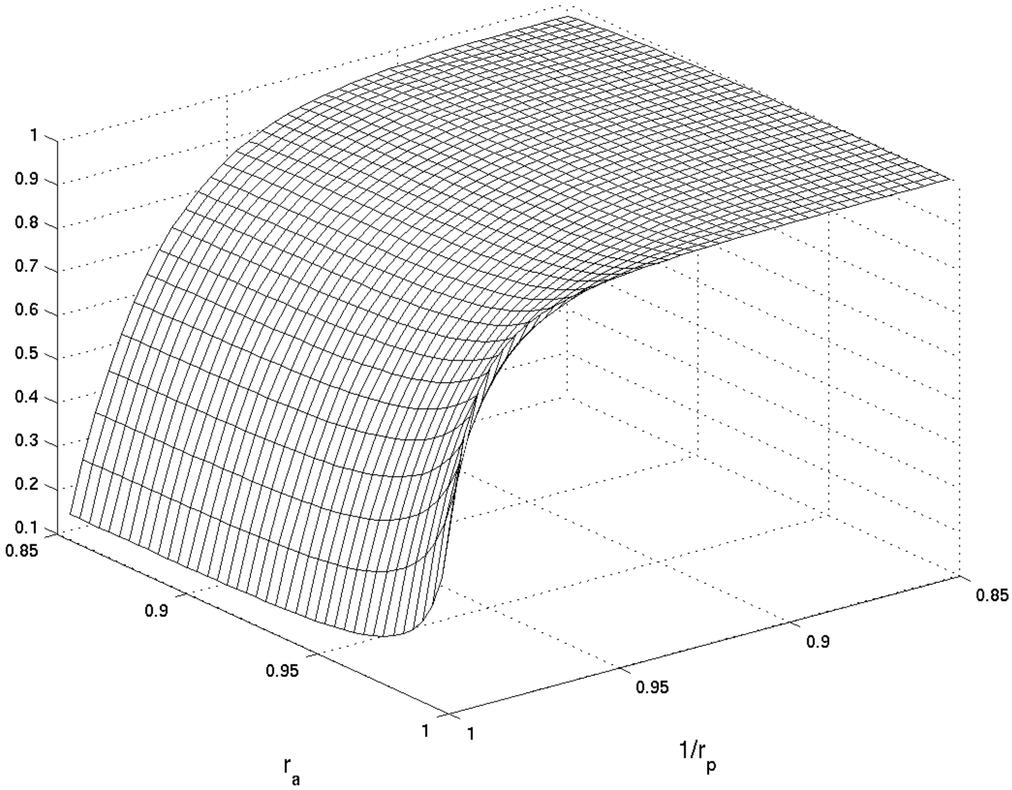


FIGURE 3. The function $P(r_a, r_p)$ is the expected fraction of times the process reaches $r_a$ before it reaches $r_p$, as a function of $(r_a, r_p)$. Parameters and processes as in Figure 2.

TABLE 3. The Nash equilibrium points for various values of $\tau_\nu$. Drift $\alpha_q = .002$ and volatility $a_q = .03$. Other parameters as in Figure 2.

| $\tau_a$ | $\tau_p$ | $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|---|---|
| .05 | .05 | .943 | 1.039 | 194.7 | 198.4 | .44 | 2.6 |
| .07 | .05 | .929 | 1.034 | 192.5 | 200.9 | .36 | 2.9 |
| .05 | .07 | .948 | 1.050 | 196.2 | 195.9 | .52 | 2.9 |
| .07 | .07 | .937 | 1.044 | 194.1 | 198.1 | .44 | 3.2 |
| .13 | .13 | .923 | 1.056 | 193.1 | 197.9 | .47 | 5.1 |
| .13 | .23 | .937 | 1.081 | 196.2 | 191.7 | .60 | 5.9 |
| .23 | .13 | .886 | 1.044 | 187.1 | 204.5 | .31 | 6.1 |
| .23 | .23 | .908 | 1.064 | 191.1 | 198.1 | .46 | 7.2 |

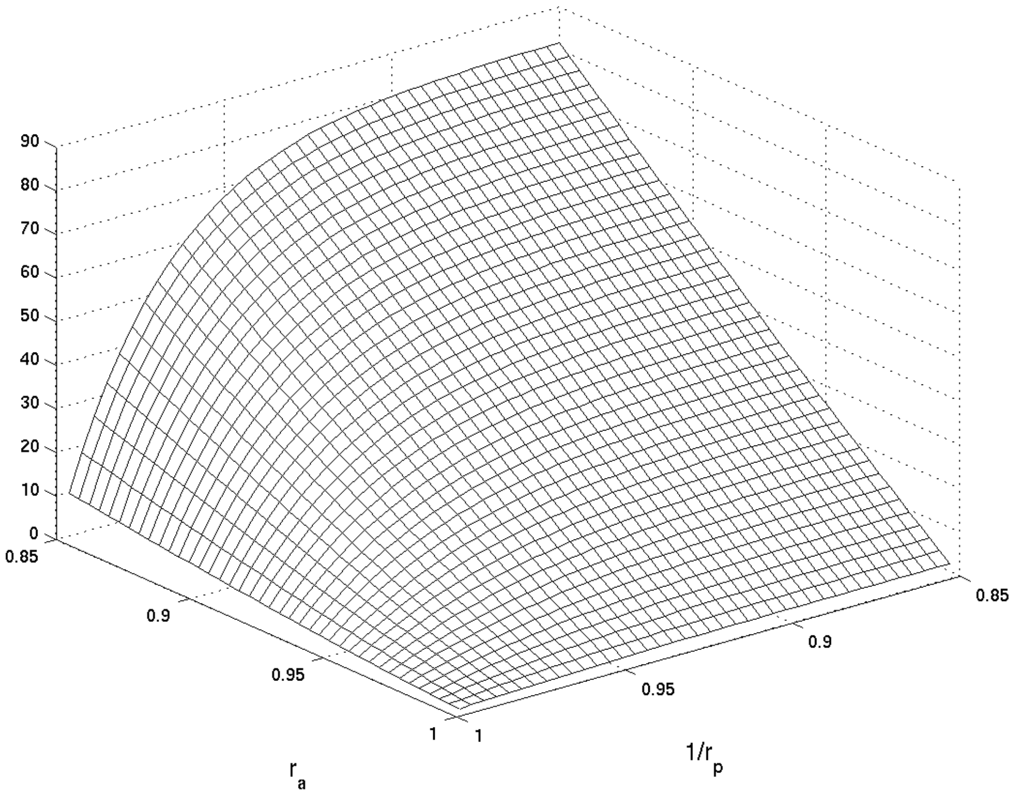


FIGURE 4. The function $E\{T(r_a, r_p, \omega)\}$ is the expected time until a renegotiation, i.e., until the process leaves the interval $(r_a, r_p)$ the first time. Parameters and processes as in Figure 2.

TABLE 4. The Nash equilibrium points for various values of $\tau_\nu$. Drift $\alpha_q = 0$ and volatility $a_q = .01$. Other parameters as in Figure 2.

| $\tau_a$ | $\tau_p$ | $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|---|---|
| .05 | .05 | .971 | 1.017 | 197.3 | 199.2 | .37 | 5.2 |
| .07 | .05 | .965 | 1.016 | 196.4 | 200.2 | .31 | 6.1 |
| .05 | .07 | .976 | 1.024 | 198.4 | 197.7 | .49 | 6.2 |
| .07 | .07 | .969 | 1.020 | 197.2 | 199.0 | .38 | 6.7 |
| .13 | .13 | .962 | 1.026 | 196.7 | 198.6 | .40 | 10.3 |
| .13 | .23 | .968 | 1.038 | 198.3 | 195.7 | .53 | 12.4 |
| .23 | .13 | .944 | 1.021 | 193.9 | 201.6 | .26 | 12.5 |
| .23 | .23 | .954 | 1.032 | 196.1 | 198.4 | .40 | 15.4 |

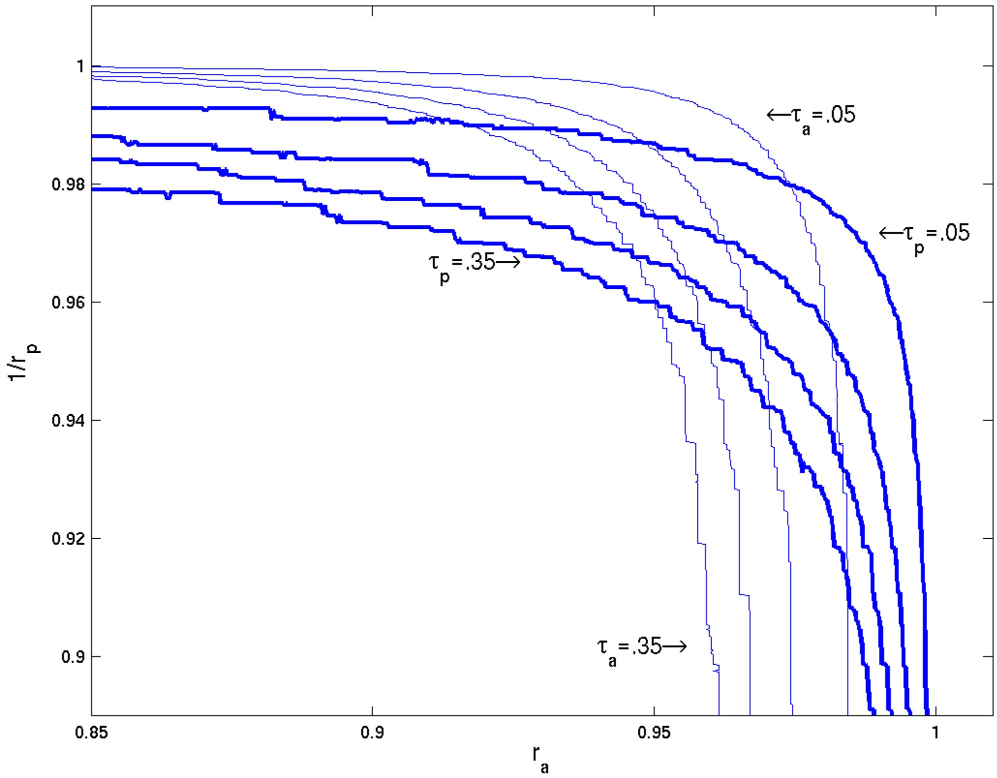


FIGURE 5. The best response function of the agent $m_a$ (thin curves) and principal $m_p$ (thick curves) for values of $\tau_a$ and $\tau_p$ from .05 to .35. Intersection between $m_a$ and $m_p$ gives the Nash equilibrium point $(r_a^e, r_p^e)$ for the particular set of $(\tau_a, \tau_p)$. Other parameters and process are as in Figure 2.

TABLE 5. The Nash equilibrium points for various values of $\tau_\nu$. Drift $\alpha_q = .006$ and volatility $a_q = .01$. Other parameters as in Figure 2.

| $\tau_a$ | $\tau_p$ | $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|---|---|
| .05 | .05 | .973 | 1.024 | 196.3 | 199.4 | .95 | 4.0 |
| .07 | .05 | .968 | 1.020 | 195.2 | 200.7 | .92 | 4.7 |
| .05 | .07 | .974 | 1.032 | 196.5 | 198.2 | .97 | 4.1 |
| .07 | .07 | .969 | 1.028 | 195.6 | 199.5 | .97 | 4.9 |
| .13 | .13 | .960 | 1.034 | 193.6 | 200.0 | .98 | 6.7 |
| .13 | .23 | .960 | 1.054 | 193.8 | 196.8 | .99 | 6.7 |
| .23 | .13 | .945 | 1.028 | 190.9 | 203.3 | .96 | 8.9 |
| .23 | .23 | .947 | 1.043 | 191.3 | 200.6 | .99 | 8.9 |

TABLE 6. The Nash equilibrium points for various values of $\tau_\nu$. Drift $\alpha_q = .002$, volatility $a_q = .01$, $\eta_a = .7$, and $\eta_p = -1.5$. Other parameters as in Figure 2.

| $\tau_a$ | $\tau_p$ | $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|---|---|
| .05 | .05 | .968 | 1.018 | 197.4 | 200.3 | .59 | 6.0 |
| .07 | .05 | .960 | 1.016 | 196.5 | 201.7 | .52 | 7.0 |
| .05 | .07 | .971 | 1.023 | 197.9 | 199.1 | .68 | 6.6 |
| .07 | .07 | .964 | 1.020 | 197.0 | 200.5 | .61 | 7.4 |
| .13 | .13 | .955 | 1.024 | 196.3 | 200.8 | .67 | 11.2 |
| .13 | .23 | .962 | 1.039 | 197.4 | 197.7 | .82 | 12.7 |
| .23 | .13 | .937 | 1.019 | 194.0 | 204.3 | .55 | 14.1 |
| .23 | .23 | .947 | 1.030 | 195.4 | 201.2 | .72 | 16.1 |

TABLE 7. The Nash equilibrium points for various values of $\tau_\nu$. Drift $\alpha_q = .002$, volatility $a_q = .01$, and $\eta_a = -\eta_p = 1.0$. Other parameters as in Figure 2.

| $\tau_a$ | $\tau_p$ | $r_a^e$ | $r_p^e$ | $u_a(r_a^e, r_p^e)$ | $u_p(r_a^e, r_p^e)$ | $P(r_a^e, r_p^e)$ | $E\{T(r_a^e, r_p^e, \omega)\}$ |
|---|---|---|---|---|---|---|---|
| .05 | .05 | .976 | 1.027 | 198.0 | 198.5 | .75 | 5.8 |
| .07 | .05 | .971 | 1.023 | 197.0 | 199.3 | .68 | 6.6 |
| .05 | .07 | .979 | 1.036 | 198.9 | 197.3 | .85 | 6.4 |
| .07 | .07 | .974 | 1.030 | 197.7 | 198.3 | .77 | 7.2 |
| .13 | .13 | .967 | 1.038 | 197.1 | 198.0 | .83 | 10.7 |
| .13 | .23 | .973 | 1.061 | 198.5 | 195.2 | .94 | 11.6 |
| .23 | .13 | .956 | 1.032 | 194.9 | 199.8 | .75 | 13.5 |
| .23 | .23 | .963 | 1.051 | 196.6 | 197.4 | .90 | 15.2 |

in contrast to the findings of Andersen and Christensen [3], where increased drift makes the principal more aggressive. Their result is probably due to their assumption of only one renegotiation; if there is drift that is disadvantagous to the principal, there will be less reason for the principal to postpone a renegotiation in the hope of a more favorable renegotiation a later stage. Indeed, in Andersen and Christensen [2], it is shown that the effect of drift is ambiguous in the case where it is allowed for many renegotiations.

The threshold of the agent is non-monotonic in the drift. This reflects two opposing effects. On the one hand, stronger drift implies that for a given threshold, renegotiations will be more frequent, so that renegotiation costs increase. To reduce the rise in renegotiation costs, the agent will be more reluctant to demand a renegotiation, thus the threshold is decreased. On the other hand, the strategic substitutability in the choice of thresholds implies that when higher drift increases the threshold of the principal, making him less aggressive, it also increases the threshold of the agent. Intuitively, the increasing threshold of the principal raises the possible gain for the agent of requiring a renegotiation, in the hope of obtaining an advantageous evolution of the contract payment.

Comparing Tables 2–3 indicates that greater volatility makes both players more reluctant to require a renegotiation, so that the threshold of the agent decreases, and the threshold of the principal increases, both further away from unity. However, the change is not so large that it prevents that the expected time between renegotiations falls. The intuition is straightforward: with greater volatility, thresholds close to unity will imply too frequent renegotiations, thus agents are less aggressive so as to reduce renegotiation costs. This result is the same as derived by Andersen and Christensen [3].

Table 6 shows the effect of reducing $\eta_a$, making the agent risk averse. Comparing with Table 2, we see that the threshold of the agent is reduced (further away from unity), while the threshold of the principal falls, i.e., becomes closer to unity. Thus, risk aversion makes the agent more reluctant to require a renegotiation. We also see that the principal obtains higher expected utility when the agent is risk averse, corresponding to the well-known result that it is advantageous to bargain with a risk averse player (see, e.g., Osborne and Rubinstein [17, p. 18]). Likewise, Table 7 shows the effect of reducing $\eta_p$, making the principal risk neutral rather than risk loving. This improves the situation for the agent, as the threshold of both players increases, making the agent more aggressive and the principal more passive, resulting in an increase in the expected utility of the agent. (Clearly, it is less relevant to consider the change in the expected utility for the player whose utility function changes.)

## 5. Discontinuities in the real renegotiation payment $Z$ or the aggregate price $Q$

If $Z$ or $Q$ are discontinuous and make occasional jumps, this may give discontinuities in the optimal responses $m_\nu$. In most cases, this will not affect the existence of Nash equilibrium with thresholds because the discontinuities in $m_\nu$ will usually be very small. However, under some circumstances jumps in $Z$ or $Q$ may imply that $m_a(r_p)$ and $m_p(r_a)$ do not intersect. Then there will be no Nash equilibrium with constant threshold strategies. However, there will exist a Nash equilibrium in mixed strategies, which is illustrated by the following stylized example.

**Example** Assume the real renegotiation payment $Z$ and the aggregated price level $Q$ are constants except for sudden jumps according to Poisson processes where the process $Q$ increases and $Z$ decreases. Let the Poisson process for $Z$ have low intensity and that $Z$ decreases with a fixed rate $1 + \rho$ in the jumps, while $Q$ has many small jumps and all jumps are according to a continuous distribution. The contract payment relative to the renegotiation payment $R/Z$ is then decreasing except for sudden jumps where it increases with the percentage $\rho$.

Consider the situation if the principal has chosen a threshold $r_p < 1 + \rho$. Then, if the real renegotiation payment $Z$ jumps immediately after a renegotiation, the contract payment relative to the renegotiation payment $R/Z$ will after the jump be above the critical threshold of the principal, inducing an immediate renegotiation. Thus, the agent will not benefit from a period with high $R/Z$ after the jump. On the other hand, if the agent let $R/Z$ fall below $r_p/(1 + \rho)$, a jump in the real renegotiation payment $Z$ will nevertheless leave $R/Z$ below the threshold of the principal. There will be no immediate renegotiation, and the agent will benefit from a period of high $R/Z$. This discontinuity at $r_p/(1 + \rho)$ will imply a discontinuity in $m_a$, i.e., in the optimal threshold of the agent.

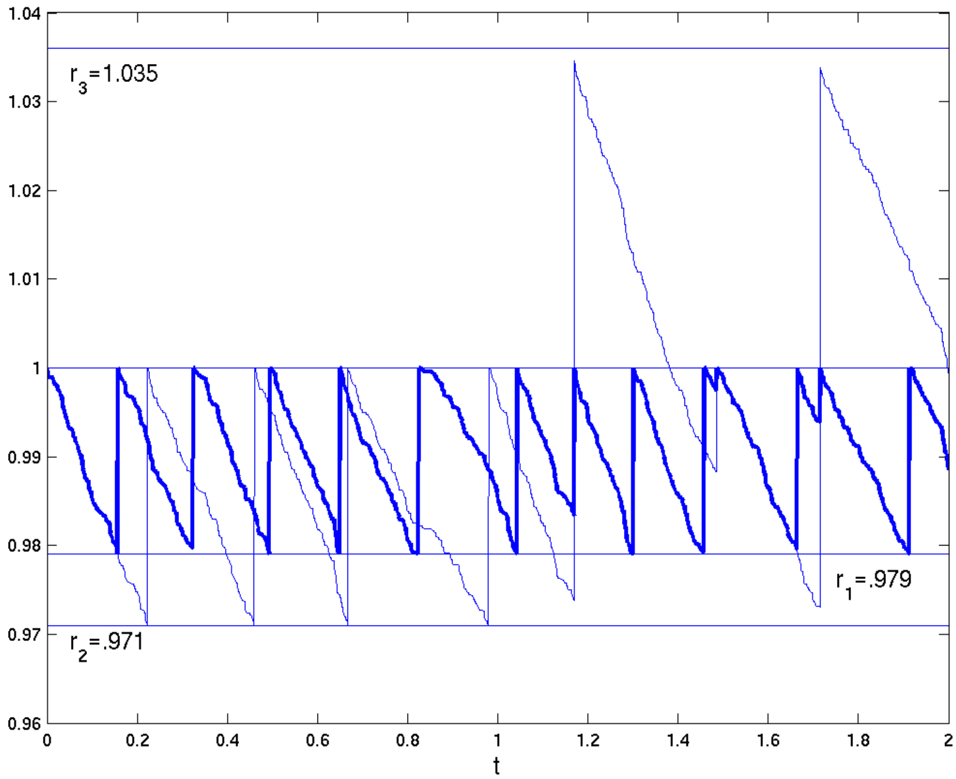


FIGURE 6. The figure shows a realization of $R(\cdot,\omega)/Z(\cdot,\omega)$ with thresholds $r_p = r_3 = 1.035$ and both $r_a = r_2 = .971$ and $r_a = r_1 = .979$. In a Nash equilibrium, the agent randomizes between these two values, the highest value gives renegotiation after every jump where $R/Z$ increases, while the lower threshold may give periods with high $R/Z$ values. The two independent Lévy processes $Z$ and $Q$ are constants except for sudden jumps according to a Poisson process where the process increases. The Poisson process for $Z$ has intensity .8 and in the jumps $Z$ increases with a factor 1.0625. The Poisson process for $Q$ has intensity approximately 1200 and in the jumps $Q$ increases according to a continuous distribution such that $E\log(Q(t,\omega)) \approx .13t$ and $\mathrm{Var}\{\log(Q(t,\omega))\} \approx .004t$. The process is modelled with time step .001.

When choosing the threshold, the agent will have to weigh the loss of allowing a low $R/Z$ (by having a low threshold) against the possible gain of a period with high $R/Z$ if there is a jump. However, if the threshold of the principal is "rather low", maintaining the possibility of reaping a period of high $R/Z$ will require a very low threshold for the agent. For sufficiently low threshold of the principal, the agent will then profit from neglecting the opportunity to benefit from a jump. At that point, the optimal threshold of the agent will make a jump, as there is now no gain to be reaped by having a low threshold.

The situation is illustrated in Figures 6 and 7. The agent mixes between two thresholds $r_1$ and $r_2$. Figure 6 shows that the real contract price relative to the renegotiation price falls monotonically, except when it is increased to unity at a renegotiation, or increased above unity when $Z$ falls. Figure 7 illustrates the strategic effects. For values of $1/r_p$ in the interval between .92 and .965, the optimal threshold of the agent, $r_a$ is reduced due to the strategic effect discussed in Section 4. However, $1/r_p > .965$, the agent is indifferent between choosing a low threshold $r_a \approx .971$, maintaining the possibility of benefitting from a jump in the real renegotiation payment, and a high threshold $r_a \approx .979$, which removes this possibility. In order to have a Nash equilibrium, the agent must mix between these two thresholds, with probabilities ensuring that it is indeed optimal for the principal to choose the threshold $1/r_p \approx .965$. For higher values of $1/r_p$, a jump in the real renegotiation payment will always induce a renegotiation request from the principal, implying that the agent sets $r_a \approx .979$.

Let us now consider the consequences of discontinuities in $Z$ and $Q$ more formally. Define $S_\nu$ as the class of strategies for a player $\nu$, where the player randomizes between two thresholds $r_1$ and $r_2$, where $r_1$ is chosen with probability $1-q$ and $r_2$ with probability $q$. Note that $S_\nu$ includes pure strategies, where $q = 0$. Let $s_\nu \in S_\nu$ denote a strategy. Furthermore, we assume that each time the contract payment relative to renegotiation payment is equal to unity or jumps from one side of unity to the other side of unity, either because the aggregate price or the real renegotiation payment fluctuates, or because a renegotiation has taken place, players select one of the two thresholds at random. This procedure ensures that past fluctuations of the aggregate price have no impact on the probability each player perceives of the thresholds of the opponent.

We extend the definition of the expected values of the objective functions $u_\nu$ to allow for randomization by both players. Furthermore, we define the optimal threshold for each player when the opponent randomizes:

$$m_a^c(s_p) = \inf\{r_a \in [0,1) \mid u_a(r_a, s_p) = \sup_{r_1} u_a(r_1, s_p)\},$$
$$m_p^c(s_a) = \sup\{r_p \in (1,\infty] \mid u_p(s_a, r_p) = \sup_{r_1} u_p(s_a, r_1)\}.$$

Thus, the function $m_p^c(s_a)$ corresponds to the usual optimal response function for the principal, $m_p(r_a)$, if the agent uses a pure strategy. However, $m_p^c(s_a)$ is also defined if the agent randomizes between thresholds $r_1$ and $r_2$, reflecting a discontinuity in $m_a(r_p)$. If $m_p^c(s_a)$ changes continuously from $m_p(r_1)$ to $m_p(r_2)$ when $q$ changes from 0 to 1, we say that $m_p^c(s_a)$ is continuous. Continuity of the function $m_a^c(s_p)$ is defined similarly.

In order to prove existence of a Nash equilibrium, we assume that $m_\nu$ is piecewise continuous and that $m_\nu^c$ is continuous in each of the discontinuities in $m_\nu$. This is a property of the stochastic processes $Z$ and $Q$, but we believe it will be fulfilled except possibly in extreme cases. For example, it will not be fulfilled if $Z$ or $Q$ only take discrete values[3]. We may then formulate the more general theorem for the existence of a Nash equilibrium.

**Theorem 5.1.** *Assume the real renegotiation payment $Z$ and the aggregate price $Q$ satisfy property $\mathcal{F}$, that $m_\nu$ is piecewise continuous and that $m_\nu^c$ is continuous in each of the discontinuities in $m_\nu$. Then there exists at least one Nash-equilibrium point $(s_a^e, s_p^e)$ with $s_a^e \in S_a$ and $s_p^e \in S_p$ such that*

$$s_a^e = \operatorname{argmax}_{s_a \in S_a}\{u_a(s_a, s_p^e)\},$$
$$s_p^e = \operatorname{argmax}_{s_p \in S_p}\{u_p(s_a^e, s_p)\},$$

*where at most one of the players randomizes.*

Let us now return to the example above, discussing the possibility of an equilibrium point $(s_a^e, r_p^e)$. Assume that $m_p(r_a)$ intersects the horizontal line $r_p = r_p'$ for $r_2 < m_p(r_3) < r_1$, where there is a discontinuity in $m_a(r_p)$. When the principal has the threshold $r_p'$, the agent gets the same value for the objective function for both $r_1$ and $r_2$, i.e., $u_a(r_1, r_p') = u_a(r_2, r_p')$. In equilibrium, the agent randomizes between $r_1$ and $r_2$, with probabilities ensuring that the optimal strategy for the principal is to have the threshold $r_p'$.

When the stochastic processes are discontinuous, we are able to construct examples where there exist multiple Nash equilibria. In Figure 8, there are two Nash equilibria with constant thresholds, and one with randomization.

## 6. Efficiency of the choice of renegotiation costs

In this section, we extend the model by allowing an additional stage of the model, taking place ahead of the basic model, where players may invest in renegotiation capacity, leading to lower renegotiation fee for the player. For example, a firm may have a large salary department, taking care of the wage negotiations. Let the costs of obtaining renegotiation fee $\tau_\nu$ be given by the function $c_\nu(\tau_\nu)$, where we assume that $c_\nu$ is differentiable and strictly decreasing, and that $c_\nu$ converges to infinity when $\tau_\nu$ converges to zero, and $c_\nu$ converges to zero when $\tau_\nu$ converges to infinity.

With some abuse of notation, let $W_\nu(\tau_a, \tau_p)$ denote the expected value of the objective function of player $\nu$, derived from Nash equilibrium in the basic model with renegotiation fees $\tau_a$ and $\tau_p$. (If the Nash equilibrium is not unique, we assume that players associate probabilities with the various Nash equilibria, and then take the expected value of the objective functions.)

When both players optimize their investment in renegotiation capacity, then the renegotiation fees are given by the first order conditions

$$\frac{\partial W_\nu}{\partial \tau_\nu} - c_\nu'(\tau_\nu) = 0, \qquad \nu = a, p. \tag{6}$$

[3]If $\mathrm{Prob}(Z(t,\omega) > r)$ and $\mathrm{Prob}(Q(t,\omega) > r)$ are continuous in $r$ for all values of $t$, then the functions $f_\nu$, $h_\nu$ are also continuous. Furthermore, $u_\nu$ is continuous by Theorem 3.1, $u_{m,\nu}$ is continuous and $m_\nu$ is well-defined, piecewise continuous and $u_a(m_a(r_p), r_p) = u_{m,a}(r_p)$ and $u_p(r_a, m_p(r_a)) = u_{m,p}(r_a)$.

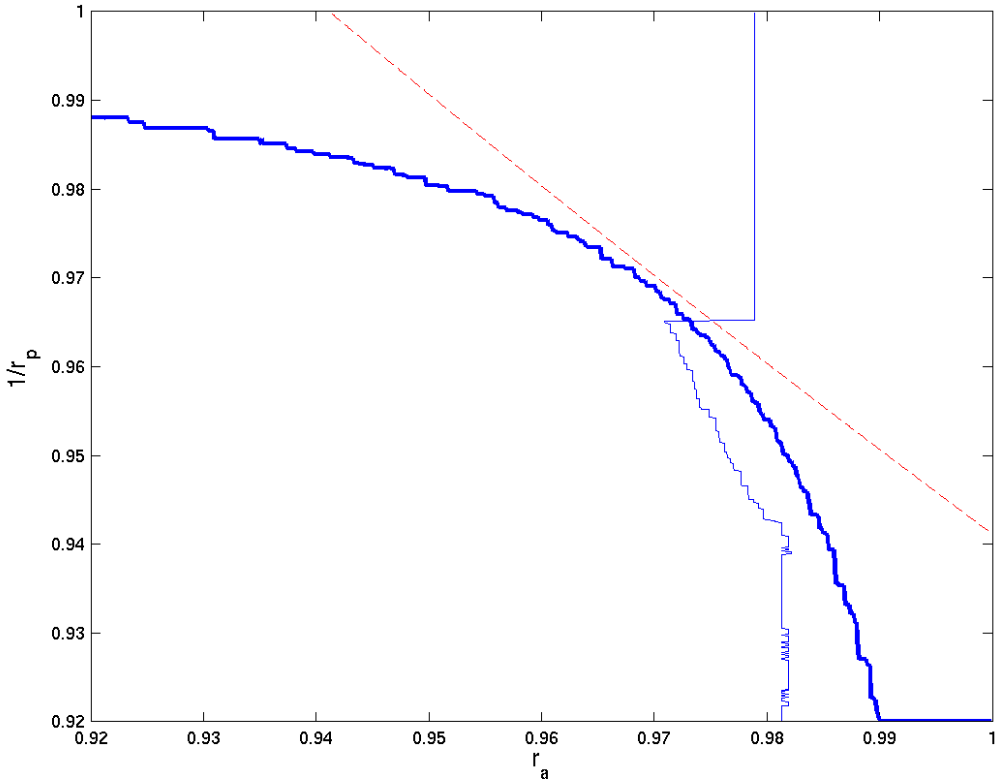


FIGURE 7. The function $m_a$ (thin curve) and $m_p$ (thick curve) for $r_a, 1/r_p \in [.92, 1]$ for values of $\tau_a = .0017$ and $\tau_p = .0057$. $R$ and $Z$ are as defined in Figure 6 and other constants are $\beta = .005$, $\eta_a = 1$, and $\eta_p = -1.5$. The dashed curve is $r_p = 1.0625 r_a$. When the thresholds satisfies $r_p < 1.0625 r_a$, then there is a renegotiation after every jump in $Z$. Therefore $m_a$ is constant above the dashed curve. It is also shown the curve $(Er_a(s_a), m_a^c(s_a))$, for $r_2 < r_a < r_1$, but it is not possible to separate this curve from the curve $(r_a, m_p(r_a))$. The strategy $s_a$ is when the agent randomizes between the values $r_2 \approx .971$ and $r_1 \approx .979$, the endpoints of the horizontal line in $m_a$ for $r_p = r'_p \approx 1/.965$. The horizontal line in $m_a(r_p)$ indicates a discontinuity where $u_a(r_1, r_p) = u_a(r_2, r_p) = u_{m,a}(r_p)$. The curve $(Er_a(s_a), m_a^c(s_a))$ intersects the horizontal line in $m_a$ at $r_a = r_3 \approx .973$. In a Nash equilibrium, the agent selects the threshold $r_1$ with probability $(r_3 - r_2)/(r_1 - r_2) \approx .25$ and else $r_2$. The randomization makes the optimal threshold for the principal equal to $r'_p$. The plot is based on $10^5$ realizations, each sampled at $10^5$ points up to time 100.

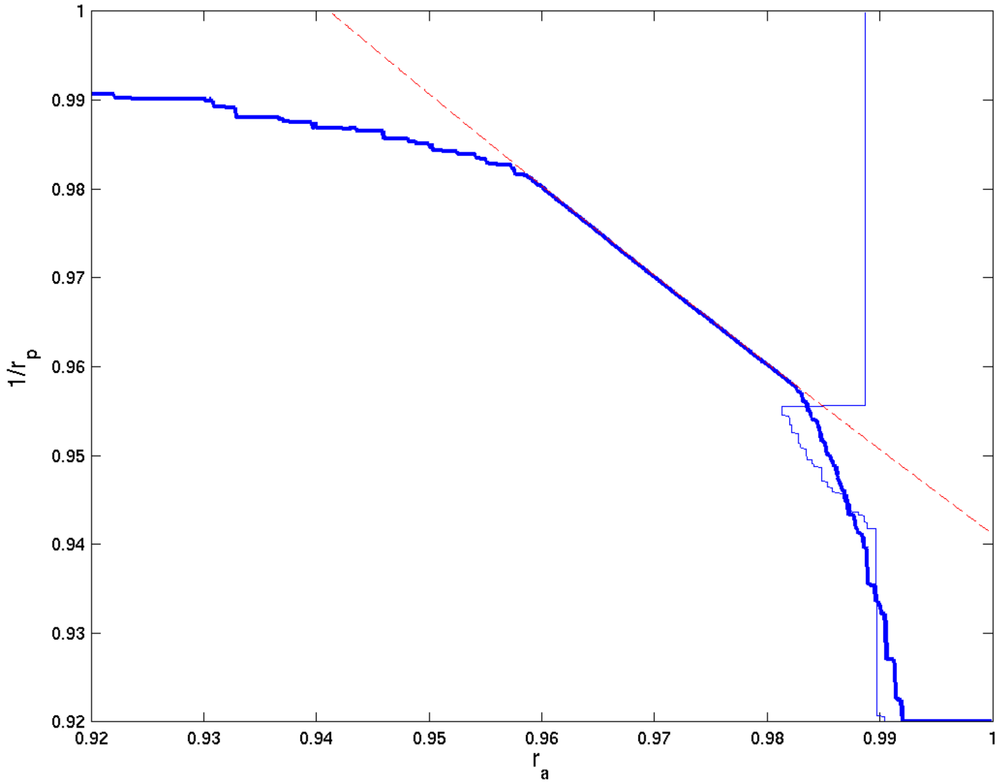


FIGURE 8. This figure shows nonuniqueness in the Nash equilibrium due to multiple crossings of the curves $m_a$ (thin curve) and $m_p$ (thick curve) for the process illustrated in Figure 7 with $\tau_a = .0005$ and $\tau_p = .0045$.

Assuming for simplicity that overall welfare can be measured by the sum of players' expected utility, the welfare maximizing levels of investment in renegotiation capacity is given by

$$\frac{\partial W_a}{\partial \tau_\nu} + \frac{\partial W_p}{\partial \tau_\nu} - c'_\nu(\tau_\nu) = 0, \qquad \nu = a, p. \tag{7}$$

Both (6) and (7) have a solution provided $c_\nu$ approaches zero sufficiently fast when $\tau_\nu$ increases. Note that $\partial W_a/\partial \tau_p > 0$ and $\partial W_p/\partial \tau_a > 0$. Then the values of $\tau_\nu$ that satisfies (6) give positive values when put into the left-hand side of the equations (7). This implies that for each solution of (6), there exists a solution of (7) with higher values of $\tau_\nu$. This implies that when each player determines the renegotiation fee from (6), there is an over-investment in renegotiation capacity compared to a solution of equations (7).

This over-investment in renegotiation capacity is due to the following. First, each of the players do not take into consideration that the contract payment in our setting is only a matter of a transfer between the players, so that what one player gains by renegotiating the

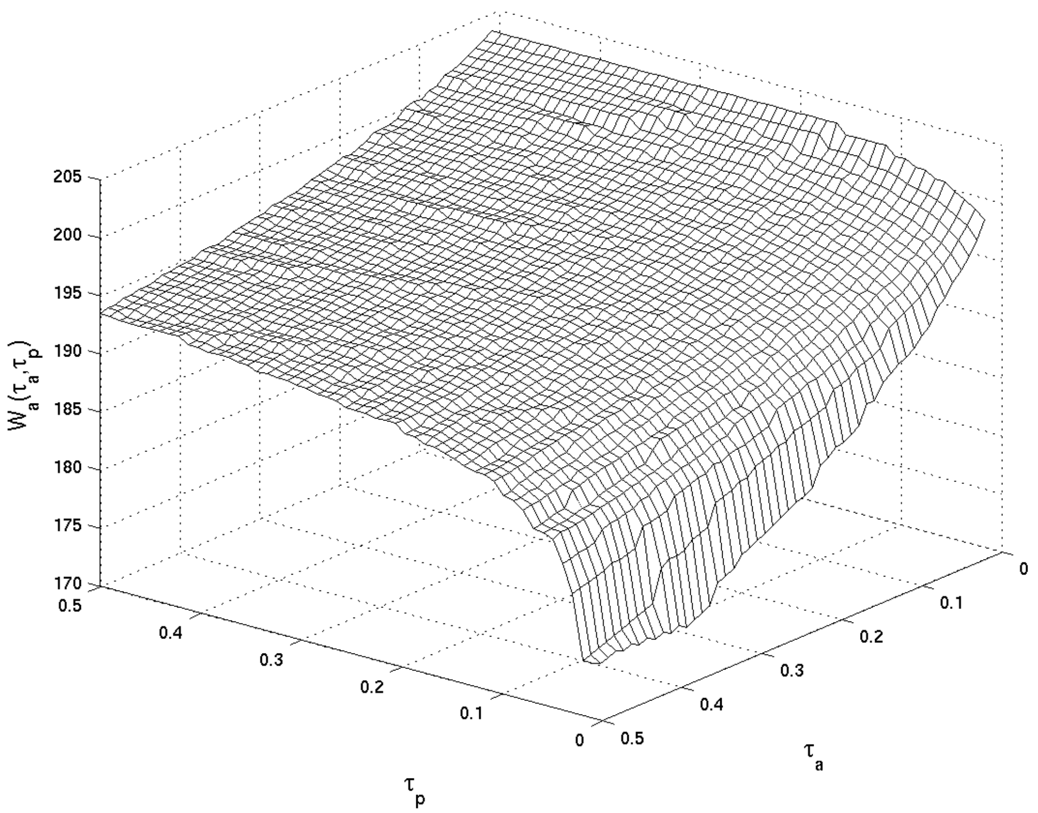


FIGURE 9. The figure shows the function $W_a(\tau_a, \tau_p)$ for $\tau_a, \tau_p \in [.01, .5]$. The data used are the same as in Table 2 and Figure 2.

price is directly linked to what the other player loses. Second, the first effect is exacerbated by the strategic substitutability in the choice of thresholds. By investing in renegotiation capacity, thus reducing the renegotiation fee, the threshold of the player is moved closer to unity, leading the other player to choose a threshold further away from unity. The player gains from both changes, i.e., both from lower own renegotiation fee, and from the opponent setting a threshold further away from unity.

## 7. The case when only one player may require renegotiation

In some real-world relationships, it may be realistic that only one of the players have the means of requiring a renegotiation. In this case the behavior of the other player can be seen as a very simple form of a Markov strategy, namely never to require renegotiation. Thus Theorem 2.3 applies and the optimal strategy for the active player is to use a critical threshold. For simplicity, we consider only the case where the agent may require a renegotiation, as the case where only the principal may require renegotiation follows directly from this analysis. This implies that $r_p = \infty$.

**Theorem 7.1.** *Assume the real renegotiation payment $Z$ and the aggregate price $Q$ are continuous and satisfy property $\mathcal{F}$ and that the principal never requires a renegotiation of the contract. Then the following holds:*
*(i) If the agent has a critical threshold, then the objective functions are given by*

$$u_\nu(r_a,\infty) = \frac{d_\nu(1-r_a^{b_\nu+\eta_\nu}) - \tau_\nu r_a^{b_\nu}}{1-r_a^{b_\nu}}, \quad \nu = a, p, \tag{8}$$

*for constants $b_\nu$ and $d_\nu$ where*

$$b_\nu = \frac{\ln(h_\nu(r_0,\infty))}{\ln(r_0)}, \tag{9}$$

*and*

$$d_\nu = \frac{f_\nu(r_0,\infty)}{1-r_0^{\eta_\nu}h_\nu(r_0,\infty)} \tag{10}$$

*for any value of $r_0 < 1$. The constants $b_\nu$ and $d_\nu$ depend only on the stochastic processes $Z$ and $Q$ and the exponents $\eta_\nu$, and they are independent of $r_0$, the discount rate $\beta$ and the contract renegotiation fees $\tau_\nu$.*
*(ii) There is a unique optimal strategy for the agent given by a threshold $r_a^e$. If $\tau_a \geq d_a$, then $r_a^e = 0$, else $r_a^e > 0$ such that $F(r_a^e) = 0$ where $F$ is defined by*

$$F(r_a) = \eta_a r_a^{\eta_a+b_a} - (\eta_a+b_a)r_a^{\eta_a} + b_a(1-\frac{\tau_a}{d_a}). \tag{11}$$

*(iii) If the optimal thresholds $r_a^e > 0$, then $r_a^e$ is decreasing in $\tau_a$ and is increasing in $d_a$.*

Part (i) of Theorem 7.1 provides explicit expressions for the objective functions, for given threshold of the agent. Part (ii) states that it is indeed optimal for the agent to have a threshold, and it derives the optimal threshold as an implicit function of the renegotiation fee $\tau_a$ and the constants $b_\nu$ and $d_\nu$.

The proof of Theorem 7.1 is based on the idea that as seen from immediately after a renegotiation, where $R = Z$, implying that $R/Z = 1$, the event that $R/Z = r_a$ so that a new renegotiation takes place, can be split into $n$ independent and identically distributed events where $R/Z$ decreases with a factor $r_0$, where $r_a = r_0^n$. This makes it possible to give an analytic expression for the expected values of the objective functions.

If we assume that the real renegotiation payment $Z$ and the aggregate price $Q$ are not stochastic, but $Z(t,\cdot) = \exp(t\alpha_z)$ and $Q(t,\cdot) = \exp(t\alpha_q)$ for constants $\alpha = \alpha_z + \alpha_q > 0$, then we get the following expressions

$$\begin{aligned} R(t,\cdot)/Z(t,\cdot) &= \exp(-t\alpha), \qquad t < t_1 \\ \alpha t_1 &= -\ln(r_a), \\ b_\nu &= \frac{\beta - \alpha_z\eta_\nu}{\alpha}, \\ d_\nu &= \frac{1}{\beta + \alpha_q\eta_\nu}. \end{aligned}$$

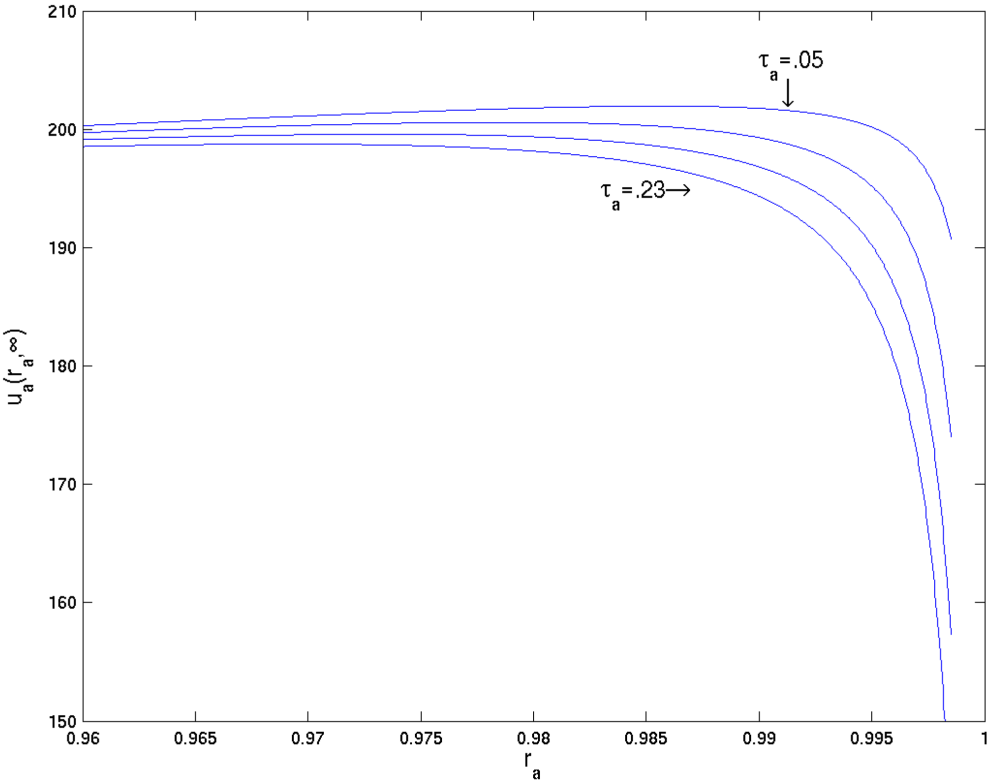


FIGURE 10. The figure shows the case when only the agent may require renegotiation. Here $Z = 1$ and $Q = \exp((\alpha_q - a^2/2)t + aB_t)$ with drift $\alpha_q = .002$, volatility $a = .01$, and $\eta_a = 1$. Note that both $u_a$ and the optimal $r_a$ value are decreasing in $\tau_a$. The constants $b_a = 2.385$ and $d_a = 144.3$ are evaluated from $f_a(.96, \infty)$ and $h_a(.96, \infty)$ based on simulations using $10^5$ realizations, each sampled at $10^5$ points up to time 100.

In the deterministic case, the approximate formula (20) from Section A is exact and the optimal threshold $r_a = m_a(r_p)$ satisfies

$$r_a^{\eta_a} = (\beta - \eta_a \alpha_z)(u_a(r_a, \infty) - \tau_a).$$

This formula is derived by assuming a fixed threshold for the other player and hence is also valid when the other player is not active.

## 8. Concluding remarks

The assumption that wages and prices are sticky in nominal terms plays a key role in macro and monetary economics. However, usually the timing of price adjustment is taken as exogenous. This has motivated a considerable literature studying the optimal

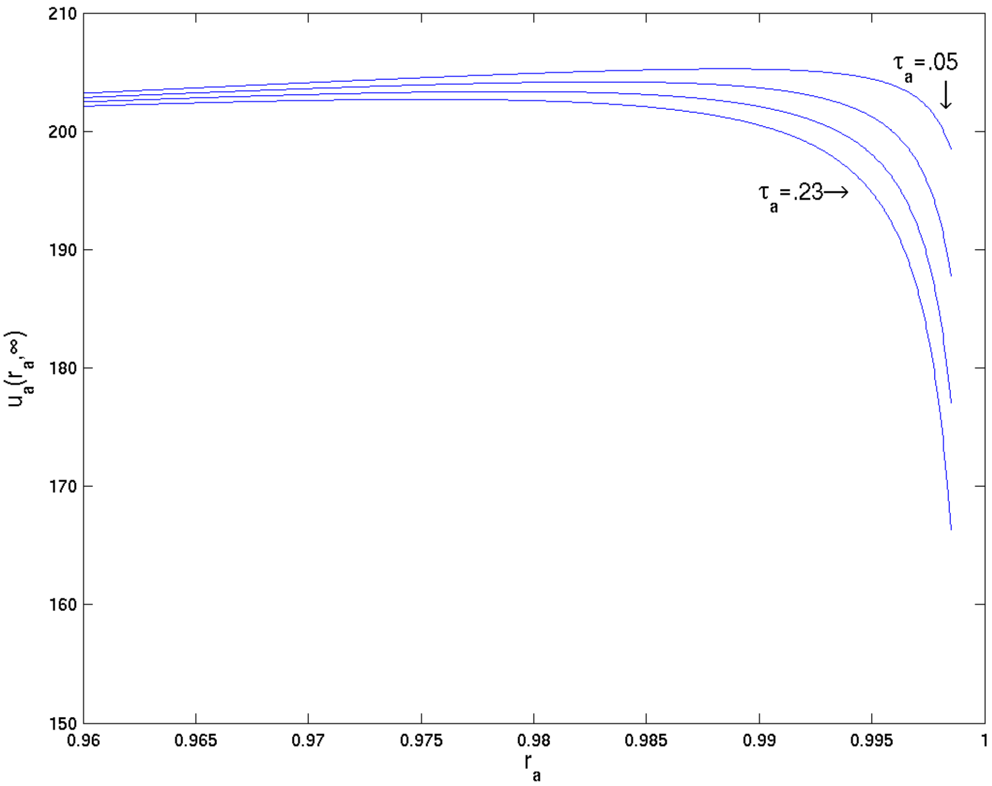


FIGURE 11. Same assumptions as in Figure 10, except $\alpha_q = .001$.

adjustment of prices under stochastic evolution of money or aggregate prices. In this paper we extend this analysis by considering bilateral adjustment, where both parties to the trade, both the seller and buyer, are allowed to require renegotiations of the contract. This follows Andersen and Christensen [3], but they focus on only one renegotiation, while we consider much more general stochastic processes, with an infinite horizon allowing for an unlimited number of renegotiations.

We show that several of the key results from the literature on unilateral price adjustment also hold in the more general case of bilateral adjustment. Optimal behavior is characterized by threshold strategies, where players demand renegotiation whenever the real contract payment is too far away relative to the real payment induced by a renegotiation of the contract. As expected, higher volatility and larger renegotiation costs make players more reluctant to demand a renegotiation, implying threshold values farther from unity.

Furthermore, we prove under rather general assumptions the existence of a Nash equilibrium in thresholds. A main result is that in equilibrium, there is strategic substitutability in players' choice of threshold: If one player becomes more aggressive, setting a threshold closer to unity, the other player becomes more passive, setting a threshold farther from unity. The strategic substitutability may exacerbate asymmetries. For example, if the renegotiation costs of the agent are reduced, the agent will respond by raising the threshold closer to unity. However, this effect will be strengthened by the principal raising his threshold, further away from unity. Numerical simulations indicate that the strategic effect may be substantial. We also find that a risk averse player will be more passive, setting a threshold farther from unity, thus benefiting the opponent.

We extend the basic model by introducing a stage ahead of the model, where players may invest in "renegotiation ability", in the sense that they may reduce their own costs of undertaking a renegotiation (e.g. by having a personell department doing the wage negotiations). We then find that players will overinvest as compared to the socially efficient level. The overinvestment arises for two reasons. First, players require a renegoation too often from a social point of view, as they do not take into consideration that their own gain from better contract terms is reflected in a loss by the opponent. By investing to lower one's own renegotiation costs, a player will require a renegotiation more often, thus hurting the other player. Secondly, the strategic substitutability mentioned above exacerbates the first effect. By reducing one's own renegotiation costs, a player becomes more aggressive. This makes the opponent more passive, which adds to the gain of the first players, as a renegotiation requested by the opponent becomes less likely.

## 9. Proofs

We have the following two technical results that are proved at the end of this section.

**Lemma 9.1.** *Assume the real renegotiation payment $Z$ and the aggregate price $Q$ are continuous and satisfy property $\mathcal{F}$. Then $v_a(r_a, r_p)$ is continuous, increasing in both variables and*

$$\frac{\partial v_a(r_a, r_p)}{\partial r_p} < (\beta - \mu_{a,z}) \frac{\partial u_a(r_a, r_p)}{\partial r_p}.$$

*Correspondingly, $v_p(r_a, r_p)$ is continuous, decreasing in both variables and*

$$\frac{\partial v_p(r_a, r_p)}{\partial r_p} > (\beta - \mu_{p,z}) \frac{\partial u_p(r_a, r_p)}{\partial r_p}.$$

**Lemma 9.2.** *If the real renegotiation payment model $Z$ is continuous and satisfies property $\mathcal{F}$, then*

$$E\{1 - Z^{\eta_\nu}(t,\omega)\exp(-\beta t)\} = (\beta - \mu_{\nu,z}) E\{\int_0^t Z^{\eta_\nu}(s,\omega)\exp(-\beta s) ds\}.$$

*Proof of Theorem 2.3.* The objective function may be written

$$\begin{aligned} U_\nu(t_1, \ldots, \omega) &= \sum_{j=0}^{\infty} \Big( \int_{t_j}^{t_{j+1}} R^{\eta_\nu}(s,\omega)\exp(-\beta s) ds - \tau_\nu Z^{\eta_\nu}(t_{j+1},\omega)\exp(- \\ &= \tau_\nu + \sum_{j=0}^{\infty} Z^{\eta_\nu}(t_j,\omega)\exp(-\beta t_j) \end{aligned}$$

$$\Big( \int_{t_j}^{t_{j+1}} \frac{Q^{\eta_\nu}(t_j,\omega)}{Q^{\eta_\nu}(s,\omega)} \exp(-\beta(s-t_j))ds - \tau_\nu \Big).$$

The expected value of integral in the last expression above is bounded due to property $\mathcal{F}$. Then $EU_\nu$ is bounded if the number of renegotiations is finite.

If there is an infinite number of renegotiations, it is in addition necessary to bound

$$E\{\sum_{j=0}^{\infty} Z^{\eta_\nu}(t_j,\omega)\exp(-\beta t_j)\}.$$

This expression is bounded due to property $\mathcal{F}$.

When both $\tau_a, \tau_p > 0$, neither player benefits from requiring renegotiation immediately all the time, e.g., have a critical threshold equal to 1. Hence the problem is well-defined.

Let $T$ satisfy $t_i < T \leq t_{i+1}$. Define $C_T$ as the contribution to the objective function for $t < T$ that cannot be changed when $t \geq T$, that is,

$$C_T = \sum_{j=0}^{i-1} \Big( \int_{t_j}^{t_{j+1}} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds - \tau_\nu Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1}) \Big) + \int_{t_i}^{T} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds \tag{12}$$

and $H_\nu(t_{i+2}-t_{i+1},\dots,\omega)$ as the contribution to the object function after $t_{i+1}$, that is,

$$H_\nu(t_{i+2}-t_{i+1},\dots,\omega) = \sum_{j=i+1}^{\infty} \Big( \int_{t_j}^{t_{j+1}} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds - \tau_\nu Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1}) \Big).$$

The function $H_\nu(t_{i+2}-t_{i+1},\dots,\omega)$ has the same distribution as $U_\nu(t_0,\dots,\omega)$. Then we may write the objective function as

$$\begin{aligned} &U_\nu(t_1,\dots,\omega) \\ &= C_T + Z^{\eta_\nu}(T,\omega)\exp(-\beta T)\Big( \frac{R^{\eta_\nu}(T,\omega)}{Z^{\eta_\nu}(T,\omega)} \int_T^{t_{i+1}} \frac{R^{\eta_\nu}(s,\omega)}{R^{\eta_\nu}(T,\omega)} \exp(-\beta(s-T))ds \\ &\quad + \frac{Z^{\eta_\nu}(t_{i+1},\omega)}{Z^{\eta_\nu}(T,\omega)} \exp(-\beta(t_{i+1}-T))(H_\nu(t_{i+2}-t_{i+1},\dots,\omega)-\tau_\nu)\Big). \end{aligned}$$

The ratios $R(s,\omega)/R(T,\omega)$ and $Z(t_{i+1},\omega)/Z(T,\omega)$ are independent of $R(T,\omega)$ and $Z(T,\omega)$ due to the Markov properties. The future contribution to the object function depends on $R(T,\omega)$ and $Z(T,\omega)$, but the optimal strategy is only a function of the ratio $R(T,\omega)/Z(T,\omega)$ and there is no memory in the game, i.e., dependencies on $t_j < T$, $R(s,\omega)$ for $s < T$ or $Z(s,\omega)$ for $s < T$.

Let $s_p$ and $s_a$ denote the strategies of the principal and the agent, respectively. With a slight abuse of notation, let $U_a(s_a, s_p, \omega)$ denote the objective function with the strategies

$s_a$ and $s_p$, respectively. Then $\sup_{s_a} E\{U_a(s_a, s_p, \omega)\}$ is well-defined and there is a sequence $s_{a,i}$ such that

$$\lim_{i\to\infty} E\{U_a(s_{a,i}, s_p, \omega)\} = \sup_{s_a} E\{U_a(s_a, s_p, \omega)\}. \tag{13}$$

Define the sequence of sets $S_i$ where $r \in S_i$ if the agent with strategy $s_{a,i}$ requires contract renegotiation for any interval for any price $Q(\,\cdot\,, \omega)$ at time $t$ where $R(t, \omega)/Z(t, \omega) = r$. Add the number 0 to $S_i$. If the renegotiation is not the first time $t$ when $R(t, \omega)/Z(s, \omega) = r$, this is not critical, since it is the contribution to the player of the objective function in the future that is critical. Since all $r \in S_i$ satisfies $0 \le r < 1$, then for any sequence $\{r_i\}_i$ with $r_i \in S_i$, there is an accumulation point $r'$ (if several, take the largest). Consider a strategy $s'$ with a critical threshold $r'$. Since the expected value of the future contribution to the objective function at time $t$ only is a function of the present $R(t, \omega)/Z(t, \omega)$, and equation (13), then

$$E\{U_a(s', s_p, \omega)\} = \sup_{s_a} E\{U_a(s_a, s_p, \omega)\}.$$

If the renegotiation payment $ZQ$ does not only change in discrete jumps, then the renegotiations will come with shorter and shorter time intervals if $r_a \to 1$. Assuming $\tau_a > 0$, then the renegotiation cost dominates the objective function which implies that the accumulation point $r' < 1$. If the price $ZQ$ only changes in discrete jumps, then the relative flow payoff can only take discrete values and $r = 1$ cannot be an accumulation point for the chain where all elements in the chain satisfies $r_i < 0$. This implies that the critical threshold may be set equal to the accumulation point $0 \le r' < 1$.

Correspondingly, if the agent has the same strategy in each time interval, then there is a corresponding argument showing that there cannot be a better strategy for the principal than what is possible to obtain with a critical threshold $r_p$. □

*Proof of Theorem 3.1.* Let $H'_\nu$ be defined as $H_\nu$ in the proof of Theorem 2.3 but with $T < t_1$ and with parameters $r_a$ and $r_p$ instead of $t_1 - t_0, \ldots$ as in (3). The definition of $U_\nu(r_a, r_p, \omega)$ in (1) and (3) implies

$$U_\nu(r_a, r_p, \omega) = \int_{t_0}^{t_1} R^{\eta_\nu}(s, \omega) \exp(-\beta s) ds + Z^{\eta_\nu}(t_1, \omega) \exp(-\beta t_1)(H'_\nu(r_a, r_p, \omega) - \tau_\nu).$$

The stochastic variables $U_\nu$ and $H'_\nu$ have similar distribution and have expectation equal to $u_\nu$. The time for the end of the first interval $t_1$ is independent of what is happening after $t_1$ due to the Markov property. Hence $Z^{\eta_\nu}(t_1, \omega) \exp(-\beta t_1)$ is independent of $H'_\nu(r_a, r_p, \omega)$. This implies that

$$u_\nu(r_a, r_p) = f_\nu(r_a, r_p) - \tau_\nu h_\nu(r_a, r_p) + h_\nu(r_a, r_p) u_\nu(r_a, r_p),$$

leading to

$$u_\nu(r_a, r_p) = \frac{f_\nu(r_a, r_p) - \tau_\nu h_\nu(r_a, r_p)}{1 - h_\nu(r_a, r_p)}.$$

This equation may also be written as

$$u_\nu(r_a, r_p) = \frac{f_\nu(r_a, r_p) - \tau_\nu}{1 - h_\nu(r_a, r_p)} + \tau_\nu. \tag{14}$$

An expression for the derivative is found using the above expression by the following calculation

$$\frac{\partial u_\nu}{\partial r_\mu} = \frac{\frac{\partial f_\nu}{\partial r_\mu}(1 - h_\nu) + (f_\nu - \tau_\nu)\frac{\partial h_\nu}{\partial r_\mu}}{(1 - h_\nu)^2} = \frac{\frac{\partial f_\nu}{\partial r_\mu} + (u_\nu - \tau_\nu)\frac{\partial h_\nu}{\partial r_\mu}}{1 - h_\nu}$$

for $\mu = a, p$. □

*Proof of Theorem 4.1.* (i) Using Lemma 9.2 we have

$$\frac{f_a}{1 - h_a} = \frac{E\{\int_0^T R^{\eta_a}(s, \omega)\exp(-\beta s)ds\}}{(\beta - \mu_{a,z})E\{\int_0^T Z^{\eta_a}(s, \omega)\exp(-\beta s)ds\}}. \tag{15}$$

Hence the fraction is a constant times the average of $R^{\eta_a}$ until the first contract renegotiation divided by the average of $Z^{\eta_a}$ in the same interval for $r_a < R/Z < r_p$. We will show that the fraction (15) is increasing in $r_p$. Let $r'_p > r_p$ and

$$T' = \inf\{s \ge T \mid R(s, \omega)/Z(s, \omega) \notin (r_a, r'_p)\}.$$

Define further $T \le T'' \le T'$ as

$$T'' = \inf\{s \ge T \mid R(s, \omega)/Z(s, \omega) \notin (1, r'_p)\}.$$

The lower endpoint is set equal to 1 since $R(0, \omega)/Z(0, \omega) = 1$. Then

$$\begin{aligned} E\{\int_0^{T'} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\} &= E\{\int_0^{T} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\} \\ &\quad + E\{\int_T^{T''} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\} \\ &\quad + E\{\int_{T''}^{T'} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\}. \end{aligned}$$

Let $P$ be the probability that there exists times $s_1$ and $s_2$ with $T < s_1 < T'' < s_2 < T'$ such that $R(s_1, \omega)/Z(s_1, \omega) \ge r_p$ and $R(s_2, \omega)/Z(s_2, \omega) \le 1$. Then

$$E\{\int_{T''}^{T'} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\} = PE\{\int_0^{T'} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\}$$

since $R(\cdot, \omega)/Z(\cdot, \omega)$ varies in the same interval $(1, r'_p)$ in both expressions. This implies that

$$\begin{aligned} E\{\int_0^{T'} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\} &= \frac{1}{1 - P}\big(E\{\int_0^{T} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\} \\ &\quad + E\{\int_T^{T''} R^{\eta_a}(s, \omega)\exp(-\beta s)ds\}\big). \end{aligned}$$

Correspondingly, we have

$$E\{\int_0^{T'} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\} = \frac{1}{1-P}\big(E\{\int_0^{T} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\} + E\{\int_T^{T''} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\}\big).$$

We have

$$\frac{E\{\int_T^{T''} R^{\eta_a}(s,\omega)\exp(-\beta s)ds\}}{E\{\int_T^{T''} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\}} > \frac{E\{\int_0^{T} R^{\eta_a}(s,\omega)\exp(-\beta s)ds\}}{E\{\int_0^{T} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\}}$$

since on the left-hand side $R(T,\omega)/Z(T,\omega) \geq r_p$, and $R(\,\cdot\,,\omega)/Z(\,\cdot\,,\omega)$ varies in the interval $(1, r_p')$ while on the right-hand side $R(0,\omega)/Z(0,\omega) = 1$, and $R(\,\cdot\,,\omega)/Z(\,\cdot\,,\omega)$ varies in the interval $(r_a, r_p)$. Then

$$\frac{E\{\int_0^{T'} R^{\eta_a}(s,\omega)\exp(-\beta s)ds\}}{E\{\int_0^{T'} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\}} > \frac{E\{\int_0^{T} R^{\eta_a}(s,\omega)\exp(-\beta s)ds\}}{E\{\int_0^{T} Z^{\eta_a}(s,\omega)\exp(-\beta s)ds\}}.$$

Hence the fraction (15) is increasing in $r_p$.

When $r_p$ increases, $ET$ increases and $h_a$ decreases due to property $\mathcal{F}$. This implies $-\tau_a/(1-h)$ increases when $r_a$ increases. Hence by using (14) we see that increasing $r_p$ increases $u_a$. The corresponding argument may be applied for $u_p(r_a, r_p)$.

(ii) Since $Z$ and $Q$ are continuous, $f_a$, $h_a$, and hence $u_a$ are continuous by Theorem 3.1. Then $u_{m,a}$ is continuous and $m_a$ is well-defined, piecewise continuous and $u_a(m_a(r_p), r_p) = u_{m,a}(r_p)$ and $u_p(r_a, m_p(r_a)) = u_{m,p}(r_a)$.

Let $r_p > 1$ be fixed. We will first prove that there is a value $0 \leq r_a < 1$ that maximizes $u_a(r_a, r_p)$. The function $u_a(r_a, r_p)$ is defined for $0 \leq r_a < 1$. We will give an argument that the maximum value is attained for $r_a$ in the closed interval $[0, 1-\epsilon]$ for $\epsilon > 0$ sufficiently small. Since the interval is closed, the maximum value will be attained for a value $r_a = m_a(r_p)$. Since $Z$ and $Q$ are continuous, then $T(r_a, r_p)$ vanishes with probability 1 when $r_a \to 1$. Then $h_a(r_a, r_p) \to 1$ and $f_a(r_a, r_p) \to 0$ when $r_a \to 1$. Further, the expression (4) for $u_a$ implies that $u_a(r_a, r_p) \to -\infty$ when $r_a \to 1$. Hence for each value for $r_p$, there is a value $r_a = m_a(r_p) < 1$ where $u_a(r_a, r_p) = u_{m,a}(r_p)$.

We will prove that the value $r_a = m_a(r_p)$ is unique, i.e., if $r_a \neq m_a(r_p)$, then $u_a(r_a, r_p) < u_{m,a}(r_p)$. Let $t_1$ be the time for the first contract renegotiation. Furthermore, let $W_1$ and $W_2$ be two new stochastic variables that are identical to $U_a$ except that a different strategy (a different limit for $r_a$) is used before $t_1$. After $t_1$, we set $r_a = m_a(r_p)$. The first contract renegotiation in $W_1$ and $W_2$, if required by the agent, is required when the relative flow payoff reaches the values $r_2 < r_1 < 1$, respectively. Let $P_{r_1}$ denote the probability that the contract payment relative to renegotiation payment $R(t,\omega)/Z(t,\omega)$ reaches $r_1$ before it reaches $r_p$. In case the ratio reaches $r_p$ first, there is no difference between $W_1$ and $W_2$. Further, let $T_1(r_1, r_p, \omega)$ denote the time of the first contract renegotiation for $W_1$. Finally, let

$$E_{r_1} = E\{\exp(-\beta T_1(r_1, r_p, \omega))\}$$

given that $r_1$ is reached. For $W_2$ there is a contract renegotiation when the contract payment relative to renegotiation payment first reaches $r_2$ or $r_p$. Assuming $r_1$ is reached, there is contract renegotiation when either the ratio decreases with a factor $r_2/r_1$ or increases with a factor $r_p/r_1$. Let $T_2(r_2/r_1, r_p/r_1, \omega)$ denote the time between $r_1$ is reached and either $r_2$ or $r_p$ are reached. Let $w_1$ and $w_2$ be the expected values of $W_1$ and $W_2$, respectively. Assuming that $r_1$ is reached before $r_p$, then

$$w_1 = EC_{T_1} + E_{r_1}E\big\{(u_{m,a}(r_p) - \tau_a)Z^{\eta_a}(T_1, \omega)\exp(-\beta T_2)\big\}$$

and

$$\begin{aligned} w_2 = EC_{T_1} + E_{r_1}E\big\{ &\int_0^{T_2} R^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds \\ &+ (u_{m,a}(r_p) - \tau_a)Z^{\eta_a}(T_1 + T_2, \omega)\exp(-\beta T_2))\big\} \end{aligned}$$

where $C_{T_1}$ is defined in (12). We have that $W_1 = W_2$ are identical except if $r_1$ is reached. Since $r_1$ is reached before $r_p$ happens with probability $P_{r_1}$, the difference is

$$\begin{aligned} w_2 - w_1 &= P_{r_1}E_{r_1}E\big\{ \int_0^{T_2} R^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds \\ &\qquad - (u_{m,a}(r_p) - \tau_a)(Z^{\eta_a}(T_1, \omega) - Z^{\eta_a}(T_1 + T_2, \omega)\exp(-\beta T_2))\big\} \\ &= P_{r_1}E_{r_1}E\big\{ \int_0^{T_2} R^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds \\ &\qquad - (\beta - \mu_{a,z})(u_{m,a}(r_p) - \tau_a)\int_0^{T_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\big\} \\ &= P_{r_1}E_{r_1}\Big(\frac{E\{\int_0^{T_2} R^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}}{E\{\int_0^{T_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}} - (\beta - \mu_{a,z})(u_{m,a}(r_p) - \tau_a)\Big) \\ &\qquad \times E\{\int_0^{T_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}. \end{aligned}$$

Lemma 9.2 is used in the second equality. Define the limit of the discounted real renegotiation payment when the lower threshold is slightly reduced by

$$L_a(r_1, r_p) = \lim_{r_2 \to r_1 -} \frac{E\{\int_0^{T_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}}{r_2 - r_1}.$$

Due to the Markov property and property $\mathcal{F}$, the nominator is monotone and hence the limit is well-defined. Letting $r_2 \to r_1$, we have

$$\frac{\partial w_1}{\partial r_1} = -P_{r_1}E_{r_1}(v_a(r_1, r_p) - (\beta - \mu_{a,z})(u_{m,a}(r_p) - \tau_a))L_a(r_1, r_p)$$

where $v_a$ is defined by (A). Hence, $\frac{\partial w_1}{\partial r_1} = 0$ when

$$v_a(r_1, r_p) = (\beta - \mu_{a,z})(u_{m,a}(r_p) - \tau_a). \qquad (16)$$

According to Theorem 3.1 and Lemma 9.1, $u_{m,a}(r_p)$ and $v_a(r_1, r_p)$ are continuous and $v_a(r_1, r_p)$ is increasing in $r_1$. Consider the function $w_1(r_1, r_p)$ with $r_p$ fixed. Then $w_1(r_1, r_p)$ reaches its maximum with respect to $r_1$ either for $r_1 = 0$ or for a value $r_1 > 0$ when $\frac{\partial w_1(r_1,r_p)}{\partial r_1} = 0$. Since $v_a(r_1, r_p)$ is increasing in $r_1$, then $\frac{\partial w_1}{\partial r_1}$ changes sign when (16) is satisfied. Hence, the maximum is unique. Since the periods between contract renegotiations are independent then the value $r_1$ that maximizes $w_1$ also maximizes $u_a(r_a, r_p)$. Hence, there is a unique value $r_a = m_a(r_p)$ that maximizes $u_a(r_a, r_p)$.

The corresponding argument may be applied for $u_p(r_a, r_p)$. However, since $r_p$ varies in an unbounded interval we should consider $u_p$ as a function of $1/r_p$ instead of $r_p$ when applying the argument. This is possible since $u_p(r_a, r_p)$ is well-defined as $r_p$ approaches $\infty$.

(iii) Above it is proved that the optimal value $m_a(r_p)$ satisfies equation

$$v_a(m_a(r_p), r_p) = (\beta - \mu_{a,z})(u_a(m_a(r_p), r_p) - \tau_a).$$

Differentiating both sides with respect to $r_p$ gives

$$\frac{\partial v_a}{\partial r_a}\frac{dm_a}{dr_p} + \frac{\partial v_a}{\partial r_p} = (\beta - \mu_{a,z})(\frac{\partial u_a}{\partial r_a}\frac{dm_a}{dr_p} + \frac{\partial u_a}{\partial r_p}).$$

Since

$$\frac{\partial v_a}{\partial r_p} < (\beta - \mu_{a,z})\frac{\partial u_a}{\partial r_p}$$

from Lemma 9.1 and $\partial u_a/\partial r_a = 0$ since $m_a(r_p)$ is the optimal value of $r_a$, this implies that

$$\frac{\partial v_a}{\partial r_a}\frac{dm_a}{dr_p} > 0.$$

Since $\partial v_a/\partial r_a > 0$, then also $dm_a/dr_p > 0$, i.e., $m_a(r_p)$ is a strictly increasing function. The proof that $m_p(r_a)$ is strictly increasing is similar.

(iv) Equation (5) may be used in order to prove that $m_a(r_p)$ decreases when $\tau_a$ increases, assuming $m_a(r_p) > 0$. The function $u_a(r_a, r_p)$ has an optimal value for $r_a = m_a(r_p) > 0$. Since $u_a$ is differentiable, then there exists a $\varepsilon > 0$ such that $\frac{\partial u_a}{\partial r_a}(r_a, r_p) > 0$ for $m_a(r_p) - \varepsilon < r_a < m_a(r_p)$ and $\frac{\partial u_a}{\partial r_a}(r_a, r_p) < 0$ for $m_a(r_p) < r_a < m_a(r_p) + \varepsilon$. If $\tau_a$ is increased, then $\frac{\partial u_a}{\partial r_a}$ is decreased which implies a reduction in the $r_a$ value where $\frac{\partial u_a}{\partial r_a} = 0$. This implies that increasing the renegotiation fee reduces the optimal threshold value $m_a(r_p)$. Correspondingly, it is proved that $m_p(r_a)$ strictly increases when $\tau_p$ increases assuming $m_p(r_a) > 0$.

(v) Since $u_{m,\nu}$ and $v_\nu$ are continuous, we infer that the functions $m_a(r_p)\colon (1, \infty] \to [0, 1)$ and $m_p(r_a)\colon [0, 1) \to (1, \infty]$ are continuous. In the infinite rectangle defined by $0 \le r_a < 1$ and $r_p > 1$, $m_a(r_p)$ gives a continuous path between the lines defined by $r_p = 0$ and $r_p = \infty$. Similarly, $m_p(r_a)$ gives a path in the same rectangle between the lines defined by $r_a = 0$ and $r_a = 1$. Hence, these two curves must intersect at least once, giving an equilibrium point. □

*Proof of Theorem 5.1.* The existence of at least one equilibrium point $(r_a^e, r_p^e)$ is proved similarly as in Theorem 4.1 where it is assumed that the price $ZQ$ is continuous, i.e., the equilibrium point is the intersection between $m_a(r_p)$ and $m_p(r_a)$. But in this case, these curves are not necessarily continuous which implies that there might not be an intersection. Define the graphs $M_a$ and $M_p$ consisting of curves $m_a(r_p)$ and $m_p(r_a)$ and in addition, where there are discontinuities in the curves, make the graph continuous by connecting the discontinuities by straight lines with constant $r_p$ and $r_a$, respectively. (See Figure 7.) Since the graphs are continuous, they must intersect. If the intersection is on the straight lines, then randomization is necessary as illustrated in Section 5. Assume $m_p(r_a)$ intersects a straight line in $M_a$ connecting the two points $(r_1, r_p^e)$ and $(r_2, r_p^e)$. Then $m_p(r_1)$ and $m_p(r_2)$ give values of $r_p$ on opposite site of $r_p^e$. We may then define a one parameter family of strategies $s_a$ where the probability for choosing $r_1$ varies in the interval $0 \leq q \leq 1$. Since $m_p(r_1)$ and $m_p(r_2)$ gives values of $r_p$ on opposite site of $r_p^e$, then also the endpoints $m_p^c(s_a)$ when $s_a$ varies in the one-parameter family gives values on the opposite side of $r_p^e$. The continuity of $m_p^c(s_a)$ ensures that there is a strategy $s_a^e$ that randomize $r_a$ between $r_1$ and $r_2$ such that $m_p^c(s_a^e) = r_p$. There is a corresponding argument if $m_a(r_p)$ intersects a straight line in $M_p$.

The continuity of $m_a^c(s_p)$ and $m_p^c(s_a)$ implies that it is not necessary that both agent and principal randomize at the same time. If $M_a$ and $M_p$ intersect with two straight lines, then there may be two Nash equilibriums defined by using $m_a^c(s_p)$ and $m_p^c(s_a)$, respectively. □

*Proof of Theorem 7.1.* Set $r_a = r_0^n$. That $R/Z$ reaches $r_a$ is the same as the ratio is decreasing $n$ times with a factor $r_0$. The changes in the ratio in different intervals are independent of each other. Hence

$$h_\nu(r_a, \infty) = h_\nu^n(r_0, \infty) = (h_\nu^{1/\ln r_0}(r_0, \infty))^{\ln r_a} = (\exp b_\nu)^{\ln r_a} = r_a^{b_\nu}$$

where it is used that $n = \ln r_a / \ln r_0$ and $b_p$ defined by (9). It is easily argued that $b_\nu$ is independent of $r_0$. One may use that if $r_a = r_0^{n_0} = r_1^{n_1}$, then both $r_0$ and $r_1$ give same value for $b_\nu$. Define the discounted flow payoff at the first contract renegotiation

$$g_\nu(r_a) = E\{Q^{-\eta_\nu}(T(r_a, \infty, \omega), \omega) \exp(-\beta T(r_a, \infty, \omega))\}.$$

Since

$$Q^{-\eta_\nu}(T(r_a, \infty, \omega), \omega) = R^{\eta_\nu}(T(r_a, \infty, \omega), \omega) = r_a^{\eta_\nu} Z^{\eta_\nu}(T(r_a, \infty, \omega), \omega),$$

we have that $g_\nu(r_a) = r_a^{\eta_\nu} h_\nu(r_a, \infty) = r_a^{\eta_\nu + b_\nu}$. The method used when finding an expression for $h_\nu$, may also be used for $f_\nu$.

$$\begin{aligned} f_\nu(r_a, \infty) &= \sum_{i=0}^{n-1} f_\nu(r_0, \infty) g_\nu^i(r_0) = f_\nu(r_0, \infty) \sum_{i=0}^{n-1} g_\nu^i(r_0) \\ &= \frac{f_\nu(r_0, \infty)(1 - g_\nu^n(r_0))}{1 - g_\nu(r_0)} = d_\nu(1 - r_a^{\eta_\nu + b_\nu}) \end{aligned}$$

where $d_\nu$ is defined by (10). It is easily argued that $d_\nu$ is independent of $r_0$. Combining these formulas and equation (4) gives

$$u_\nu(r_a, \infty) = \frac{d_\nu(1 - r_a^{\eta_\nu + b_\nu}) - \tau_\nu r_a^{b_\nu}}{1 - r_a^{b_\nu}}.$$

The object function (8) is an analytic expression that obtain its maximum value for $0 \le r_a < 1$. The optimal action for the agent is to choose $r_a = 0$ or equal to the value that makes the derivative vanish. The derivative is

$$\begin{aligned}
&\frac{du_a(r_a,\infty)}{dr_a}\\
&\quad = \frac{1}{(1-r_a^{b_a})^2}\big(d_a\eta_a r_a^{2b_a+\eta_a-1} - d_a(\eta_a+b_a)r_a^{\eta_a+b_a-1} + b_a(d_a-\tau_a)r_a^{b_a-1}\big)\\
&\quad = \frac{d_a r_a^{b-1}}{(1-r_a^{b_a})^2}F(r_a)
\end{aligned}$$

where $F$ is defined in (11). Note that $dF/dr_a$ and $du_a/dr_a$ have the same sign. We have that $b_a, d_a > 0$ and that $r_a^{b_a} < 1$ due to property $\mathcal{F}$. Further, we have that $(\eta_a+b_a)r_a^{\eta_a} > \eta_a r_a^{\eta_a+b_a}$ and hence $F(r_a)$ is monotone decreasing in $(0,1)$.

If $\tau_a \ge d_a$, then $F(0) < 0$, which implies that $r_a = r_a^e = 0$ is the unique optimal value. If $\tau_a < d_a$, then $F(0) > 0$. Since $F(1) < 0$, and F is monotone decreasing, then there is an optimal value $r_a = r_a^e > 0$ where $F(r_a^e) = 0$.

(iii) When $\tau_a/d_a$ increases, then $F(0)$ decreases but $dF/dr_a$ is not changed. This implies that $r_a$ decreases in $\tau_a/d_a$. □

*Proof of Lemma 9.1.* Assume $R(T_1,\omega)/Z(T_1,\omega) = r_1$. Define $T_3(c, r_p/r_1)$ as the first time after $T_1$ where either $R(T_1+T_3,\omega)/Z(T_1+T_3,\omega) = cr_1$ or $R(T_1+T_3,\omega)/Z(T_1+T_3,\omega) = r_p$ for a constant $0 < c < 1$. Define

$$v_a^c(r_1,r_p) = \frac{E\{\int_0^{T_3} R^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\}}{E\{\int_0^{T_3} Z^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\}},$$

i.e., $\lim_{c\to 1-} v_a^c(r_1,r_p) = v_a(r_1,r_p)$. Let $r_2 < cr_1$ and consider the function $v_a^c(r_2,r_p)$. Assume $R(T_2,\omega)/Z(T_2,\omega) = r_2$ and define $T_4(c, r_p/r_2)$ as the first time after $T_2$ where either $R(T_2+T_4,\omega)/Z(T_2+T_4,\omega) = cr_2$ or $R(T_2+T_4,\omega)/Z(T_2+T_4,\omega) = r_p$. The interval $(T_2, T_2+T_4)$ may consist of several intervals $(t_i^1, t_i^2)$, i.e., $T_2 < t_i^1 < t_i^2 \le T_2+T_4$ that satisfies the properties of a $(T_1, T_1+T_3)$ interval, i.e., $R(t_i^1,\omega)/Z(t_i^1,\omega) = r_1$, $cr_1 < R(s,\omega)/Z(s,\omega) < r_p$ for $t_i^1 < s < t_i^2$ and $R(t_i^2,\omega)/Z(t_i^2,\omega) = cr_1$ or $R(t_i^2,\omega)/Z(t_i^2,\omega) = r_p$. Let $\Omega_1 = \cup_i(t_i^1, t_i^2)$ and $\Omega_2 = (T_2, T_2+T_4) \setminus \Omega_1$. Then

$$\begin{aligned}
&v_a^c(r_2,r_p)\\
&\quad = \frac{E\{\int_{\Omega_1} R^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\} + E\{\int_{\Omega_2} R^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\}}{E\{\int_{\Omega_1} Z^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\} + E\{\int_{\Omega_2} Z^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\}}.
\end{aligned}$$

We have that

$$v_a^c(r_1,r_p) = \frac{E\{\int_{\Omega_1} R^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\}}{E\{\int_{\Omega_1} Z^{\eta_a}(T_1+s,\omega)\exp(-\beta s)ds\}},$$

since each interval $(t_i^1, t_i^2)$ has the same properties as $(T_1, T_1 + T_3)$. Furthermore, we have that

$$E\{\int_{\Omega_2} R^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\} = \gamma_c E\{\int_{\Omega_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}$$

for a value $(cr_2)^{\eta_a} < \gamma_c < r_1^{\eta_a}$ since $\gamma_c$ is a weighted average of $R^{\eta_a}(T_1 + s, \omega)$ divided by the weighted average of $Z^{\eta_a}(T_1 + s, \omega)$ where the ratio $R/Z$ is varying in the same interval for each $\omega$ and pointwise in the integral. This implies that

$$\begin{aligned} v_a^c(r_2, r_p) = & \Big[v_a^c(r_1, r_p)E\{\int_{\Omega_1} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\} \\ & + \gamma_c E\{\int_{\Omega_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}\Big] \\ & \times \Big[E\{\int_{\Omega_1} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\} \\ & + E\{\int_{\Omega_2} Z^{\eta_a}(T_1 + s, \omega)\exp(-\beta s)ds\}\Big]^{-1} \\ = & (1-d)v_a^c(r_1, r_p) + d\gamma_c, \end{aligned}$$

for $0 < d < 1$. Note that $\gamma_c < r_1^{\eta_a}$ and $d > 0$ also when $c \to 0$ since $\Omega_2$ cannot be empty due to the first interval $0 < s < t_1^1$ where $cr_2 < R(T_1+s, \omega)/Z(T_1+s, \omega) < r_1$. This implies that $v_a^c(r_2, r_p) < v_a^c(r_1, r_p)$. Since $\lim_{c\to 1-} v_a^c(r_1, r_p) = v_a(r_1, r_p)$, then $v_a(r_1, r_p)$ is increasing in $r_1$. The function $v_a(r_1, r_p)$ is increasing and continuous in $r_p$ since this makes it possible with high $R(t, \omega)/Z(t, \omega)$ values and the probability for reaching $r_p$ changes continuously. From the definition of $v_a$ we have that $cv_a(r_1, r_p) = v_a(cr_1, cr_p)$. This implies that $v_a(r_1, r_p)$ also is continuous in $r_1$. The function $v_p(r_a, r_p)$ has similar properties. This function is decreasing in both arguments since $\eta_p < 0$.

The proof that

$$\frac{\partial v_a}{\partial r_p} < (\beta - \mu_{a,z})\frac{\partial u_a}{\partial r_p}$$

is quite similar to the above argument. Assume $R(T_5, \omega)/Z(T_5, \omega) = r_1/c$ where $0 < c < 1$. Define $T_6(c, r_p/(cr_1))$ as the first time after $T_5$ where either $R(T_5+T_6, \omega)/Z(T_5+T_6, \omega) = r_1$ or $R(T_5+T_6, \omega)/Z(T_5+T_6, \omega) = r_p$. Consider $v_a^c(r_1/c, r_p)$ and compare this with $u_a(r_1, r_p)$ where $R(s, \omega)/Z(s, \omega)$ starts at 1 and ends at either $r_1$ or $r_p$. Define $T_7 > 0$ as the first time $R(T_5 + T_7, \omega)/Z(T_5 + T_7, \omega) = 1$, if this value is obtained. Define $\Omega_1 = (T_7, T_6)$ and $\Omega_2 = (T_5, T_6) \setminus \Omega_1$. Note that $\Omega_1$ may be empty. Then

$$\begin{aligned} & v_a^c(r_1/c, r_p) \\ & = \frac{E\{\int_{\Omega_1} R^{\eta_a}(T_5 + s, \omega)\exp(-\beta s)ds\} + E\{\int_{\Omega_2} R^{\eta_a}(T_5 + s, \omega)\exp(-\beta s)ds\}}{E\{\int_{\Omega_1} Z^{\eta_a}(T_5 + s, \omega)\exp(-\beta s)ds\} + E\{\int_{\Omega_2} Z^{\eta_a}(T_5 + s, \omega)\exp(-\beta s)ds\}}. \end{aligned}$$

Using (4) assuming for a moment that $\tau_a = 0$ and using Lemma 9.2 gives

$$(\beta - \mu_{a,z})\frac{f_a}{1-h_a} = \frac{E\{\int_{\Omega_1} R^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\}}{E\{\int_{\Omega_1} Z^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\}},$$

since the interval $(T_7, T_6)$ has the same properties as in the variation of $u_a$. Furthermore, we have that

$$E\{\int_{\Omega_2} R^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\} = \gamma_c E\{\int_{\Omega_2} Z^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\}$$

for a value $r_2^{\eta_a} < \gamma_c < 1$ by the same argument as above since in this case $r_2 < R/Z < 1$. This implies that

$$\begin{aligned} v_a^c(r_2, r_p) = &\Big[(\beta - \mu_{a,z})\frac{f_a}{1-h_a}E\{\int_{\Omega_1} Z^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\} \\ &+ \gamma_c E\{\int_{\Omega_2} Z^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\}\Big] \\ &\times \Big[E\{\int_{\Omega_1} Z^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\} \\ &+ E\{\int_{\Omega_2} Z^{\eta_a}(T_5+s,\omega)\exp(-\beta s)ds\}\Big]^{-1} \\ = &(1-d)(\beta - \mu_{a,z})\frac{f_a}{1-h_a} + d\gamma_c, \end{aligned}$$

for $0 < d < 1$. Note that $\gamma_c$ is independent of $r_p$ and $d > 0$ since $\Omega_2$ cannot be empty due to the first interval $0 < s < T_7$. Since $\lim_{c\to 1-} v_a^c(r_1/c, r_p) = v_a(r_1, r_p)$ and $\partial h_a/\partial r_p < 0$, we have

$$\frac{\partial v_a}{\partial r_p} < (\beta - \mu_{a,z})\frac{\partial}{\partial r_p}\Big(\frac{f_a}{1-h_a}\Big) < (\beta - \mu_{a,z})\frac{\partial}{\partial r_p}\Big(\frac{f_a - \tau_a}{1-h_a}\Big) = (\beta - \mu_{a,z})\frac{\partial u_a}{\partial r_p}.$$

Correspondingly, it is proved that

$$\frac{\partial v_p}{\partial r_a} > (\beta - \eta_p \alpha_z)\frac{\partial u_p}{\partial r_a}.$$

Note that since $\eta_p < 0$ both expressions above are negative. □

*Proof of Lemma 9.2.* Let $t_i = it/n$ and $Z(0,\omega) = 1$. Then

$$\begin{aligned} E\{1 - &Z^{\eta_\nu}(t,\omega)\exp(-\beta t)\} \\ &= E\{\sum_{i=0}^{n}\big(Z^{\eta_\nu}(t_i,\omega)\exp(-\beta t_i) - Z^{\eta_\nu}(t_{i+1},\omega)\exp(-\beta t_{i+1})\big)\} \\ &= E\{\sum_{i=0}^{n} Z^{\eta_\nu}(t_i,\omega)\exp(-\beta t_i)(1 - \frac{Z^{\eta_\nu}(t_{i+1},\omega)}{Z^{\eta_\nu}(t_i,\omega)}\exp(-\beta(t_{i+1}-t_i))\} \end{aligned}$$

$$= \frac{1}{t_1} E\{1 - Z^{\eta_\nu}(t_1, \omega) \exp(-\beta t_1)\} E\{\sum_{i=0}^{n} Z^{\eta_\nu}(t_i, \omega) \exp(-\beta t_i) t_1\}.$$

We have that

$$\begin{aligned} \lim_{t \to 0} E\{\frac{1 - Z^{\eta_\nu}(t, \omega) \exp(-\beta t)}{t}\} &= \lim_{t \to 0} \frac{1 - E\{Z^{\eta_\nu}(t, \omega)\} \exp(-\beta t)}{t} \\ &= \lim_{t \to 0} \frac{1 - \exp(t(\mu_{\nu,z} - \beta))}{t} \\ &= \beta - \mu_{\nu,z} \end{aligned}$$

and

$$\lim_{n \to \infty} E\{\sum_{i=0}^{n} Z^{\eta_\nu}(t_i, \omega) \exp(-\beta t_i) t_1\} = E\{\int_0^t Z^{\eta_\nu}(s, \omega) \exp(-\beta s) ds\}.$$

Combining these three calculations proves the lemma. □

## Appendix A. Approximate formulas

Given the weak assumptions we impose on the stochastic processes, explicit formulas are difficult to obtain. However, we can derive some approximate formulas that may provide useful intuition for how the model works, and to get some sense of the numerical magnitudes that are involved.

We will first explore the effect on the payoff of a player from a marginal reduction in his threshold. Let the principal and agent have threshold $r_p$ and $r_1$ respectively, and consider the situation at $T_1$ when $R(T_1, \omega)/Z(T_1, \omega) = r_1$. If the agent sticks to the threshold $r_1$, there will be an immediate renegotiation at $T_1$. In contrast, if the agent adopts a new threshold $r_2 < r_1$, there will be a renegotiation at $T_2$, where $T_2$ denotes the first time after $T_1$ where either the real contract payment relative to the real renegotiation payment has decreased by a factor $r_2/r_1$, or increased by a factor $r_p/r_1$. Formally

$$T_2 = \inf\{s > 0 \mid R(T_1 + s, \omega)/Z(T_1 + s, \omega) \notin (r_2, r_p)\}.$$

Considering the payoffs associated with $r_2$, when we let $r_2$ converge towards $r_1$ from below, we obtain the effect of a marginal reduction in $r_1$. The limit of the average ratio of the real contract payment to the real renegotiation payment, is then

$$v_a(r_1, r_p) = \lim_{r_2 \to r_1 -} \frac{E\{\int_0^{T_2} R^{\eta_a}(T_1 + s, \omega) \exp(-\beta s) ds\}}{E\{\int_0^{T_2} Z^{\eta_a}(T_1 + s, \omega) \exp(-\beta s) ds\}}.$$

Define $v_p(r_a, r_1)$ correspondingly.

Assume there is a Gaussian component in either $Z$ or $Q$, i.e., that either $a_z > 0$ or $a_q > 0$. Then a well-known property of Gaussian processes implies that when $r_1$ is reached, the probability of reaching $r_p$ before $r_2$ vanishes when $r_2 \to r_1-$ and that

$$\lim_{r_2 \to r_1 -} E\{T_2(r_2/r_1, r_p/r_1, \omega)\} = 0. \tag{17}$$

Equation (17) implies that

$$v_a(r_1, r_p) \approx r_1^{\eta_a} \tag{18}$$

and quite insensitive with respect to variation of $r_p$. We do not have equality in the limit when $r_2 \to r_1$, since with probability zero, the time $T_2(r_2/r_1, r_p/r_1, \omega)$ is positive and in this time period we have that $R(s,\omega)/Z(s,\omega) > r_2$ and $R(s,\omega)/Z(s,\omega)$ may reach $r_p$ before $r_2$. In the approximation we neglect the possibility that $T_2$ does not vanish in the limit. The more volatile the ratio $R(t,\omega)/Z(t,\omega)$ is, and the closer $r_1$ and $r_p$ are to 1, the more $v_a(r_1, r_p)$ is sensitive with respect to variation of $r_p$. By a similar argument, we have $v_p(r_a, r_p) \approx r_p^{\eta_p}$. The function $v_p(r_a, r_p)$ is decreasing in $r_p$ since $\eta_p < 0$.

In the proof of Theorem 4.1, equation (16), it is shown that the optimal threshold satisfies

$$v_a(m_a(r_p), r_p) = (\beta - \mu_{a,z})(u_a(m_a(r_p), r_p) - \tau_a) \tag{19}$$

if $u_a(m_a(r_p), r_p) > \tau_a$. Correspondingly, if $u_p(r_a, m_p(r_a)) > \tau_p$, then

$$v_p(r_a, m_p(r_a)) = (\beta - \mu_{p,z})(u_p(r_a, m_p(r_a)) - \tau_p).$$

Lemma 9.2 gives an interpretation of the coefficient on the right-hand side.

When combining (18) and (19) we get the approximations

$$u_a(m_a(r_p), r_p) \approx \frac{1}{\beta - \mu_{a,z}} m_a^{\eta_a}(r_p) + \tau_a, \tag{20}$$

$$u_p(r_a, m_p(r_a)) \approx \frac{1}{\beta - \mu_{p,z}} m_p^{\eta_p}(r_a) + \tau_p. \tag{21}$$

It is assumed that the optimal thresholds satisfy $r_a > 0$ and $r_p < \infty$. Comparing with the numerical simulations in Section 4, these approximations underestimate $u_a$ and $u_p$ by about 2 percent. The approximation is better the smaller the volatility.

To obtain more intuition for the expression, consider the case with time invariant renegotiation payment, $Z = 1$, implying that $\mu_{\nu,z} = 0$. If in addition, $\eta_a = 1$ and $\eta_p = -1$, then (20) and (21) read $u_a \approx r_a/\beta + \tau_a$ and $u_p \approx 1/(r_p\beta) + \tau_p$, which can be rearranged to $r_a \approx (u_a - \tau_a)\beta$ and $1/r_p \approx (u_p - \tau_p)\beta$.

The following heuristic argument explains these expressions: By renegotiating the contract, a player incurs the renegotiation fee, and then obtains the expected utility after a renegotiation, $u_\nu$. Multiplying by the discount rate $\beta$, we obtain the equivalent flow payoff. A player should demand a renegotiation when the real contract payment equals the flow payoff from requiring a renegotiation, i.e., the critical thresholds are given by these formulas.

These approximations imply that the volatility only influences the thresholds through the expected objective functions $u_\nu$. These relations may be useful in order to find the optimal thresholds.

**Acknowledgment** We are grateful to Karl Ove Moene for valuable comments to an earlier draft, and to Geir Asheim, Fred Espen Benth, and Bernt Øksendal for helpful discussions.

## References


[1] P. Aghion, M. Dewatripont, and P. Rey. Renegotiation design with unverifiable information. *Econometrica* **62**, (1994) 257–282.

[2] T. Andersen and M. Stampe Christensen. Contract renegotiation under uncertainty. Working paper No 1997-5, Dep. of Economics, School of Economics and Management, University of Aarhus.

[3] T. Andersen and M. Stampe Christensen. Contract renewal under uncertainty. *J. Economic Dynamics & Control* **26**, (2002) 637–652.

[4] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion — Facts and Formulae.* Birkhäuser, Basel (1996).

[5] A. Caplin and J. Leahy. Aggregation and optimisation with state-dependent pricing. *Econometrica* **65**, (1997) 601-625.

[6] A. Caplin and D. F. Spulber. Menu costs and the neutrality of money. *Quarterly Journal of Economics* **CII**, (1987) 703-725.

[7] L. Danziger. Price adjustment with stochastic inflation. *International Economic Review* **24**, (1983) 699–707.

[8] E. Dockner, S. Jørgensen, N. Van Long, and G. Sorger. *Differential Games in Economics and Management Science.* Cambridge University Press, Cambridge (2000).

[9] R. Gibbons. *A Primer in Game Theory.* Prentice Hall, UK (1992).

[10] P. Grout. Investment and wages in the absence of binding contracts. A Nash Bargaining Approach *Econometrica* **52**, (1984) 449–460.

[11] O. Hart and J. Moore. Incomplete contracts and renegotiation. *Econometrica* **56**, (1988) 755–785.

[12] S. Holden. Renegotiation and the efficiency of investments. *Rand Journal of Economics* **30**, (1999) 106–119.

[13] S. Holden. Wage bargaining, holdouts and inflation. *Oxford Economic Papers* **49**, (1997) 235–255.

[14] R. Isaacs. *Differential Games.* Wiley, New York (1965).

[15] D. E. Lebow, R. E. Saks, and B. A. Wilson. Downward nominal wage rigidity: Evidence from the Employment cost index. *Advances in Macroeconomics* **3**, (2003), Issue 1, Article 2.

[16] W. B. MacLeod, and J. M. Malcomson. Investment, holdup, and the form of market contracts. *American Economic Review* **37**, (1993) 343–354.

[17] M. J. Osborne and A. Rubinstein. *Bargaining and Markets.* San Diego, Academic Press (1990).

[18] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge (1999).

[19] E. Sheshinsky and Y. Weiss. Inflation and costs of price adjustment. *Review of Economic Studies* **50**, (1983) 513–529.



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