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The semantics of preference-based belief operators*

Geir B. Asheim† and Ylva Søvik‡

February 21, 2003

Abstract

We show how different kinds of belief operators derived from preferences can be defined in terms an accessibility relation of epistemic priority, and characterized by means of a vector of nested accessibility relations. The semantic structure is used to reconcile and compare certain non-standard notions of belief that have recently been used in epistemic analyses of games.

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1 Introduction

In epistemic analyses of games, it is common to use subjective belief operators. There are numerous examples where KD45 operators like 'belief with probability 1' (e.g., Tan & Werlang [25]), 'belief with primary probability 1' (Brandenburger [10]) and 'conditional belief with probability 1' (Ben-Porath [7]) are applied. More recently, Brandenburger & Keisler [13] and Battigalli & Siniscalchi [6] have proposed non-monotonic subjective belief operators called 'assumption' and 'strong belief', respectively. These operators all have in common that they are based on subjective probabilities — arising from a probability distribution, a lexicographic probability

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system, or a conditional probability system — that represent the preferences of the player as a decision maker. Thus, in game theory there is a prevalence of preference-based belief operators.

While all the above contributions use subjective probabilities to define the epistemic operators, Morris [19] observes that it is unnecessary to go via subjective probabilities to derive subjective belief operators from the preferences of a decision maker. This suggestion has been followed by Asheim & Dufwenberg [4] and Asheim [3], who consider epistemic conditions for forward induction and backward inductions without the use of subjective probabilities. In the case of Asheim & Dufwenberg's [4] it is necessary for the characterization of forward induction to use incomplete preferences that cannot be represented by subjective probabilities, while Asheim [3] points to the possibility of characterizing backward induction without the use of subjective probabilities since one can convincingly argue that subjective probabilities play no role in the backward induction argument.

When deriving belief operators from preferences, it is essential that the preferences determine 'subjective possibility' as well as 'epistemic priority'. As we shall see, preferences need not satisfy completeness to determine 'subjective possibility' and 'epistemic priority'. We intend to show how belief operators corresponding to those used in the literature can be derived from preferences that need not be complete.[1]

After presenting the decision-theoretic framework in Sect. 2, we show in Sect. 3 how a binary accessibility relation of epistemic priority Q can be derived from preferences that satisfy conditions that are weaker than those usually applied in the Anscombe–Aumann [2] framework. The properties of this priority relation are similar to but more general than those found, e.g., in Lamarre & Shoham [18] and Stalnaker [23, 24] in that reflexivity is not required. Furthermore, we show how preferences give rise to a vector of nested binary accessibility relations $(R_1, \dots, R_L)$, where, for each k, R_k fulfills the usual properties of Kripke representations of beliefs; i.e., they are serial, transitive and euclidean. Finally, we establish that the two kinds of accessibility relations yield two equivalent representations of 'subjective possibility' and 'epistemic priority'.

[1] The term 'epistemic priority' will here be used to refer to what elsewhere is sometimes referred to as 'plausibility' or 'prejudice'; see, e.g., Friedman & Halpern [14] and Lamerre & Shoham [18]. This is similar to 'preference' among states (or worlds) in nonmonotonic logic (cf. Shoham [22] and Kraus et al. [17]), leading agents towards some states and away from others. In contrast, we use the term 'preferences' in the decision-theoretic sense of a binary relation on the set of functions ('acts') from states to outcomes; see Sect. 2.

From Sect. 4 on we take the accessibility relation of epistemic priority Q as the point of departure. We first define the following belief operators:

- *Certain belief* corresponds to what Morris [19] calls 'Savage-belief' and means that the complement of the event is subjectively impossible.
- *Conditional belief* is a generalization of 'conditional belief with probability 1'.
- *Full belief* corresponds to what Stalnaker [24] calls 'absolutely robust belief'.

We then in Sect. 5 show how these operators can be characterized by means of the vector of nested binary accessibility relations $(R_1, \ldots, R_L)$, while we in Sect. 6 establish properties of these belief operators. In particular, we show that the full belief operator (while poorly behaved) is bounded by certain and conditional belief, which are KD45 operators.

In Sect. 7 we interpret our one-agent decision-theoretic framework in terms of the n-agent decision-theoretic framework encountered in games, and note how the characterization of full belief corresponds to the primitive definition of this operator in Asheim & Dufwenberg [4] as well as Brandenburger & Keisler's [13] concept of 'assumption'. In Sect. 8 we amend the decision-theoretic framework to be able to handle systems of conditional preferences used in analyses of extensive form games and show how Battigalli & Siniscalchi's [6] concept of 'strong belief' is related to full belief. We thereby reconcile and compare these non-standard notions of belief which have recently been used in epistemic analyses of games. We conclude in Sect. 9.

2 Decision-theoretic Framework

Consider a decision maker under uncertainty. Let W be a finite set of states (or possible worlds), where the decision maker is uncertain about what the true state is. Let Z be a finite set of outcomes. In the tradition of Anscombe & Aumann [2], the decision maker has preferences over the set of functions that assign an objective randomization over outcomes to any state. Any such function $\mathbf{x} : W \to \Delta(Z)$ is called an *act* on W. If the true state is a, then the preferences of the decision maker is a binary relation $\succeq^a$ on the set of acts, with $\succ^a$ and $\sim^a$ denoting the asymmetric and symmetric parts, respectively. For any $a \in W$, $\succeq^a$ is assumed to be

- *reflexive* and *transitive*, but not necessarily *complete*.
- *nontrivial* in the sense that there exist $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \succ^a \mathbf{y}$.

- *objectively independent* in the sense that $\mathbf{x}' \succ^a$ (respectively $\sim^a$) $\mathbf{x}''$ iff $\gamma\mathbf{x}' + (1-\gamma)\mathbf{y} \succ^a$ (respectively $\sim^a$) $\gamma\mathbf{x}'' + (1-\gamma)\mathbf{y}$, whenever $0 < \gamma < 1$ and $\mathbf{y}$ is arbitrary.

If $(\emptyset \neq)\ \phi \subseteq W$, let $\mathbf{x}_\phi$ denote the restriction of $\mathbf{x}$ to ϕ. Define the *conditional* binary relation $\succeq^a_\phi$ by $\mathbf{x}' \succeq^a_\phi \mathbf{x}''$ if, for some $\mathbf{y}$, $(\mathbf{x}'_\phi, \mathbf{y}_{\neg\phi}) \succeq^a (\mathbf{x}''_\phi, \mathbf{y}_{\neg\phi})$, where $\neg\phi$ denotes $W\backslash\phi$. By objective independence this definition does not depend on $\mathbf{y}$. Say that the state $b \in W$ is *Savage-null* at a if $\mathbf{x} \sim^a_{\{b\}} \mathbf{y}$ for all acts $\mathbf{x}$ and $\mathbf{y}$ on W. Say that b is deemed *infinitely more likely* than c at a ($b \gg^a c$; cf. Blume et al. [8, Def. 5.1]) if $b \neq c$, b is not Savage-null at a, and $\mathbf{x} \succ^a_{\{b\}} \mathbf{y}$ implies $\mathbf{x} \succ^a_{\{b,c\}} \mathbf{y}$. According to this definition, c may, but need not, be Savage-null at a if $b \gg^a c$. For any $a \in W$, $\succeq^a$ is assumed to satisfy

- *conditional completeness*; i.e., $\forall b \in W$, $\succeq^a_{\{b\}}$ is complete.
- *conditional continuity*; i.e., $\forall b \in W$, there exist $0 < \gamma < \delta < 1$ such that $\delta\mathbf{x}' + (1-\delta)\mathbf{x}'' \succ^a_{\{b\}} \mathbf{y} \succ^a_{\{b\}} \gamma\mathbf{x}' + (1-\gamma)\mathbf{x}''$ whenever $\mathbf{x}' \succ^a_{\{b\}} \mathbf{y} \succ^a_{\{b\}} \mathbf{x}''$.
- *partitional priority*; i.e., if $b \gg^a c$, then, $\forall b' \in W$, $b \gg^a b'$ or $b' \gg^a c$.
- *non-null state independence*; i.e., $\mathbf{x} \succ^a_{\{b\}} \mathbf{y}$ iff $\mathbf{x} \succ^a_{\{c\}} \mathbf{y}$ whenever b and c are not Savage-null at a and $\mathbf{x}$ and $\mathbf{y}$ satisfy $\mathbf{x}(b) = \mathbf{x}(c)$ and $\mathbf{y}(b) = \mathbf{y}(c)$.

Let W be partitioned into equivalence classes, where $a \approx b$ denotes that a and b are in the same equivalence class, with $\approx$ being a reflexive, transitive and symmetric binary relation. Write $\tau^a := \{b \in W | \, a \approx b\}$. Let κ^a denote the set of states that are *not* Savage-null at a. Since $\succeq^a$ is nontrivial, $\kappa^a \neq \emptyset$. Assume that, for any $a \in W$, $\kappa^a \subseteq \tau^a$, and $\forall a' \in \tau^a$, $\mathbf{x} \succeq^{a'} \mathbf{y}$ iff $\mathbf{x} \succeq^a \mathbf{y}$. This assumption will ensure that the preference-based operators satisfy positive and negative introspection; it corresponds to "being aware of one's own type".

Definition 1 A *preference system* $\{\succeq^a \,|\, a \in W\}$ consists of

(1) a finite set of states W that is partitioned into equivalence classes by $\approx$, and

(2) for each $a \in W$, a reflexive and transitive binary relation $\succeq^a$ on the set of acts (where each act is a function $\mathbf{x} : W \to \Delta(Z)$ and Z is a finite set of outcomes), depending only on to which equivalence class a belongs, and satisfying non-triviality, objective independence, conditional completeness, conditional continuity, partitional priority, non-null state independence, and that $a \approx b$ if b is not Savage-null at a.

Since, as shown by the following result, partitional priority is implied if conditional completeness is strengthened to completeness, Def. 1 generalizes the decision-theoretic framework considered by Blume et al. [8].

Proposition 1 *If $\succeq^a$ is complete, then the assumption of partitional priority is redundant.*

Proof. We must show that, under completeness, if $b \gg^a c$, then, $\forall b' \in W$, $b \gg^a b'$ or $b' \gg^a c$. Clearly, $b \gg^a c$ entails $b \in \kappa^a$, implying that $b \gg^a b'$ or $b' \gg^a c$ if $b' \notin \kappa^a$ or $c \notin \kappa^a$. The case where $b' = b$ or $b' = c$ is trivial. The case where $b' \neq b$, $b' \neq c$, $b' \in \kappa^a$ and $c \in \kappa^a$ remains. Assume that $b \gg^a b'$ does not hold, which by completeness entails the existence of $\mathbf{x}'$ and $\mathbf{y}'$ such that $\mathbf{x}' \preceq^a_{\{b,b'\}} \mathbf{y}'$ and $\mathbf{x}' \succ^a_{\{b\}} \mathbf{y}'$. It suffices to show that $b' \gg^a c$ is obtained; i.e., $\mathbf{x} \succ^a_{\{b'\}} \mathbf{y}$ implies $\mathbf{x} \succ^a_{\{b',c\}} \mathbf{y}$.

Let $\mathbf{x} \succ^a_{\{b'\}} \mathbf{y}$. Assume w.l.o.g. that $\mathbf{x}(a') = \mathbf{y}(a')$ for $a' \neq b'$, c, and $\mathbf{x}'(a') = \mathbf{y}'(a')$ for $a' \neq b$, b'. By transitivity, $\mathbf{x}' \preceq^a_{\{b,b'\}} \mathbf{y}'$ and $\mathbf{x}' \succ^a_{\{b\}} \mathbf{y}'$ imply $\mathbf{x}' \prec^a_{\{b'\}} \mathbf{y}'$. However, since $\succeq^a$ satisfies objective independence and conditional continuity, $\exists \gamma \in (0,1)$ such that $\gamma\mathbf{x} + (1-\gamma)\mathbf{x}' \succ^a_{\{b'\}} \gamma\mathbf{y} + (1-\gamma)\mathbf{y}'$. Moreover, $\mathbf{x}(b) = \mathbf{y}(b)$ and $\mathbf{x}' \succ^a_{\{b\}} \mathbf{y}'$ entail that $\gamma\mathbf{x} + (1-\gamma)\mathbf{x}' \succ^a_{\{b\}} \gamma\mathbf{y} + (1-\gamma)\mathbf{y}'$ by objective independence, which implies that $\gamma\mathbf{x} + (1-\gamma)\mathbf{x}' \succ^a_{\{b,c\}} \gamma\mathbf{y} + (1-\gamma)\mathbf{y}'$ since $b \gg^a c$. Hence, by transitivity, $\gamma\mathbf{x} + (1-\gamma)\mathbf{x}' \succ^a_{\{b,b',c\}} \gamma\mathbf{y} + (1-\gamma)\mathbf{y}'$ — or equivalently, $\gamma\mathbf{x} + (1-\gamma)\mathbf{x}' \succ^a \gamma\mathbf{y} + (1-\gamma)\mathbf{y}$. Now, $\mathbf{y}' \succeq^a_{\{b,b'\}} \mathbf{x}'$ means that $\gamma\mathbf{x} + (1-\gamma)\mathbf{y}' \succeq^a \gamma\mathbf{x} + (1-\gamma)\mathbf{x}'$ by objective independence, implying that $\gamma\mathbf{x} + (1-\gamma)\mathbf{y}' \succ^a \gamma\mathbf{y} + (1-\gamma)\mathbf{y}'$ by transitivity, and $\mathbf{x} \succ^a \mathbf{y}$ — or equivalently, $\mathbf{x} \succ^a_{\{b',c\}} \mathbf{y}$ — by objective independence. Thus, $\mathbf{x} \succ^a_{\{b'\}} \mathbf{y}$ implies $\mathbf{x} \succ^a_{\{b',c\}} \mathbf{y}$, meaning that $b' \gg^a c$. ■

Say that $\succeq^a$ is *conditionally represented* by a vNM utility function $u^a : Z \to \mathbb{R}$ (writing $u^a(x) = \sum_{z \in Z} x(z) u^a(z)$ whenever $x \in \Delta(Z)$ is an objective randomization) if (1) $\succeq^a$ is nontrivial and (2) $\mathbf{x} \succeq^a_{\{b\}} \mathbf{y}$ iff $u^a(\mathbf{x}(b)) \geq u^a(\mathbf{y}(b))$ whenever b is not Savage-null at a. By the properties of the bullet points it follows directly from the vNM theorem on expected utility representation that there, for any $a \in W$, exists a vNM utility function u^a such that $\succeq^a$ is conditionally represented by u^a. If $A \subseteq W$, say that $\mathbf{x}_A$ *weakly dominates* $\mathbf{y}_A$ at a if, $\forall b \in A$, $u^a(\mathbf{x}_A(b)) \geq u^a(\mathbf{y}_A(b))$, with strict inequality for some $c \in A$. Say that $\succeq^a$ *is admissible* on A if A is non-empty and $\mathbf{x} \succ^a \mathbf{y}$ whenever $\mathbf{x}_A$ weakly dominates $\mathbf{y}_A$ at a.

The following connection between admissibility on subsets and the infinitely-more-likely relation is important for relating the accessibility relations derived from preferences in the next section (cf. Sect. 3.3).

Proposition 2 *Let $A \neq \emptyset$ and $\neg A \neq \emptyset$. $\succeq^a$ is admissible on A iff $b \in A$ and $c \in \neg A$ imply $b \gg^a c$.*

Proof. *Only if.* Assume that $\succeq^a$ is admissible on A. Let $b \in A$ and $c \in \neg A$. It now follows directly that b is not Savage-null at a and that $\mathbf{x} \succ^a_{\{b\}} \mathbf{y}$ implies $\mathbf{x} \succ^a_{\{b,c\}} \mathbf{y}$. *If.* Assume that $b \in A$ and $c \in \neg A$ imply $b \gg^a c$. Let $\mathbf{x}$ and $\mathbf{y}$ satisfy that $\mathbf{x}_A$ weakly dominates $\mathbf{y}_A$ at a. Then there exists $b_0 \in A$ such that $u^a(\mathbf{x}(b_0)) > u^a(\mathbf{y}(b_0))$. Write $\neg A = \{c_1, \dots, c_n\}$. Let, for $m \in \{0, \dots, n\}$,

$$\mathbf{x}^m(a') = \begin{cases} \frac{n+1-m}{n+1}\mathbf{x}(a') + \frac{m}{n+1}\mathbf{y}(a') & \text{if } a' = b_0 \\ \mathbf{x}(a') & \text{if } a' \in A \backslash b_0 \\ \mathbf{y}(a') & \text{if } a' = c_{m'} \text{ and } m' \in \{1, \dots, m\} \\ \mathbf{x}(a') & \text{if } a' = c_{m'} \text{ and } m' \in \{m+1, \dots, n\}. \end{cases}$$

Then $\mathbf{x} = \mathbf{x}^0$, $\mathbf{x}^{m-1} \succ^a \mathbf{x}^m$ for all $m = \{1, \dots, n\}$ (since $b \in A$ and $c \in \neg A$ imply that $b \gg^a c$), and $\mathbf{x}^n \succ^a \mathbf{y}$ (since $\mathbf{x}^n$ weakly dominates $\mathbf{y}$ at a with $u^a(\mathbf{x}^n(b_0)) > u^a(\mathbf{y}(b_0))$). By transitivity of $\succeq^a$, it follows that $\mathbf{x} \succ^a \mathbf{y}$. ■

3 From Preferences to Accessibility Relations

The purpose of this section is to show how two different kinds of accessibility relations (see, e.g., Lamarre & Shoham [18] and Stalnaker [23, 24]) can be derived from preferences. The one kind is based on the infinitely-more-likely relation, while the other is based on admissibility on subsets.

3.1 Accessibility relation of epistemic priority.

Consider the following definition of the accessibility relation Q.

Definition 2 aQb ("a does not have higher epistemic priority than b") if
(1) $a \approx b$,
(2) b is not Savage-null at a, and
(3) a is not deemed infinitely more likely than b at a.

Proposition 3 *The relation Q is serial,*[2] *transitive, and satisfies forward linearity*[3] *and quasi-backward linearity.*[4]

[2]$\forall a$, $\exists b$ such that aQb.

[3]aQb and aQc imply bQc or cQb.

[4]If $\exists a' \in W$ such that $a'Qb$, then aQc and bQc imply aQb or bQa.

Proof. *(Q serial.)* If a is Savage-null at a, then there exists $b \in \tau^a$ such that b is not Savage-null at a since $\succeq^a$ is nontrivial. Clearly, a is not infinitely more likely than b at a, and aQb. If a is not Savage-null at a, then aQa since a is not infinitely more likely than itself at a.

(Q transitive.) We must show that aQb and bQc imply aQc. Clearly, aQb and bQc imply $a \approx b \approx c$, and that c is not Savage-null at a. It remains to be shown that $a \gg^a c$ does not hold if aQb and bQc. Suppose to the contrary that $a \gg^a c$. It suffices to show that aQb contradicts bQc. Since c is not Savage-null at $a \approx b$, $b \gg^a c$ is needed to contradict bQc. This follows from partitional priority because aQb entails that $a \gg^a b$ does not hold.

(Q satisfies forward linearity.) We must show that aQb and aQc imply bQc or cQb. From aQb and aQc it follows that $a \approx b \approx c$ and that both b and c are not Savage-null at $b \approx c$. Since $b \gg^b c$ and $c \gg^c b$ cannot both hold, we have that bQc or cQb.

(Q satisfies quasi-backward linearity.) We must show that aQc and bQc imply aQb or bQa if $\exists a' \in W$ such that $a'Qb$. From aQc and bQc it follows that $a \approx b \approx c$, while $a'Qb$ implies that b is not Savage-null at $a' \approx a \approx b$. If a is Savage-null at a, then $a \gg^a b$ cannot hold, implying that aQb. If a is not Savage-null at $a \approx b$, then $a \gg^a b$ and $b \gg^b a$ cannot both hold, implying that aQb or bQa. ■

3.2 A vector of nested accessibility relations.

Consider the collection of all sets A satisfying that $\succeq^a$ is admissible on A. Since $\succeq^a$ is admissible on κ^a, it follows that the collection is non-empty as it is contains κ^a. Furthermore, since any $b \in A$ is not Savage-null at a if $\succeq^a$ is admissible on A, it follows that any set in this collection is a subset of κ^a. Finally, since $b \gg^a c$ implies that $c \gg^a b$ does not hold, it follows from Prop. 2 that $A' \subseteq A''$ or $A'' \subseteq A'$ if $\succeq^a$ is admissible on both A' and A'', implying that the sets in the collection are nested. Hence, there exists a vector of nested sets, $(\rho_1^a, \dots, \rho_{L^a}^a)$, on which $\succeq^a$ is admissible, satisfying: $\emptyset \neq \rho_1^a \subset \cdots \subset \rho_k^a \subset \cdots \subset \rho_{L^a}^a = \kappa^a \subseteq \tau^a$ (where $\subset$ denotes $\subseteq$ and $\neq$). Let $L := \max_{a \in W} L^a$. If, for some $a \in W$, $L^a < L$, let $\rho_{L^a}^a = \rho_k^a = \kappa^a$ for $k \in \{L^a + 1, \dots, L\}$. The collection of sets, $\{\rho_k^a | \, a \in W\}$, defines an accessibility relation, R_k.

Definition 3 aR_kb ("at a, b is deemed possible at the epistemic level k") if $b \in \rho_k^a$.

Proposition 4 *The vector of relations, $(R_1, \ldots, R_L)$, has the following properties: For each $k \in \{1, \ldots, L\}$, R_k is serial, transitive, and euclidian.*[5] *For each $k \in \{1, \ldots, L-1\}$, (i) aR_kb implies $aR_{k+1}b$ and (ii) ($\exists c$ such that $aR_{k+1}c$ and $bR_{k+1}c$) implies ($\exists c'$ such that aR_kc' and bR_kc').*

Proof. *(R_k serial.)* For all $a \in W$, $\rho_k^a \neq \emptyset$.

(R_k transitive.) We must show that aR_kb and bR_kc imply aR_kc. Since aR_kb implies that $a \approx b$, we have that $\rho_k^a = \rho_k^b$. Now, bR_kc (i.e., $c \in \rho_k^b$) implies aR_kc (i.e., $c \in \rho_k^a$).

(R_k euclidian.) We must show that aR_kb and aR_kc imply bR_kc. Since aR_kb implies that $a \approx b$, we have that $\rho_k^a = \rho_k^b$. Now, aR_kc (i.e., $c \in \rho_k^a$) implies bR_kc (i.e., $c \in \rho_k^b$).

(aR_kb implies $aR_{k+1}b$.) This follows from the property that $\rho_k^a \subseteq \rho_{k+1}^a$.

($\exists c$ such that $aR_{k+1}c$ and $bR_{k+1}c$) implies ($\exists c'$ such that aR_kc' and bR_kc'). Since $aR_{k+1}c$ implies that $a \approx c$ and $bR_{k+1}c$ implies that $b \approx c$, we have that $a \approx b$ and $\rho_k^a = \rho_k^b$. Hence, by the non-emptiness of this set, $\exists c'$ such that aR_kc' and bR_kc'. ■

3.3 The correspondence between Q and $(R_1, \ldots, R_L)$

Below we prove a correspondence between the relations Q and $(R_1, \ldots, R_L)$.

Proposition 5 *(i) aQa iff aR_La. (ii) (aQb and not bQa) iff ($\exists k \in \{1, \ldots, L\}$ such that aR_kb and not bR_ka).*

Proof. *(i) (aQa is equivalent to a being not Savage-null at a.)* If aQa, then it follows directly from Def. 2 that a is not Savage-null at a. If a is not Savage-null at a, then by Def. 2 it follows that aQa since $a \approx a$ and not $a \gg^a a$. *(aR_La is equivalent to a being not Savage-null at a.)* By Def. 3, aR_La iff $a \in \rho_L^a = \kappa^a$, which directly establishes the result.

(ii) Only if. Assume that aQb and not bQa. From aQb it follows that $a \approx b$ and b is not Savage-null at a, i.e. $b \in \kappa^a (\subseteq \tau^a)$. Consider $A := \{b' \in W \,|\, bQb'\}$. Clearly, $b \in A \subseteq \kappa^a (\subseteq \tau^a)$ and $a \in \tau^a \backslash A \neq \emptyset$. If $b' \in A$ and $c \in \tau^a \backslash A$, then not $b'Qc$, since otherwise it would follow from bQb' and the transitivity of Q that bQc, thereby contradicting $c \notin A$. If, on the one hand, $c \in \kappa^a \backslash A$, then $b' \gg^a c$ since c is not Savage-null at $a \approx b'$ and $b'Qc$ does not hold. If, on the other hand, $c \notin \kappa^a$,

[5] aR_kb and aR_kc imply bR_kc.

then $b' \gg^a c$ since c is Savage-null at a and b' is not. Hence, $b' \in A$ and $c \in \neg A$ imply $b' \gg^a c$. By Prop. 2, $\succeq^a$ is admissible on A, entailing that $\exists k \in \{1, \ldots, L\}$ such that $\rho_k^a = A$. By Def. 3, $aR_k b$ and not $bR_k a$ since $b \in A$ and $a \in \tau^a \backslash A$.

If. Assume that $\exists k \in \{1, \ldots, L\}$ such that $aR_k b$ and not $bR_k a$. From $aR_k b$ it follows that $a \approx b$ and $b \in \rho_k^a (\subseteq \kappa^a)$; in particular, b is not Savage-null at a. Since $bR_k a$ does not hold, however, $a \notin \rho_k^b = \rho_k^a$. By construction, $\succeq^a$ is admissible on ρ_k^a, and it now follows from Prop 2 that $b \gg^a a$. Furthermore, $b \gg^a a$ implies that $a \gg^a b$ does not hold. Hence, aQb since $a \approx b$, b is not Savage-null at a and $a \gg^a b$ does not hold, while not bQa since $b \gg^a a$. ■

That a is not Savage-null at a can be interpreted as a being deemed subjectively possible (at some epistemic level) at any state in the same equivalence class. By Prop. 5(i), a being not Savage-null at a has two equivalent representations in terms of accessibility relations: aQa and $aR_L a$. Likewise, $b \gg^a a$ can be interpreted as b having a higher epistemic priority than a. By Prop. 5(ii), $b \gg^a a$ have two equivalent representations: (aQb and not bQa) and ($\exists k \in \{1, \ldots, L\}$ such that $aR_k b$ and not $bR_k a$). Thus, both Q and $(R_1, \ldots, R_L)$ capture 'subjective possibility' and 'epistemic priority' as implied by the preferences of the preference system.

If conditional continuity is strengthened to *continuity* — i.e., $\forall b \in W$, there exist $0 < \gamma < \delta < 1$ such that $\delta \mathbf{x}' + (1-\delta)\mathbf{x}'' \succ^a \mathbf{y} \succ^a \gamma \mathbf{x}' + (1-\gamma)\mathbf{x}''$ whenever $\mathbf{x}' \succ^a \mathbf{y} \succ^a \mathbf{x}''$ — then b being deemed infinitely more likely than c at a implies that c is Savage-null. Hence, $L = 1$, and by Defs. 2 and 3, $Q = R_1$. Hence, we are left with a unique serial, transitive, and euclidian accessibility relation if preferences are continuous.

4 Defining Belief Operators

In the following Sects. 4–6 our point of departure will be an accessibility relation of epistemic priority having the properties that are specified in Prop. 3. We shall first show how equivalence classes can be derived, and then demonstrate how various belief operators can be defined and characterized, on the basis of this accessibility relation. Since – as shown through Def. 2 and Prop. 3 – such an accessibility relation can be derived from preferences of a preference system that satisfy the minimum requirements of Def. 1, these belief operators can be considered to be preference-based.

We first restate the properties of Prop. 3 as a primitive definition before deriving equivalence classes and defining belief operators.

Definition 4 The relation Q is serial, transitive, and satisfies forward linearity and quasi-backward linearity.

We will interpret aQa as a being deemed subjectively possible at any state in the same equivalence class, and (aQb and not bQa) as b having a higher epistemic priority than a. Hence, when interpreted in terms of preferences, aQa corresponds to a not being Savage-null at a, and (aQb and not bQa) corresponds to b being infinitely more likely than a at a.

4.1 Deriving equivalence classes

Define the relation $\approx$ as follows.

Definition 5 $a \approx b$ if $\exists c \in W$ such that aQc and bQc.

We must show that $\approx$ is an equivalence relation.

Proposition 6 *The relation $\approx$ is reflexive, transitive, and symmetric.*

Proof. *($\approx$ reflexive.)* Since Q is serial, $\forall a$, $\exists c$ such that aQc. Hence, $a \approx a$.

($\approx$ transitive.) We must show that $a \approx b$ and $b \approx c$ imply $a \approx c$. From $a \approx b$ it follows that $\exists b'$ such that aQb' and bQb'. From $b \approx c$ it follows that $\exists c'$ such that bQc' and cQc'. It now follows from forward linearity that either (i) $b'Qc'$, which by transitivity of Q implies aQc' and cQc', or (ii) $c'Qb'$, which by transitivity of Q implies aQb' and cQb'. In either case we have established that $a \approx c$.

($\approx$ symmetric.) Obvious. ■

Write $\tau^a := \{b \in W |\ a \approx b\}$. The following observation is useful.

Lemma 1 *If aQb, then $a \approx b$ and bQb.*

Proof. Let aQb. By seriality, $\exists c$ such that bQc and, by transitivity, aQc. Hence, $\exists c$ such that bQc and aQc, which by Def. 5 yields $a \approx b$. Furthermore, by forward linearity, aQb implies that bQb. ■

4.2 Certain, conditional, and full belief

Let

$$\kappa^a := \{b \in \tau^a | \exists c \text{ such that } cQb\} = \{b \in \tau^a | bQb\},$$

denote the set of states that are deemed subjectively possible at a, where $\kappa^a \neq \emptyset$ since Q is serial, and where the last equality follows from Lemma 1. Define 'certain belief' as follows.

Definition 6 At a the decision maker *certainly believes* A if $a \in \mathrm{K}A$, where $\mathrm{K}A := \{a \in W | \kappa^a \subseteq A\}$.

Hence, at a an event A is certainly believed if the complement is deemed subjectively impossible at a. This corresponds to what Morris [19] calls 'Savage-belief'.

'Conditional belief' is defined conditionally on sets that are subjectively possible at any state; i.e., sets in the following collection:

$$\Phi := \{\phi \in 2^W \backslash \{\emptyset\} | \, \forall a \in W, \, \kappa^a \cap \phi \neq \emptyset\}.$$

In particular, $W \in \Phi$ and, $\forall \phi \in \Phi$, $\emptyset \neq \phi \subseteq W$.

Since every $\phi \in \Phi$ is subjectively possible at any state, it follows that, $\forall \phi \in \Phi$,

$$\beta^a(\phi) := \{b \in \tau^a \cap \phi | \forall c \in \tau^a \cap \phi, cQb\}$$

is nonempty, as demonstrated by the following lemma.

Lemma 2 *If $\kappa^a \cap \phi \neq \emptyset$, then $\exists b \in \tau^a \cap \phi$ such that $\forall c \in \tau^a \cap \phi, cQb$.*

Proof. It follows from the definition of κ^a that $\exists b_1 \in \tau^a \cap \phi$ such that b_1Qb_1 if $\kappa^a \cap \phi \neq \emptyset$. Either, $\forall c \in \tau^a \cap \phi$, cQb_1 – in which case we are through – or not. In the latter case, $\exists b_2 \in \tau^a \cap \phi$ such that b_2Qb_1 does not hold. Since $b_1, b_2 \in \tau^a$, $\exists b_2' \in \tau^a$ such that b_1Qb_2' and b_2Qb_2'. Since b_1Qb_1 and not b_2Qb_1 it now follows from quasi-backward linearity that b_1Qb_2. Moreover, not b_2Qb_1 implies $b_2 \neq b_1$. Either $\forall c \in \tau^a \cap \phi$, cQb_2 – in which case we are through – or not. In the latter case we can, by repeating the above argument and invoking transitivity, show the existence of some $b_3 \in \tau^a \cap \phi$ such that b_1Qb_3, b_2Qb_3, and $b_3 \neq b_1, b_2$. Since $\tau^a \cap \phi$ is finite, this algorithm converges to some b satisfying, $\forall c \in \tau^a \cap \phi, cQb$. ■

Define 'conditional belief' as follows.

Definition 7 At a the decision maker *believes* A *conditional* on ϕ if $a \in \mathrm{B}(\phi)A$, where $\mathrm{B}(\phi)A := \{a \in W | \beta^a(\phi) \subseteq A\}$.

Hence, at a an event A is believed conditional on ϕ if A contains any state in $\tau^a \cap \phi$ with at least as high epistemic priority as any other state in $\tau^a \cap \phi$. This way of defining conditional belief is in the tradition of, e.g., Grove [15], Katsuno & Mendelzon [16], Boutilier [9], and Lamerre & Shoham [18].

Let Φ^A be the collection of subjectively possible events ϕ having the property that A is subjectively possible conditional on ϕ whenever A is subjectively possible:

$$\Phi^A := \{\phi \in 2^W \backslash \{\emptyset\} | \, \forall a \in W, \, \kappa^a \cap \phi \neq \emptyset, \text{ and } A \cap \kappa^a \cap \phi \neq \emptyset \text{ if } A \cap \kappa^a \neq \emptyset\} \, .$$

Note that Φ^A is a subset of Φ that satisfies $W \in \Phi^A$; hence, $\emptyset \neq \Phi^A \subseteq \Phi$.

Define 'full belief' as follows.

Definition 8 At a the decision maker *fully believes* A if $a \in \mathrm{B}^0 A$, where $\mathrm{B}^0 A := \cap_{\phi \in \Phi^A} \mathrm{B}(\phi) A$.

Hence, at a an event A is fully believed if A is believed conditional on any ϕ that does not make A subjectively impossible. This corresponds to what Stalnaker [24] calls 'absolutely robust belief'. The relation between this belief operator and the operators 'assumption' and 'strong belief', introduced by Brandenburger & Keisler [13] and Battigalli & Siniscalchi [6] respectively, will be discussed in Sects. 7 and 8.

5 Characterizing Belief Operators

As in the previous section, our point of departure is the accessibility relation of epistemic priority, Q, having the properties that are specified in Def. 4. We now show how the certain, conditional, and full belief operators – which were defined in Defs. 6–8 directly from Q – can be characterized by means a vector of nested accessibility relations $(R_1, \dots, R_L)$ derived from Q. Hence, we first derive $(R_1, \dots, R_L)$ from Q and then show how $(R_1, \dots, R_L)$ characterizes the belief operators.

5.1 Deriving the vector of relations $(R_1, \dots, R_L)$

Define a vector of nested sets $(\rho_1^a, \dots, \rho_{L^a}^a)$ as follows.

Definition 9 $\rho_1^a := \{b \in \tau^a | \forall c \in \tau^a, \, cQb\}$

$\rho_2^a := \rho_1^a \cup \{b \in \tau^a | \forall c \in \tau^a \backslash \rho_1^a, \, cQb\}$

...

$\rho_k^a := \rho_{k-1}^a \cup \{b \in \tau^a | \forall c \in \tau^a \backslash \rho_{k-1}^a, \, cQb\}$

$\dots$

$\rho^a_{L^a} := \rho^a_{L^a-1} \cup \{b \in \tau^a | \forall c \in \tau^a \backslash \rho^a_{L^a-1},\, cQb\}$,

where L^a is determined such that $\rho^a_{L^a-1} \neq \rho^a_{L^a}$, while $\rho^a_{L^a} = \rho^a_{L^a+1}$ if $\rho^a_{L^a+1}$ were defined by $\rho^a_{L^a+1} := \rho^a_{L^a} \cup \{b \in \tau^a | \forall c \in \tau^a \backslash \rho^a_{L^a},\, cQb\}$.

When interpreted in terms of preference, $\{\rho^a_1, \dots, \rho^a_{L^a}\}$ corresponds to the collection of sets on which the preferences at a are admissible.

Lemma 3 $\emptyset \neq \rho^a_1 \subset \cdots \subset \rho^a_k \subset \cdots \subset \rho^a_{L^a} \subseteq \tau^a$.

Proof. It is sufficient to show that, $\forall a$, $\rho^a_1 \neq \emptyset$; i.e., $\forall a$, $\exists b$ such that, $\forall c \in \tau^a$, cQb. This follows directly from Lemma 2 by setting $\phi = W$ since $\kappa^a \neq \emptyset$. ■

Let $L := \max_{a \in W} L^a$. If, for some $a \in W$, $L^a < L$, let $\rho^a_{L^a} = \rho^a_k$ for $k \in \{L^a + 1, \dots, L\}$. The collection of sets, $\{\rho^a_k | a \in W\}$, defines a relation, R_k.

Definition 10 aR_kb if $b \in \rho^a_k$.

The following observation holds for any $k \in \{1, \dots, L\}$, implying that $\tau^a := \{b \in W | a \approx b\}$ equals $\{b \in W | \exists c$ such that aR_kc and $bR_kc\}$.

Lemma 4 *$a \approx b$ iff $\exists c$ such that aR_kc and bR_kc.*

Proof. The lemma follows directly from Lemma 3 and Def. 10. ■

Proposition 7 *The vector of relations, $(R_1, \dots, R_L)$, has the following properties: For each $k \in \{1, \dots, L\}$, R_k is serial, transitive, and euclidian. For each $k \in \{1, \dots, L-1\}$, (i) aR_kb implies $aR_{k+1}b$ and (ii) ($\exists c$ such that $aR_{k+1}c$ and $bR_{k+1}c$) implies ($\exists c'$ such that aR_kc' and bR_kc').*

Proof. This proof is identical to the proof of Prop. 4. ■

To show that the correspondence between the relations Q and $(R_1, \dots, R_L)$ as derived in the present section is the same as the correspondence between the preference-based relations Q and $(R_1, \dots, R_L)$ of Sect. 3 (cf. Prop. 5), we need the following result. In particular, by comparing Prop. 5(ii) and Prop. 8(ii), it follows that $(R_1, \dots, R_L)$ as derived from preferences by means of Def. 3 coincides with $(R_1, \dots, R_L)$ as derived from preferences through Defs. 2, 9, and 10.

Proposition 8 *(i) aQa iff aR_La. (ii) (aQb and not bQa) iff ($\exists k \in \{1, \dots, L\}$ such that aR_kb and not bR_ka).*

The following characterization result is helpful for establishing Prop. 8.

Lemma 5 *($\exists c \in \tau^a$ such that $A = \{b \in W | cQb\}$) iff ($\exists k \in \{1, \ldots, L\}$ such that $A = \rho_k^a$).*

Proof. *Only if.* Assume $\exists c \in \tau^a$ such that $A = \{b \in W | cQb\}$. Either $c \in \rho_k^a \backslash \rho_{k-1}^a$ for some $k \in \{1, \ldots, L^a\}$ (writing $\rho_0^a = \emptyset$) or $c \in \tau^a \backslash \rho_{L^a}^a$. Consider first the case where $c \in \rho_k^a \backslash \rho_{k-1}^a$ for some k. It follows directly from Def. 9 that $A \supseteq \rho_k^a$ since, $\forall b \in \rho_k^a$, cQb. The converse, $A \subseteq \rho_k^a$, follows also from Def. 9 since cQb and transitivity combined with, $\forall c' \in \tau^a \backslash \rho_{k-1}^a$, $c'Qc$ imply $b \in \rho_k^a$. Consider then the case where $c \in \tau^a \backslash \rho_{L^a}^a$. It follows directly from Def. 9 that $A \supseteq \rho_{L^a}^a$ since, $\forall b \in \rho_{L^a}^a$, cQb. To establish the converse, suppose $A \backslash \rho_{L^a}^a \neq \emptyset$; i.e. $\exists b_0 \in \tau^a \backslash \rho_{L^a}^a$ such that cQb_0. Then by quasi-backward linearity, $\forall c' \in \tau^a \backslash \rho_{L^a}^a$, $c'Qb_0$ or b_0Qc'. Either, $\forall c' \in \tau^a \backslash \rho_{L^a}^a$, $c'Qb_0$, thus establishing a contradiction to $\{b \in \tau^a | \forall c' \in \tau^a \backslash \rho_{L^a}^a, c'Qb\} \backslash \rho_{L^a}^a = \emptyset$ (cf. Def. 9), or not. In the latter case, $\exists b_1 \in \tau^a \backslash \rho_{L^a}^a$ such that b_0Qb_1 and not b_1Qb_0. Clearly, b_0Qb_1 and $b_1 \neq b_0$. Either, $\forall c' \in \tau^a \backslash \rho_{L^a}^a$, $c'Qb_1$, thus establishing a contradiction to $\{b \in \tau^a | \forall c' \in \tau^a \backslash \rho_{L^a}^a, c'Qb\} \backslash \rho_{L^a}^a = \emptyset$, or not. In the latter case, we can show the existence of some $b_2 \in \tau^a \backslash \rho_{L^a}^a$ such that b_0Qb_2, b_1Qb_2, and $b_2 \neq b_0, b_1$. Since $\tau^a \backslash \rho_{L^a}^a$ is finite, this algorithm will converge to some $b \in \tau^a \backslash \rho_{L^a}^a$ satisfying, $\forall c' \in \tau^a \backslash \rho_{L^a}^a$, $c'Qb$, thus contradicting $\{b \in \tau^a | \forall c' \in \tau^a \backslash \rho_{L^a}^a, c'Qb\} \backslash \rho_{L^a}^a = \emptyset$. Hence, $A \backslash \rho_{L^a}^a = \emptyset$, or equivalently, $A \subseteq \rho_{L^a}^a$.

If. Assume $\exists k \in \{1, \ldots, L^a\}$ such that $A = \rho_k^a$. Since by Lemma 3, $\rho_k^a \backslash \rho_{k-1}^a \neq \emptyset$ (writing $\rho_0^a = \emptyset$), it is sufficient to show that, $\forall c \in \rho_k^a \backslash \rho_{k-1}^a$, $A = \{b \in W | cQb\}$. Let $c \in \rho_k^a \backslash \rho_{k-1}^a$. It follow directly from Def. 9 that $A \subseteq \{b \in W | cQb\}$ since, $\forall b \in A$, cQb. The converse, $A \supseteq \{b \in W | cQb\}$, follows also from Def. 9 since cQb and transitivity combined with, $\forall c' \in \tau^a \backslash \rho_{k-1}^a$, $c'Qc$ imply $b \in A$. ■

Proof of Prop. 8. *(i) Only if.* Assume that aQa. Consider $A := \{b \in W | \, aQb\}$. Clearly, $a \in A$. Lemma 5 implies that $\exists k \in \{1, \ldots, L\}$ such that $\rho_k^a = A$. This in turn implies that $a \in \rho_k^a \subseteq \rho_L^a$ and aR_La by Def. 10.

If. Assume that aR_La. By Def. 10, $a \in \rho_L^a$, and by Def. 9, aQa.

(ii) Only if. Assume that aQb and not bQa. By Lemma 1 it follows from aQb that $b \in \tau^a$ and bQb. Consider $A := \{b' \in W | \, bQb'\}$. We have that $b \in A$ since bQb, while $a \in \tau^a \backslash A$ since not bQa. It follows from Lemma 5 that $\exists k \in \{1, \ldots, L\}$ such that $\rho_k^a = A$. By Def. 10, aR_kb and not bR_ka since $b \in A$ and $a \in \tau^a \backslash A$.

If. Assume that $\exists k \in \{1, \ldots, L\}$ such that aR_kb and not bR_ka. W.l.o.g. we may assume that $k \in \{1, \ldots, L^a\}$, implying that $\rho_{k-1}^a \neq \rho_k^a$. From aR_kb it follows that $a \approx b$ and $b \in \rho_k^a$. Since $bR_k a$ does not hold, however, $a \notin \rho_k^b = \rho_k^a$. It follows directly from Def. 9 that aQb. Suppose bQa. Then, since by Def. 9, $\forall c \in \tau^a \backslash \rho_{k-1}^a$, cQb, it follows by the transitivity of Q that $\forall c \in \tau^a \backslash \rho_{k-1}^a$, cQa. This leads to the contradiction that $a \in \rho_k^a$. ■

5.2 Certain, conditional, and full belief

The purpose of this subsection is to show that the certain, conditional, and full belief operators can be characterized by means of the vector of relations $(R_1, \ldots, R_L)$.

Proposition 9 $\mathrm{K}A = \{a \in W | \rho_L^a \subseteq A\}$.

Proof. It suffices to show that $\rho_L^a = \kappa^a$. By the definition of κ^a and Lemma 1, this follows directly from Prop. 8(i). ■

Proposition 10 *$\forall \phi \in \Phi$, $\mathrm{B}(\phi)A = \{a \in W | \exists k \in \{1, \ldots, L\}$ such that $\emptyset \neq \rho_k^a \cap \phi \subseteq A\}$.*

To prove Prop. 10 it suffices to show the following lemma.

Lemma 6 *If $\phi \in \Phi$, then $\rho_\ell^a \cap \phi = \beta^a(\phi)$, where $\ell := \min\{k \in \{1, \ldots, L\} | \rho_k^a \cap \phi \neq \emptyset\}$.*

Proof. ($\beta^a(\phi) \subseteq \rho_\ell^a \cap \phi$) Assume that $(\tau^a \cap \phi) \backslash \rho_\ell^a \neq \emptyset$. Let $b \in (\tau^a \cap \phi) \backslash \rho_\ell^a$. Since $\rho_\ell^a \cap \phi \neq \emptyset$, $\exists c \in \rho_\ell^a \cap \phi$. Then, by Def. 10 $bR_\ell c$ and not $cR_\ell b$, which by Prop. 8(ii) implies bQc and not cQb. Hence, $b \in (\tau^a \cap \phi) \backslash \beta^a(\phi)$, and $\rho_\ell^a \cap \phi = (\tau^a \cap \phi) \cap \rho_\ell^a \supseteq (\tau^a \cap \phi) \cap \beta^a(\phi) = \beta^a(\phi)$. Assume then that $(\tau^a \cap \phi) \backslash \rho_\ell^a = \emptyset$. In this case, $\rho_\ell^a \cap \phi = (\tau^a \cap \phi) \cap \rho_\ell^a = \tau^a \cap \phi \supseteq \beta^a(\phi)$.

($\rho_\ell^a \cap \phi \subseteq \beta^a(\phi)$) Let $b \in \rho_\ell^a \cap \phi$. If $c \in \rho_\ell^a \cap \phi$, then $cR_L c$ since $\rho_\ell^a \subseteq \rho_L^a$ by Lemma 3, and cQc by Prop. 8(i). Since $b, c \in \tau^a$ and cQc, it follows by quasi-backward linearity of Q that cQb or bQc. However, since by construction, $\forall k \in \{1, \ldots \ell - 1\}$, $\rho_k^a \cap \phi = \emptyset$, there is no $k \in \{1, \ldots \ell - 1\}$ such that $cR_k b$ and not $bR_k c$ or vice versa, and Prop. 8(ii) implies that both cQb and bQc must hold. In particular, cQb. If, on the other hand, $c \in (\tau^a \cap \phi) \backslash \rho_\ell^a$, then by Def. 10 $cR_\ell b$ and not $bR_\ell c$, implying by Prop. 8(ii) that cQb. Since thus, $\forall c \in \tau^a \cap \phi$, cQb, it follows that $b \in \beta^a(\phi)$. ■

Proposition 11 $\mathrm{B}^0 A = \{a \in W | \exists k \in \{1, ..., L\}$ such that $\rho_k^a = A \cap \kappa^a\}$.

Proof. Recall that $\mathrm{B}^0 A := \cap_{\phi \in \Phi^A} \mathrm{B}(\phi)A$, where $\Phi^A := \{\phi \in 2^W \backslash \{\emptyset\} | \forall a \in W, \kappa^a \cap \phi \neq \emptyset$ and $A \cap \kappa^a \cap \phi \neq \emptyset$ if $A \cap \kappa^a \neq \emptyset\}$ is a non-empty collection.

(If $\exists k \in \{1, ..., L\}$ such that $\rho_k^a = A \cap \kappa^a$, then $a \in \mathrm{B}^0 A$.) Let $\rho_k^a = A \cap \kappa^a$ and consider any $\phi \in \Phi^A$. We must show that $a \in \mathrm{B}(\phi)A$. By the definition of Φ^A, $A \cap \kappa^a \cap \phi \neq \emptyset$ since $\phi \in \Phi^A$ and $A \cap \kappa^a = \rho_k^a \neq \emptyset$. Since $\rho_k^a \cap \phi = A \cap \kappa^a \cap \phi$, it follows that $\emptyset \neq \rho_k^a \cap \phi \subseteq A$, so by Prop. 10, $a \in \mathrm{B}(\phi)A$.

(If $a \in \mathrm{B}^0 A$, then $\exists k \in \{1, ..., L\}$ such that $\rho_k^a = A \cap \kappa^a$.) Let $a \in \mathrm{B}^0 A$; i.e., $\forall \phi \in \Phi^A$, $a \in \mathrm{B}(\phi)A$. We first show that $\rho_1^a \subseteq A$. Consider some $\phi' \in \Phi^A$ satisfying $\tau^a \cap \phi' = (A \cap \tau^a) \cup \rho_1^a$. Since $a \in \mathrm{B}(\phi')A$, $\exists k \in \{1, ..., L\}$ such that $\emptyset \neq \rho_k^a \cap \phi' = \rho_k^a \cap (A \cup \rho_1^a) \subseteq A$. By Lemma 3, $\rho_1^a \subseteq A$. Let $\ell = \max\{k | \rho_k^a \subseteq A\}$. If $\ell = L$, then $\rho_\ell^a = \kappa^a$, and $\rho_\ell^a \subseteq A$ implies $\rho_\ell^a = A \cap \kappa^a$. If $\ell < L$, then, by Lemma 3, $\rho_\ell^a = \rho_\ell^a \cap \kappa^a \subseteq A \cap \kappa^a$. To show that $\rho_\ell^a = A \cap \kappa^a$ also in this case, suppose instead that $(A \cap \kappa^a) \backslash \rho_\ell^a \neq \emptyset$, and consider some $\phi'' \in \Phi^A$ satisfying $\tau^a \cap \phi'' = ((A \cap \kappa^a) \cup \rho_{\ell+1}^a) \backslash \rho_\ell^a$. It follows from $\rho_\ell^a \cap \phi'' = \emptyset$ and Lemma 3 that, $\forall k \in \{1, .., \ell\}$, $\rho_k^a \cap \phi'' = \emptyset$. Since by construction, $\rho_\ell^a \subseteq A$, while $\rho_{\ell+1}^a \subseteq A$ does not hold, $\rho_{\ell+1}^a \cap \phi'' = \rho_{\ell+1}^a \backslash \rho_\ell^a$ is not included in A. By Lemma 3 there is no $k \in \{0, \dots, L\}$ such that $\emptyset \neq \rho_k^a \cap \phi'' \subseteq A$, contradicting by Prop. 10 that $a \in \mathrm{B}(\phi'')A$. Hence, $\rho_\ell^a = A \cap \kappa^a$. ■

In combination with Prop. 8(ii) Prop. 11 means that A is fully believed iff any subjectively possible state in A has higher epistemic priority than any state in the same equivalence class outside A.

6 Properties of Belief Operators

In the present section we establish some properties of certain, conditional, and full belief operators. We do not seek to establish sound and complete axiomatic systems for these operators; this should, however, be standard for the certain and conditional belief operators, while harder to establish for the full belief operator. Rather, our main goal is to show how the poorly behaved full belief operator is bounded by the two KD45 operators certain and conditional belief. While the results of Sects. 6.1 and 6.2 are included for comprehensiveness and as a background for the results of Sect. 6.3, the latter findings shed light on non-standard notions of belief recently used in epistemic analyses of games.

6.1 Properties of certain and conditional belief

We start by showing that certain belief implies conditional belief.

Proposition 12 *For any $\phi \in \Phi$, $\mathrm{K}A \subseteq \mathrm{B}(\phi)A$.*

Proof. The result follows from Defs. 6 and 7 since $\beta^a(\phi) \subseteq \kappa^a \cap \phi$. ■

Secondly, we show a result that combined with Prop. 12 implies that both operators K and B(ϕ) correspond to KD45 systems.

Proposition 13 *For any $\phi \in \Phi$, the following properties hold:*

$$\begin{aligned} &\mathrm{K}A \cap \mathrm{K}A' = \mathrm{K}(A \cap A') && \mathrm{B}(\phi)A \cap \mathrm{B}(\phi)A' = \mathrm{B}(\phi)(A \cap A') \\ &\mathrm{K}W = W && \mathrm{B}(\phi)\emptyset = \emptyset \\ &\mathrm{K}A \subseteq \mathrm{KK}A && \mathrm{B}(\phi)A \subseteq \mathrm{KB}(\phi)A \\ &\neg\mathrm{K}A \subseteq \mathrm{K}(\neg\mathrm{K}A) && \neg\mathrm{B}(\phi)A \subseteq \mathrm{K}(\neg\mathrm{B}(\phi)A). \end{aligned}$$

Proof. ($\mathrm{K}A \cap \mathrm{K}A' = \mathrm{K}(A \cap A')$) To prove $\mathrm{K}A \cap \mathrm{K}A' \subseteq \mathrm{K}(A \cap A')$, let $a \in \mathrm{K}A$ and $a \in \mathrm{K}A'$. Then, by Def. 6, $\kappa^a \subseteq A$ and $\kappa^a \subseteq A'$ and hence, $\kappa^a \subseteq A \cap A'$, implying that $a \in \mathrm{K}(A \cap A')$. To prove $\mathrm{K}A \cap \mathrm{K}A' \supseteq \mathrm{K}(A \cap A')$, let $a \in \mathrm{K}(A \cap A')$. Then $\kappa^a \subseteq A \cap A'$ and hence, $\kappa^a \subseteq A$ and $\kappa^a \subseteq A'$, implying that $a \in \mathrm{K}A$ and $a \in \mathrm{K}A'$.

($\mathrm{B}(\phi)A \cap \mathrm{B}(\phi)A' = \mathrm{B}(\phi)(A \cap A')$) Using Def. 7 the proof of conjunction for B(ϕ) is identical to the one for K except that $\beta^a(\phi)$ is substituted for κ^a.

($\mathrm{K}W = W$) $\mathrm{K}W \subseteq W$ is obvious. That $\mathrm{K}W \supseteq W$ follows from Def. 6 since, $\forall a \in W$, $\kappa^a \subseteq \tau^a \subseteq W$.

($\mathrm{B}(\phi)\emptyset = \emptyset$) This follows from Def. 7 since, $\forall a \in W$, $\beta^a(\phi) \neq \emptyset$, implying that there exists no $a \in W$ such that $\beta^a(\phi) \subseteq \emptyset$.

($\mathrm{K}A \subseteq \mathrm{KK}A$) Let $a \in \mathrm{K}A$. By Def. 6, $a \in \mathrm{K}A$ is equivalent to $\kappa^a \subseteq A$. Since $\forall b \in \tau^a$, $\kappa^b = \kappa^a$, it follows that $\tau^a \subseteq \mathrm{K}A$. Hence, $\kappa^a \subseteq \tau^a \subseteq \mathrm{K}A$, implying by Def. 6 that $a \in \mathrm{KK}A$.

($\mathrm{B}(\phi)A \subseteq \mathrm{KB}(\phi)A$) Let $a \in \mathrm{B}(\phi)A$. By Def. 7, $a \in \mathrm{B}(\phi)A$ is equivalent to $\beta^a(\phi) \subseteq A$. Since $\forall b \in \tau^a$, $\beta^b(\phi) = \beta^a(\phi)$, it follows that $\tau^a \subseteq \mathrm{B}(\phi)A$. Hence, $\kappa^a \subseteq \tau^a \subseteq \mathrm{B}(\phi)A$, implying by Def. 6 that $a \in \mathrm{KB}(\phi)A$.

($\neg\mathrm{K}A \subseteq \mathrm{K}(\neg\mathrm{K}A)$) Let $a \in \neg\mathrm{K}A$. By Def. 6, $a \in \neg\mathrm{K}A$ is equivalent to $\kappa^a \subseteq A$ not holding. Since $\forall b \in \tau^a$, $\kappa^b = \kappa^a$, it follows that $\tau^a \subseteq \neg\mathrm{K}A$. Hence, $\kappa^a \subseteq \tau^a \subseteq \neg\mathrm{K}A$, implying by Def. 6 that $a \in \mathrm{K}(\neg\mathrm{K}A)$.

($\neg\mathrm{B}(\phi)A \subseteq \mathrm{K}(\neg\mathrm{B}(\phi)A)$) Let $a \in \neg\mathrm{B}(\phi)A$. By Def. 7, $a \in \neg\mathrm{B}(\phi)A$ is equivalent to $\beta^a(\phi) \subseteq A$ not holding. Since $\forall b \in \tau^a$, $\beta^b(\phi) = \beta^a(\phi)$, it follows that $\tau^a \subseteq \neg\mathrm{B}(\phi)A$. Hence, $\kappa^a \subseteq \tau^a \subseteq \neg\mathrm{B}(\phi)A$, implying by Def. 6 that $a \in \mathrm{K}(\neg\mathrm{B}(\phi)A)$. ■

Note that $\mathrm{K}\emptyset = \emptyset$, $\mathrm{B}(\phi)W = W$, $\mathrm{B}(\phi)A \subseteq \mathrm{B}(\phi)\mathrm{B}(\phi)A$ and $\neg\mathrm{B}(\phi)A \subseteq \mathrm{B}(\phi)(\neg\mathrm{B}(\phi)A)$ follow from Prop. 13 since $\mathrm{K}A \subseteq \mathrm{B}(\phi)A$.

Since an event can be certainly believed even though the true state is an element of the complement of the event, it follow that neither operator satisfies the truth axiom (i.e. $\mathrm{K}A \subseteq A$ and $\mathrm{B}(\phi)A \subseteq A$ need not hold).

6.2 Belief revision

We show in this section that $\mathrm{B}(\phi)$ satisfies the usual properties for belief revision as given by Stalnaker [24] (see also Alchourrón et al. [1]). To show this we must define the set, β^a, that determines the decision maker's unconditional belief at the state a:

$$\beta^a := \{b \in \tau^a | \forall c \in \tau^a, cQb\},$$

i.e. $\beta^a = \beta^a(W)$. Then the following result can be established.

Proposition 14 *1. $\beta^a(\phi) \subseteq \phi$.*

2. If $\beta^a \cap \phi \neq \emptyset$, then $\beta^a(\phi) = \beta^a \cap \phi$.

3. If $\phi \in \Phi$, then $\beta^a(\phi) \neq \emptyset$.

4. If $\beta^a(\phi) \cap \phi' \neq \emptyset$, then $\beta^a(\phi \cap \phi') = \beta^a(\phi) \cap \phi'$.

Proof. *(1.)* $\beta^a(\phi) \subseteq \phi$ follows by definition since, $\forall b \in \beta^a(\phi)$, $b \in \phi$.

(2.) By Def. 9, $\beta^a = \rho^a_1$. Hence, $\beta^a \cap \phi \neq \emptyset$ implies $\rho^a_1 \cap \phi \neq \emptyset$ and $\min\{k | \rho^a_k \cap \phi \neq \emptyset\} = 1$. By Lemma 6, $\beta^a(\phi) = \rho^a_1 \cap \phi = \beta^a \cap \phi$.

(3.) This follows directly from Lemma 2, since $\phi \in \Phi$ implies that, $\forall a \in W$, $\kappa^a \cap \phi \neq \emptyset$.

(4.) Let $\beta^a(\phi) \cap \phi' \neq \emptyset$. By Lemma 6, $\beta^a(\phi) = \rho^a_\ell \cap \phi \neq \emptyset$ where $\ell := \min\{k | \rho^a_k \cap \phi \neq \emptyset\}$. Likewise, $\beta^a(\phi \cap \phi') = \rho^a_{\ell'} \cap \phi \cap \phi'$, where $\ell' := \min\{k | \rho^a_k \cap \phi \cap \phi' \neq \emptyset\}$. It suffices to show that $\ell = \ell'$. Obviously, $\ell \leq \ell'$. However, $\emptyset \neq \beta^a(\phi) \cap \phi' = (\rho^a_\ell \cap \phi) \cap \phi' = \rho^a_\ell \cap \phi \cap \phi'$ implies that $\ell' \leq \ell$. ■

6.3 Properties of full belief

It is straightforward to show that certain belief implies full belief, which in turn implies (unconditional) belief.

Proposition 15 *$\mathrm{K}A \subseteq \mathrm{B}^0 A \subseteq \mathrm{B}(W)A$.*

Proof. That $\mathrm{K}A \subseteq \mathrm{B}^0 A$ follows from Def. 6 and Props. 8 and 11 since $\kappa^a \subseteq A$ implies that $\rho^a_L = \kappa^a = \kappa^a \cap A$. That $\mathrm{B}^0 A \subseteq \mathrm{B}(W)A$ follows from Def. 8 since $W \in \Phi^A$. ■

Even though full belief is thus bounded by two KD45 operators, full belief is not itself a KD45 operator.

Proposition 16 *The following properties hold:*

$$\mathrm{B}^0 A \cap \mathrm{B}^0 A' \subseteq \mathrm{B}^0(A \cap A')$$
$$\mathrm{B}^0 A \subseteq \mathrm{K}\mathrm{B}^0 A$$
$$\neg\mathrm{B}^0 A \subseteq \mathrm{K}(\neg\mathrm{B}^0 A).$$

Proof. ($\mathrm{B}^0 A \cap \mathrm{B}^0 A' \subseteq \mathrm{B}^0(A \cap A')$) Let $a \in \mathrm{B}^0 A$ and $a \in \mathrm{B}^0 A'$. Then, by Prop. 11, there exist k such that $\rho_k^a = A \cap \kappa^a$ and k' such that $\rho_{k'}^a = A' \cap \kappa^a$. By Lemma 3, either $\rho_k^a \subseteq \rho_{k'}^a$ or $\rho_k^a \supseteq \rho_{k'}^a$, or equivalently, $A \cap \kappa^a \subseteq A' \cap \kappa^a$ or $A \cap \kappa^a \supseteq A' \cap \kappa^a$. Hence, either $\rho_k^a = A \cap \kappa^a = A \cap A' \cap \kappa^a$ or $\rho_{k'}^a = A' \cap \kappa^a = A \cap A' \cap \kappa^a$, implying by Prop. 11 that $a \in \mathrm{B}^0(A \cap A')$.

($\mathrm{B}^0 A \subseteq \mathrm{K}\mathrm{B}^0 A$) Let $a \in \mathrm{B}^0 A$. By Prop. 11, $a \in \mathrm{B}^0 A$ is equivalent to $\exists k \in \{1, ..., L\}$ such that $\rho_k^a = A \cap \kappa^a$. Since $\forall b \in \tau^a$, $\rho_k^b = \rho_k^a$ and $\kappa^b = \kappa^a$, it follows that $\tau^a \subseteq \mathrm{B}^0 A$. Hence, $\kappa^a \subseteq \tau^a \subseteq \mathrm{B}^0 A$, implying by Def. 6 that $a \in \mathrm{K}\mathrm{B}^0 A$.

($\neg\mathrm{B}^0 A \subseteq \mathrm{K}(\neg\mathrm{B}^0 A)$) Let $a \in \neg\mathrm{B}^0 A$. By Prop. 11, $a \in \neg\mathrm{B}^0 A$ is equivalent to there not existing $k \in \{1, ..., L\}$ such that $\rho_k^a = A \cap \kappa^a$. Since $\forall b \in \tau^a$, $\rho_k^b = \rho_k^a$ and $\kappa^b = \kappa^a$, it follows that $\tau^a \subseteq \neg\mathrm{B}^0 A$. Hence, $\kappa^a \subseteq \tau^a \subseteq \neg\mathrm{B}^0 A$, implying by Def. 6 that $a \in \mathrm{K}(\neg\mathrm{B}^0 A)$. ■

Note that $\mathrm{B}^0\emptyset = \emptyset$, $\mathrm{B}^0 W = W$, $\mathrm{B}^0 A \subseteq \mathrm{B}^0\mathrm{B}^0 A$ and $\neg\mathrm{B}^0 A \subseteq \mathrm{B}^0(\neg\mathrm{B}^0 A)$ follow from Props. 13 and 16 since $\mathrm{K}A \subseteq \mathrm{B}^0 A \subseteq \mathrm{B}(W)A$. However, even though the operator B^0 satisfies $B^0 A \subseteq \neg B^0 \neg A$ as well as positive and negative introspection, it does not satisfy monotonicity since $A \subseteq A'$ does not imply $B^0 A \subseteq B^0 A'$. To see this let $\rho_1^a = \{a\}$ and $\rho_2^a = \kappa^a = \{a, b, c\}$ for some $a \in W$. Now let $A = \{a\}$ and $A' = \{a, b\}$. Clearly, $A \subseteq A'$, and since $\rho_1^a = A \cap \kappa^a$ we have $a \in B^0 A$. However, since neither $\rho_1^a = A' \cap \kappa^a$ nor $\rho_2^a = A' \cap \kappa^a$, $a \notin B^0 A'$.

7 Epistemic analysis of strategic form games

The purpose of this section is two-fold: Firstly, to describe how our analysis – having been derived from a one-person decision-theoretic framework – can be interpreted in terms of the n-person decision-theoretic framework encountered in games. Secondly, to show how the full belief operator corresponds to Brandenburger & Keisler's [13] concept of 'assumption'. For notational simplicity, and following Brandenburger & Keisler [13], we limit the discussion to two-person games.

7.1 Preferences in games

Let S_i denote player i's finite set of *pure strategies*, and let $z : S \to Z$ map strategy vectors into outcomes, where $S = S_1 \times S_2$ is the set of strategy vectors and Z is the finite set of outcomes. Then (S_1, S_2, z) is a finite *strategic* two-person *game form.*

For each player i, any of i's strategies is an act from strategy choices of his opponent j to outcomes. The uncertainty faced by a player i in a strategic game form concerns j's strategy choice, j's preferences over acts from i's strategy choices to outcomes, and so on (see Tan & Werlang [25]). A type of a player i corresponds to preferences over acts from j's strategy choices, preferences over acts from j's preferences over acts from i's strategy choices, and so on.

For each type of any player i, the type's decision problem is to choose one of i's strategies. As the type is not uncertain of his own choice, the type's preferences over acts from i's strategy choices is not relevant and can be ignored.

By adding subscript i to the framework introduced in Sect. 2, the finite set of states (or possible worlds) for player i, W_i, can be interpreted as

$$W_i = T_i \times S_j \times T_j \, ,$$

where, for each $a \in W_i$, the set of states in the same equivalence class as a equals

$$\tau_i^a = \{t_i^a\} \times S_j \times T_j \, ,$$

with t_i^a denoting the type of i determined by the state a. The property that all three belief operators satisfy positive and negative introspection, corresponds to the property that at any state $a \in W_i$ (and any conditioning event), player i certainly believes/conditionally believes/fully believes that he is of type t_i^a.

Definition 11 An *interactive preference structure* for the strategic game form (S_1, S_2, z) is a structure

$$(S_1, S_2, \{\succeq_1^a \mid a \in T_1 \times S_2 \times T_2\}, \{\succeq_2^a \mid a \in T_2 \times S_1 \times T_1\}) \, ,$$

where, for each i, $\{\succeq_i^a \mid a \in T_i \times S_j \times T_j\}$ satisfies Def. 1 with, for all $a \in T_i \times S_j \times T_j$, $\tau_i^a = \{t_i^a\} \times S_j \times T_j$.

It is a property of an interactive preference structure that any $t_i \in T_i$ corresponds to preferences over acts from j's strategy choices, preferences over acts from j's preferences over acts from i's strategy choices, and so on.

7.2 The 'assumption' operator

Asheim & Dufwenberg [4] employ an interactive preference structure like the one described in Def. 11. They say that an event A is fully believed at a if the preferences at a are admissible on the set of states in A that are deemed subjective possible at a. By interpreting the characterization result, Prop. 11, in terms of preferences (cf. Sect. 3.2), this corresponds to full belief as defined in Def. 8.

Brandenburger & Keisler [13] consider an interactive preference structure that is

– more general than the one that we consider in Def. 11, since T_1 and T_2 – and thus, $W_1 = T_1 \times S_2 \times T_2$ and $W_2 = T_2 \times S_1 \times T_1$ – need not be finite, and

– more special than ours, since, for each i and all $a \in W_i$, conditional completeness, conditional continuity and partitional priority of $\succeq_i^a$ are strengthened to completeness and partitional continuity.

The latter property means that, for any $a \in W$, $\succeq^a$ is assumed to satisfy

- *partitional continuity*; i.e., there is a partition $\{\pi_1^a, \ldots, \pi_{L^a}^a\}$ of κ^a such that
  (a) for each $k \in \{1, \ldots, L^a\}$, there exist $0 < \gamma < \delta < 1$ such that $\delta\mathbf{x}' + (1-\delta)\mathbf{x}'' \succ_{\pi_k^a}^a \mathbf{y} \succ_{\pi_k^a}^a \gamma\mathbf{x}' + (1-\gamma)\mathbf{x}''$ whenever $\mathbf{x}' \succ_{\pi_k^a}^a \mathbf{y} \succ_{\pi_k^a}^a \mathbf{x}''$, and
  (b) for each $k \in \{1, \ldots, L^a - 1\}$, $\mathbf{x} \succ_{\pi_k^a}^a \mathbf{y}$ implies $\mathbf{x} \succ_{\pi_k^a \cup \pi_{k+1}^a}^a \mathbf{y}$,

where we again refer to the decision maker without using the subscript i.

Within our setting with a finite set of states, W, it now follows from Blume et al. [8, Thm. 5.3] that $\succeq^a$ is represented by u^a and a *lexicographic conditional probability system* (LCPS) – i.e., a hierarchy of subjective probability distributions with non-overlapping supports where the support of the k-level probability distribution p_k^a equals π_k^a (cf. [8, Def. 5.2]). Brandenburger & Keisler [13, Appendix B] employ an LCPS to define the preferences in their setting with an infinite set of states.

Provided that completeness and partitional continuity are satisfied, Brandenburger & Keisler [13, Def. B1] introduce the following belief operator.

Definition 12 (Brandenburger & Keisler [13]) At a the decision maker *assumes* A if $\succeq_A^a$ is nontrivial and $\mathbf{x} \succ_A^a \mathbf{y}$ implies $\mathbf{x} \succ^a \mathbf{y}$.

Proposition 17 *Assume that* $\succeq^a$ *is complete and satisfies partitional continuity. Then* A *is assumed at* a *iff* $a \in B^0 A$.

Lemma 7 *Assume that $\succeq^a$ is complete and satisfies partitional continuity, and let $k, k' \in \{1, \dots, L^a\}$ satisfy $k < k'$. Then $\mathbf{x} \succ^a_{\pi^a_k} \mathbf{y}$ implies $\mathbf{x} \succ^a_{\pi^a_k \cup \pi^a_{k'}} \mathbf{y}$.*

Proof. This follows from the representation result of Blume et al. [8, Thm. 5.3]. ■

Proof of Prop. 17. *(If A is assumed at a, then $a \in B^0 A$.)* Let A be assumed at a. Then it follows that $\succeq^a_A$ is nontrivial; hence, $A \cap \kappa^a \neq \emptyset$. Assume that $\mathbf{x}_{A\cap\kappa^a}$ weakly dominates $\mathbf{y}_{A\cap\kappa^a}$ at a. Since $A \cap \kappa^a \neq \emptyset$, we have that $\mathbf{x} \succ^a_A \mathbf{y}$. Hence, it follows from the premise (viz., that A is assumed at a) that $\mathbf{x} \succ^a \mathbf{y}$. This shows that $\succeq^a$ is admissible on $A \cap \kappa^a$, and, by Prop. 11, $a \in B^0 A$.

(If $a \in B^0 A$, then A is assumed at a.) Let $a \in B^0 A$, or by Prop. 11, $\succeq^a$ is admissible on $A \cap \kappa^a$ ($\neq \emptyset$). Hence, by Prop. 2, $b \in A \cap \kappa^a$ and $c \in \neg(A \cap \kappa^a)$ implies $b \gg^a c$. By the partitional continuity of $\succeq^a$ this in turn implies that $\exists \ell$ such that

$$A \cap \kappa^a = \bigcup\nolimits_{k=1}^{\ell} \pi^a_k \,,$$

since property (a) – the continuity of $\succeq^a$ within each partitional element – rules out that b and c are in the same element of the partition $\{\pi^a_1, \dots, \pi^a_{L^a}\}$ if $b \gg^a c$.

Assume that $\mathbf{x} \succ^a_A \mathbf{y}$. Then $\mathbf{x} \succ^a_{A\cap\kappa^a} \mathbf{y}$, and, by the above argument,

$$\mathbf{x} \succ^a_{\bigcup_{k=1}^{\ell} \pi^a_k} \mathbf{y} \,.$$

By completeness and partitional continuity, Lemma 7 entails that $\exists \ell' \in \{1, \dots, \ell\}$ such that $\mathbf{x} \succ^a_{\pi^a_{\ell'}} \mathbf{y}$ and, $\forall k \in \{1, \dots, \ell' - 1\}$, $\mathbf{x} \sim^a_{\pi^a_k} \mathbf{y}$. By Lemma 7, $\mathbf{x} \succ^a \mathbf{y}$ since $\bigcup_{k=1}^{L^a} \pi^a_k = \kappa^a$. Hence, $\mathbf{x} \succ^a_A \mathbf{y}$ implies $\mathbf{x} \succ^a \mathbf{y}$. Moreover, $\succeq^a_A$ is nontrivial since $A \cap \kappa^a \neq \emptyset$, and it follows from Def. 12 that A is assumed at a. ■

Proposition 17 shows that the 'assumption' operator coincides with full belief (and thus with Stalnaker's [24] 'absolutely robust belief') under completeness and partitional continuity. However, if partitional continuity is weakened to conditional continuity, then this equivalence is not obtained. To see this, let $\kappa^a = \{a, b, c\}$, and let the preferences $\succeq^a$, in addition to the properties listed in Def. 1, also satisfy completeness. It then follows from Blume et al. [8, Thm. 3.1] that $\succeq^a$ is represented by u^a and a *lexicographic probability system* (LPS) – i.e., a hierarchy of subjective probability distributions with possibly overlapping supports. Consider the example of Blume et al. [8, Sect. 5] of a two-level LPS, where the primary probability distribution, p^a_1, is given by $p^a_1(a) = 1/2$ and $p^a_1(b) = 1/2$, and the secondary probability distribution, p^a_2, used to resolve ties, is given by $p^a_2(a) = 1/2$ and $p^a_2(c) = 1/2$. Consider the acts $\mathbf{x}$ and $\mathbf{y}$, where $u^a(\mathbf{x}(a)) = 2$, $u^a(\mathbf{x}(b)) = 0$, and $u^a(\mathbf{x}(c)) = 0$, and where $u^a(\mathbf{y}(a)) = 1$, $u^a(\mathbf{y}(b)) = 1$, and $u^a(\mathbf{y}(c)) = 2$. Even though $\succeq^a$ is admissible on $\{a, b\}$, and thus $\{a, b\}$ is fully believed at a, it follows that $\{a, b\}$ is not 'assumed' at a since $\mathbf{x} \succ^a_{\{a,b\}} \mathbf{y}$, while $\mathbf{x} \prec^a \mathbf{y}$. Brandenburger & Keisler [13] do not indicate that their definition – as stated in Def. 12 – should be used outside the realm of partionally continuous preferences. Hence, our definition of full belief – combined with the characterization result of Prop. 11 and its interpretation in term of admissibility – yields a preference-based generalization of Brandenburger & Keisler's [13] operator (in our setting with a finite set of states) to preferences that need only satisfy the properties of Def. 1.

8 Epistemic analysis of extensive form games

In the setting of extensive form games, Battigalli & Siniscalchi [6] have recently suggested a non-monotonic 'strong belief' operator. In this section we show how their 'strong belief' operator is related to full belief (and thereby, to Stalnaker's [24] 'absolutely robust belief' and Brandenburger & Keisler's [13] 'assumption'). This, however, requires that we amend the decision-theoretic framework to be able to handle systems of conditional preferences used in analyses of extensive form games.

8.1 The multi-stage game form

Inspired by Osborne & Rubinstein [21, Ch. 6], a finite extensive two-person game form of almost perfect information with $M - 1$ stages can be described as follows. The sets of *histories* is determined inductively: The set of histories at the beginning of the first stage 1 is $H^1 = \{\emptyset\}$. Let H^m denote the set of histories at the beginning of stage m. At $h \in H^m$, let, for each player i, i's action set be denoted $A_i(h)$, where i is inactive at h if $A_i(h)$ is a singleton. Write $A(h) := A_1(h) \times A_2(h)$. Define the set of histories at the beginning of stage $m + 1$ as follows: $H^{m+1} := \{(h, a) \,|\, h \in H^m \text{ and } a \in A(h)\}$. This concludes the induction. Let $H := \bigcup_{m=1}^{M-1} H^m$ denote the set of subgames and let $Z := H^M$ denote the set of outcomes.

A pure strategy for player i is a function s_i that assigns an action in $A_i(h)$ to any $h \in H$. Let S_i denote player i's finite set of pure strategies, and write $S := S_1 \times S_2$. Let $z : S \to Z$ map strategy vectors into outcomes. Then (S_1, S_2, z) is a finite strategic two-person game form. For any $h \in H$, let $S(h) = S_1(h) \times S_2(h)$ denote the set of strategy vectors that are *consistent* with h being reached. Note that $S(\emptyset) = S$.

Thus, also in a multi-stage game, the set of states (or possible worlds) for each player i, W_i, can be interpreted as $W_i = T_i \times S_j \times T_j$ with, for all $a \in W_i$, $\tau_i^a = \{t_i^a\} \times S_j \times T_j$. Let Φ_i^H denote the collection of subsets that correspond to subgames:

$$\Phi_i^H := \{\phi \in 2^{W_i} \backslash \{\emptyset\} | \exists h \in H \text{ s.t. } \phi = T_i \times S_j(h) \times T_j\} .$$

Assume that any subgame is subjectively possible at any state: For all $a \in W_i$ and $h \in H$, $\kappa_i^a \cap (T_i \times S_j(h) \times T_j) \neq \emptyset$, implying that $\Phi_i^H \subseteq \Phi_i$.

A system of conditional preferences in an extensive form game need not consist of conditional binary relations derived from a single binary relation over acts on W_i, as implied by Def. 11. Indeed, a concept like sequential equilibrium requires a system of conditional preferences that cannot be made up of conditional binary relations derived from a single binary relation over acts on W_i (cf., e.g., Asheim & Perea [5, Sect. 2]). To show how belief operators derived from a system of conditional preferences relate to the belief operators defined in Sect. 4, we must indicate how a system of conditional preferences is isomorphic to preferences that are consistent with Def. 1. This is the purpose of the following subsection.

8.2 A system of conditional preferences

We again refer to a decision maker without using the subscript i. As before, for any $a \in A$, let τ^a denote the equivalence class to which a belongs, let κ^a denote the set of subjectively possible states at a, and let Φ denote the collection of sets that are subjectively possible at any state. For each $a \in W$, consider a *system of conditional preferences*, $\{\succeq_\phi^a | \phi \in \Phi\}$, in the following sense: For any $\phi \in \Phi$, the preferences of the decision maker conditional on ϕ is a binary relation $\succeq_\phi^a$ on the set of acts on ϕ; i.e., on the set of functions $\mathbf{x}_\phi : \phi \to \Delta(Z)$.

When one considers the above setting, Battigalli & Siniscalchi [6] and Ben-Porath [7] in effect invoke the following assumptions:

- For each $a \in W$ and $\phi \in \Phi$, $\succeq_\phi^a$ is complete and transitive and satisfies non-triviality, objective independence, and continuity.

- For each $a \in W$, $\{\succeq_\phi^a | \phi \in \Phi\}$ satisfies non-null state independence in the sense that if $\mathbf{x}_\phi \succ_\phi^a \mathbf{y}_\phi$ iff $\mathbf{x}_{\phi'} \succ_{\phi'}^a \mathbf{y}_{\phi'}$ whenever $\kappa^a \cap \phi = \{b\}$ and $\kappa^a \cap \phi' = \{b'\}$, and $\mathbf{x}$ and $\mathbf{y}$ satisfy $\mathbf{x}(b) = \mathbf{x}(b')$ and $\mathbf{y}(b) = \mathbf{y}(b')$.

- For each $a \in W$ and any ϕ, $\phi' \in \Phi$, if it holds that $\phi' \subseteq \phi$ and the conditional binary relation $\succeq_{\phi|\phi'}^a$ is nontrivial, then $\mathbf{x}_{\phi'} \succeq_{\phi'}^a \mathbf{x}_{\phi'}$ iff $\mathbf{x}_\phi \succeq_{\phi|\phi'}^a \mathbf{x}_\phi$.

Under these assumptions a system of conditional preferences, $\{\succeq^a_\phi \mid \phi \in \Phi\}$, is isomorphic to a vNM utility function u^a and an LCPS, $(p^a_1, \ldots, p^a_{L^a})$ — where, for each $k \in \{1, \ldots, L^a\}$, $\text{supp} p^a_k = \pi^a_k$ and $\{\pi^a_1, \ldots, \pi^a_{L^a}\}$ is a partition of κ^a — in the following sense (cf. Blume et al. [8, Sect. 5]): For any $\phi \in \Phi$, $\succeq^a_\phi$ is represented by u^a and the probability distribution p^a_ϕ, where is p^a_ϕ is the conditional of p^a_ℓ on ϕ, with $\ell := \min\{k \in \{1, \ldots, L^a\} | \, \pi^a_k \cap \phi \neq \emptyset\}$.

The system of probability distributions, $\{p^a_\phi | \, \phi \in \Phi\}$, is called a *conditional probability system* (cf. Myerson [20]). Battigalli & Siniscalchi [6] and Ben-Porath [7] employ a CPS to define the system of conditional preferences.

8.3 The 'strong belief' operator

Battigalli & Siniscalchi [6] and Ben-Porath [7] define conditional belief with probability 1: At a the decision maker believes A conditional on ϕ if $\text{supp} p^a_\phi \subseteq A$.

If the preferences $\succeq^a$ are represented by u^a and an LCPS, then it follows by the definition of an LCPS and Lemma 7 that, $\forall \ell \in \{1, \ldots, L^a\}$, $\rho^a_\ell = \bigcup_{k=1}^{\ell} \pi^a_k$. If we consider the LCPS that is isomorphic to a system of conditional preferences, as considered by Battigalli & Siniscalchi [6] and Ben-Porath [7], it is now a consequence of Lemma 6 that $\beta^a(\phi) = \text{supp} p^a_\phi$. Hence, conditional belief with probability 1, as defined by Battigalli & Siniscalchi [6] and Ben-Porath [7], can be captured through our conditional belief operator by considering the LCPS that is isomorphic to the system of conditional preferences that they consider.

Given that the conditional belief operator of Battigalli & Siniscalchi [6] coincides with the $B(\phi)$ operator of the present paper, we can define their 'strong belief' operator as follows, where $\Phi^H \cap \Phi^A$ is the collection of subgames ϕ having the property that A is subjectively possible conditional on ϕ whenever A is subjectively possible.

Definition 13 (Battigalli & Siniscalchi [6]) At a the decision maker *strongly believes* A if $a \in \bigcap_{\Phi^H \cap \Phi^A} B(\phi)$.

Since $\Phi^A \supseteq \Phi^H \cap \Phi^A \supseteq \{W\}$, it follows that the 'strong belief' operator is bounded by the full belief and (unconditional) belief operators.

Proposition 18 *If $a \in B^0(A)$, then A is strongly believed at a. If A is strongly believed at a, then $a \in B(W)A$.*

It can be shown that the 'strong belief' operator shares the properties of full belief: Also 'strong belief' satisfies the properties of Prop. 16, but is not monotonic.

9 Concluding Remarks

We have presented a model with (i) a serial, transitive, forwardly linear and quasi-backwardly linear epistemic priority relation Q, and, equivalently, (ii) a vector of nested, serial, transitive and euclidean accessibility relations $(R_1, \ldots, R_L)$. The two kinds of relations give two equivalent representations of the notions of 'subjective possibility' and 'epistemic priority'. We have shown how both Q and $(R_1, \ldots, R_L)$ can be derived from preferences that need not be complete and thus representable by subjective probabilities. The model thus provides semantics for preference-based belief operators.

On the basis of the epistemic priority relation Q we have defined the concepts of certain, conditional, and full belief. Certain belief and conditional belief (for a given conditioning event) are both KD45 operators, whereas full belief is not, as it does not satisfy monotonicity. We have shown how full belief not only corresponds to 'absolutely robust belief' in the sense of Stalnaker [24], but also coincides with Brandenburger & Keisler's [13] 'assumption', and is related to Battigalli & Siniscalchi's [6] 'strong belief'.

If preferences were required to be continuous, then certain belief, (unconditional) belief, and full belief would coincide. This in turn would mean that there would be no scope for an interesting theory of belief revision. Hence, it is essential to allow for discontinuous preferences when modelling the belief revision of a decision maker, for instance in the context of extensive form games.

A feature of this model is that it does not require that the epistemic priority relation is reflexive. The decision maker may be subjectively unable to distinguish between two objectively possible states, while deeming (at the lowest epistemic level) that one is subjectively possible and the other not. Because Q lacks reflexivity, not even the certain belief operator obeys the truth axiom; thus, we allow that the decision maker holds the true state as subjectively impossible (even at the lowest epistemic level).

The distinction between subjectively possible and subjectively impossible events can be illustrated within an interactive preference structure for the strategic game form of a multi-stage game (cf. Def. 11 and Sect. 8.1). If each player considers any opponent strategy to be subjectively possible, then any $\phi \in \Phi^H$ (the collection of subsets of states that correspond to subgames) will be subjectively possible, as well as potentially observable (cf. Brandenburger [11]). The player can still deem it subjectively impossible that the opponent holds particular preferences, as the preferences of the opponent are not directly observable. Brandenburger & Keisler [12] show that there need not exist a preference-complete interactive preference structure when preferences are not representable by subjective probabilities. This result makes models that do not require a decision maker to hold all objectively possible opponent preferences as subjectively possible particularly relevant in game-theoretic applications.

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