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Imperfect loss offset and the after-tax expected rate of return to equity, with an application to rent taxation

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Imperfect loss offset and the after-tax expected rate of return to equity, with an application to rent taxation

By Diderik Lund

Abstract

Non-neutral taxation, and in particular imperfect loss offset, is shown to have a strong effect not only on investment decisions, but also on required expected after-tax rates of return to equity. Systematic risk is valued according to the CAPM, while non-linear taxes are valued by option valuation methods.

Imperfect loss offset and the after-tax expected rate of return to equity, with an application to rent taxation*

It is well known that a profits tax without full loss offset is a heavier burden for a corporation than a similar tax with full loss offset. Only if losses can be carried forward with interest, and the eventual offset is certain, then a loss carry-forward provision can be of equal value to an immediate offset.[1]

Uncertainty plays a major role in the valuation of the burden of imperfect loss offset. Ball and Bowers (1982) suggested to use option valuation, and this was further explored by Majd and Myers (1985) and MacKie-Mason (1990). As seen from a point in time before the tax year, the cash flow to a tax claim with no loss offset at all is equivalent to a European call option, $t \cdot \max(p, 0)$, when t is the tax rate, and p is profits. With many periods and imperfect loss carry-forward provisions the option is more complicated, and numerical methods must be used to analyze it. This was done by Lund (1991), who also showed the impact of the tax on value-maximizing decisions by a corporation subject to a rent tax on petroleum extraction.

The effect on the decision is also a topic in the present paper. The main new topic is an analysis of the systematic risk of the after-tax cash flow to equity in such a corporation. This is important for several reasons. Some corporations use a risk-adjusted discount rate and make decisions based on present values of expected net cash flows. While some of the deficiencies of such a method may be known, their magnitudes are not well known. In particular, this is relevant for the practice of using the same discount rate across projects and jurisdictions. The text-book warning against this is mainly focused on the differences in risks in the pre-tax cash flows. One might believe that projects with the same output could be evaluated by the same discount rate. This is not so. Furthermore the risk-adjusted discounting could be used to estimate equity values, and a more correct method would be useful.

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The numerical examples below concern rent taxation at very high marginal tax rates. Many countries apply such taxes in natural resource related industries. An excellent, detailed discussion of such taxes is given in Boadway et al. (1989). A main reason why tax design becomes critical for rent taxes, is that the distortive effects of taxes in general depends on the fraction $(1-t_1)/(1-t_2)$, where t_1 is the marginal effective tax rate on (gross) income, and t_2 is the marginal effective tax rate against which costs are deductible. As long as they are below one hundred percent, then the higher the rates, the stronger is the effect of their difference.

The model is suited to show the effects of several variables:

- leverage,
- tax rates,
- tax depreciation rates,
- the probability of being in tax position,
- the ratio of operating (gross) income to operating costs,
- the beta values of product and factor prices.

In what follows we restrict attention to the four first of these. The conclusion is that the systematic risk depends strongly on all of them. There are also lessons to be learned for tax authorities, who should be concerned about loss offset provisions. This is in line with the recommendation of Boadway et al. (1989).

In order to arrive at analytical results without too complicated expressions, several simplifying assumptions have been made. The most important are:

- The corporation produces only one commodity using only one variable input factor.
- Investment takes place in period 0, production in period 1.
- There are no more periods. Depreciation deductions are divided between the two periods.

[1] In theory it is possible to preserve the value of a loss carry-forward even when it is uncertain, given that it accumulates at a risk-adjusted interest rate. In practice the uncertainty will be specific to each

- The ratio of (gross) operating income to operating costs in period 1 is exogenous to the corporation.
- The debt of the corporation is risk free.
- The leverage, i.e., the ratio of interest carrying debt to the total financing need in period zero, is some given number, independent of the investment decision.
- When there is uncertainty about the corporation's tax position in period 1, we have used an additional simplifying assumption. The factor price is assumed to be perfectly correlated with the product price, or not risky at all. This reduces the problem to a one-dimensional option valuation problem, which makes it analytically much easier to handle.
- If the corporation is out of tax position in period 1, there is no loss carry-forward or carry-back.
- In order to value the after-tax cash flow in period 1, as seen from period 0, we use the standard CAPM and the standard Black and Scholes option valuation, even though the underlying assumptions of the two are not identical.
- Inflation is neglected.

The simplifications mean that we are not able to give a really realistic analysis. In our judgment, the numerical examples show important aspects of important effects of the tax system. To be more realistic one has to resort to multi-period models and numerical methods.

The assumption of risk free debt may be realistic when the tax paying corporation is owned by a larger corporation which implicitly guarantees the debt payments. The assumption of a fixed leverage ratio may also be a reasonable approximation in that situation, when the ratio is not determined by the usual considerations of solidity.

Before going into the corporation's decision problem, we define the beta values of the product and factor prices. For a traded security the return (i.e., one plus the rate of return) between period 0 and 1 is S_1 / S_0, where S_T denotes the value at the end of period T, and S_1 includes dividends, if any. The CAPM states that

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situation, so the appropriate rate cannot be found.

$$E(S_1 / S_0) = R_f + b\lfloor E(R_m) - R_f \rfloor,$$

where $R_m \equiv 1 + r_m$ is the return on the market portfolio, the risk free rate is r_f, and we let $R_f \equiv 1 + r_f$. The beta value is

$$b = \frac{\text{cov}(S_1 / S_0, R_m)}{\text{var}(R_m)}.$$

Solving for S_0 gives

$$S_0 = \frac{1}{R_f}[E(S_1) - l\,\text{cov}(S_1, R_m)],$$

where $l = \lfloor E(R_m) - R_f \rfloor / \text{var}(R_m)$ is independent of the specific security. This gives a general expression for the market valuation in period 0 of the claim to S_1,

$$j\,(S_1) \equiv \frac{1}{R_f}[E(S_1) - l\,\text{cov}(S_1, R_m)].$$

In the context of investment projects, however, there is little reason to believe that output or input prices behave like security prices. Except perhaps for precious metals, a commodity price is likely to have what McDonald and Siegel (1984) called an (expected-)rate-of-return shortfall. If P_T denotes the output price at time T (T = 0,1), one would expect

$$P_0 < j\,(P_1) \equiv \frac{1}{R_f}[E(P_1) - l\,\text{cov}(P_1, R_m)].$$

In what follows, P_0 plays no role, while $j\,(P_1)$ is important. This is the market value at time 0 of a claim to receiving one unit of the output one period later, which may be viewed as a prepaid forward contract. The return $P_1 / j\,(P_1)$ does not have a rate-of-return shortfall, contrary to P_1/P_0. The beta value of the price P must be defined in relation to this return,

$$b_P = \frac{\operatorname{cov}(P_1 / j(P_1), R_m)}{\operatorname{var}(R_m)},$$

and similarly for the input price,

$$b_W = \frac{\operatorname{cov}(W_1 / j(W_1), R_m)}{\operatorname{var}(R_m)}.$$

It is possible to express, e.g., b_P more explicitly, without the detour via $j(P_1)$, namely as[2]

$$b_P = \frac{R_f}{E(P_1)\dfrac{\operatorname{var}(R_m)}{\operatorname{cov}(P_1, R_m)} - \left[E(R_m) - R_f\right]}. \tag{1}$$

In the analysis which follows, the corporation chooses its real investment in period 0 in order to maximize its market value at that time. The period-0 market value of the net operating income in period 1 is an increasing concave function of the investment. The first-order condition for optimal investment is that the last dollar invested creates a net operating income with the period-0 market value of one dollar. Thus there is no (after-tax) rent to be earned at the margin, and it is the return on the last dollar invested which satisfies the CAPM market equlibrium condition.

1 Case 1: No leverage, cash flow tax or no tax

Consider first *case 1*, a fully equity financed corporation, not subject to a cash flow tax or no tax at all. Let V_1 (subscript denoting case 1) be the cash flow of the corporation in period 1, in the no-tax case equal to

[2] This equation is also the correct expression for “project beta,” i.e., the beta of a potential investment project producing P_1 in period 1.

$$V_1 = PQ - WL\,,$$

where Q is the output quantity, L is the factor quantity. There is no need for a time subscript on P or W, since we are hereafter only interested in their values in period 1. Using the definition above, the beta value of the equity is

$$b_{V1} = \frac{Qj\,(P)}{j\,(V_1)}\,b_P - \frac{Lj\,(W)}{j\,(V_1)}\,b_W\,.$$

The simplifying assumption is that whatever investment level the corporation chooses, the ratio of $Qj\,(P)$ to $Lj\,(W)$ is fixed. We fix the value at 2, but the model is suited to investigate consequences of different values, each being exogenous to the corporation and independent of investment.

The marginal investment I is characterized by the first-order condition described above. Let $g_1 I$ (the subscript denotes case 1) be the marginal product, in the form of market value as seen from period 0. This is the difference between

$$Qj\,(P) = 2g_1 I$$

and

$$Lj\,(W) = g_1 I\,.$$

Here, and in each of the subsequent cases 2–5, we solve for that g which satisfies the first-order condition. In this very simple case, $g_1 I$ must be exactly equal to I, which must be equal to the necessary equity for investment,

$$g_1 I = I = j\,(V_1)\,. \tag{2}$$

Thus g_1 equals 1, and $Qj\,(P)/j\,(V_1) = 2$, and $Lj\,(W)/j\,(V_1) = 1$. These weights allow us to find the equity's beta value, given beta values for P and W.

Example: If $b_P = 0.48$, $b_W = 0.36$, then $b_{V1} = 0.60$.

Below the same method will be used to see how the equity's after-tax beta value is affected by, first, leverage, then tax, then tax with loss carry-forward, and finally a tax with imperfect loss carry-forward. The method also gives a g for each case, which indicates how the investment decision is distorted as compared to a no-tax situation.

The effects of a cash flow tax are easily analyzed. We shall show that g and b are unaffected, but introduce temporarily the notation $1t$ for this case. The after-tax cash flow in period 1 is

$$V_{1t} = (PQ - WL)(1 - t),$$

and the financing equation is

$$g_{1t} I(1 - t) = I(1 - t) = j(V_{1t}).$$

Thus we get $g_{1t} = 1 = g_1$, and

$$b_{V1t} = \frac{Qj(P)(1 - t)}{j(V_{1t})} b_P - \frac{Lj(W)(1 - t)}{j(V_{1t})} b_W = b_1.$$

Obviously, the cash flow tax acts, at least in standard financial models, as just another shareholder.

When looking at the numerical examples in each case, the deviations from case 1 can be interpreted as follows:

- A high b_{Vi}, greater than $b_{V1} = 0.60$, means that the equity has a higher systematic risk than in case 1.

- A high g_i, greater than $g_1 = 1$, means that the corporation requires a higher pre-tax marginal productivity than in case 1. This will imply that less is invested.

Of course, low values of b_{Vi} and g_i have the opposite interpretations. We shall see that there is no simple, monotonous relation between b and g. But that would not be expected, since b is related to the expected rate of return on equity after tax, while g is related to the expected rate of return on the total investment before tax. We make more complicated changes than, e.g., a monotonous change in a tax rate.

2 Case 2: Leverage, but no tax

In *case 2* we consider the effect of financing part of the investment by debt in period 0. We assume the whole debt is paid back with interest with full certainty in period 1. The cash flow to equity in period 1 is

$$V_2 = PQ - WL - (1 + r_f)B, \tag{3}$$

where B is the debt. The period-0 value of equity will be

$$j(V_2) = Qj(P) - Lj(W) - B. \tag{4}$$

We assume 50 per cent debt financing independent of the investment decision. Different fixed numbers may easily be analyzed. Let $g_2 I$ be the marginal product of the marginal investment, viewed as period-0 market value,

$$Qj(P) = 2g_2 I,$$
$$Lj(W) = g_2 I.$$

Solving for g_2 like before,

$$g_2 I = I = j(V_2) + B. \qquad (5)$$

This gives g_2 equal to 1, and $j(V_2) = 0{,}5I$. With no tax the debt finance does not affect the real investment decision. But the beta value of equity is affected. We find,

$$\frac{Q j(P)}{j(V_2)} = 4,$$

$$\frac{L j(W)}{j(V_2)} = 2,$$

and again we may find the equity beta based on b_P and b_W.

Example: If $b_P = 0.48$, $b_W = 0.36$, then $b_{V2} = 1.20$.

This effect of leverage on the equity beta is well known.

3 Case 3: Two types of profits tax

Inspired by the Norwegian petroleum tax system we now assume to types of tax on the corporation's net income (or profits). One is an ordinary profits tax with net financial costs deductible. Tax depreciation allowances are straight-line over six years in the actual system, starting in year 0. In order to represent this in a two-year model, we assume the remainder is deductible in year 1. However, in order to keep the two-period model as realistic as possible, we represent the actual sequence of remaining depreciation over six years with its present value 0.7312, calculated at a 7 percent discount rate.

The base for the other tax, a special tax intended to capture rent, has only one feature different from the first. There is an extra deduction proportional to the investment, an uplift, of 5 per cent in each of six years. We compress it in the same manner. We use the actual tax rates, and show these and the deductions in the table.

	Income tax	Special tax
Tax rate	t = 0.28	t = 0.50
Depreciation allowance and uplift (if any), year 0	a = 0.1667	b = 0.2167
Depreciation allowance and uplift (if any), year 1	c = 0.7312	d = 0.9506

In case 3 we assume that the corporation with certainty will be in tax position in both years. This is close to realistic at the margin for a corporation with many ongoing rent-yielding activities. (In the Norwegian case, the tax base is all activities of the corporation in the sector.) The cash flow to equity in year 1 will be

$$V_3 = (PQ - WL)(1 - t - t) - B(1 + r_f) + Br_f(t + t) + I(tc + td)\,. \qquad (6)$$

The market value of this as viewed from year 0 is

$$j\,(V_3) = \left[Qj\,(P) - Lj\,(W)\right](1 - t - t) - B + \frac{r_f\,(t + t)B}{R_f} + \frac{(tc + td)I}{R_f}\,. \qquad (7)$$

Just as in the previous cases, the expression in square brackets is set equal to $g_3 I$, and then the equation is solved for g_3.

In order to find a useful solution, all terms must be written as proportional to I. This is achieved here by considering the division in year 0 of the financing need between equity and debt:

$$j\,(V_3) + B + taI + tbI = I\,, \qquad (8)$$

which gives, when divided equally,

$$j\,(V_3) = B = \frac{I(1 - ta - tb)}{2}\,. \qquad (9)$$

Thus we find

$$g_3 = \frac{1}{1-t-t}\left[1 - ta - tb - \frac{r_f\,(t+t\,)(1-ta-tb)/2 + tc + td}{R_f}\right]. \tag{10}$$

The beta value of equity is now

$$b_{V3} = \frac{Qj\,(P)(1-t-t\,)}{j\,(V_3)}\,b_P - \frac{Lj\,(W)(1-t-t\,)}{j\,(V_3)}\,b_W\,. \tag{11}$$

Introducing the numbers for tax rates and deductions gives $j\,(V_3) = 0.4225I$, and, with a risk free rate of 7 per cent, $g_3 = 0.8542$. This is somewhat below the g values of cases 1 and 2, indicating that the tax system contributes to reducing the required pre-tax productivity of the investment. This is due to the generous deductions. It must be stressed, however, that this is a stylized version of the actual tax system.

Based on these numbers we can find

$$\frac{Qj\,(P)(1-t-t\,)}{j\,(V_3)} = 0.8896,$$

$$\frac{Lj\,(W)(1-t-t\,)}{j\,(V_3)} = 0.4448\,.$$

Example: If $b_P = 0.48$, $b_W = 0.36$, then $b_{V3} = 0.2669$.

The example illustrates that the tax system can counteract the effect leverage has on the beta value of equity. Relative to a neutral cash flow tax, this tax system dictates a “loan” from the corporation to the tax authorities. In our stylized version the “loan” is

$$I\left[t(1-a) + t\,(1-b)\right],$$

as compared to receiving $I(t + t\,)$ in tax rebate in period 0. The “loan” is repaid with

$$I(tc + td)\,,$$

the last term in (6). The “interest rate” paid by the tax system is

$$\frac{tc + td}{t(1 - a) + t(1 - b)} - 1\,,$$

which for the given numbers is 8.8 percent. Thus it is favorable under these assumptions.

4 Case 4: Out of tax position in investment year

In *case 4*, we consider the same situation as in case 3, except that the corporation is out of tax position in the investment year. But we assume that the corporation pays both taxes with full certainty in the production year. For some corporations this may be more realistic than case 3. Case 4 is also a good basis for comparison with case 5.

We assume deductions for the investment are the same, but that they become effective in year 1. The cash flow to equity in year 1 will be

$$V_4 = (PQ - WL)(1 - t - t) - B(1 + r_f) + Br_f(t + t) + I[t(a + c) + t(b + d)]. \qquad (12)$$

The period-0 market value of this is

$$j(V_4) = [Qj(P) - Lj(W)](1 - t - t) - B + \frac{r_f(t + t)B}{R_f} + \frac{[t(a + c) + t(b + d)]I}{R_f}\,. \qquad (13)$$

The financing need, which is larger than that in case 4, is split between equity and debt,

$$j(V_4) + B = I\,, \qquad (14)$$

which gives, when the division is equal,

$$j\,(V_4) = B = \frac{I}{2}\,. \tag{15}$$

This gives

$$g_{\,4} = \frac{1}{1-t-t}\left[1-\frac{r_f\,(t+t\,)/2+t(a+c)+t\,(b+d)}{R_f}\right]. \tag{16}$$

The beta value of equity is

$$b_{V4} = \frac{Q\,j\,(P)(1-t-t\,)}{j\,(V_4)}\,b_P - \frac{L\,j\,(W)(1-t-t\,)}{j\,(V_4)}\,b_W\,. \tag{17}$$

Introducing the numbers for taxes, deductions, and the risk free rate gives $g_{\,4} = 0.8823$. This is somewhat above the g value from case 3, but still below unity. The tax system does not encourage as low-productive investment as in case 3. Based on the numbers we also find

$$\frac{Q\,j\,(P)(1-t-t\,)}{j\,(V_4)} = 0.7764\,,$$

$$\frac{L\,j\,(W)(1-t-t\,)}{j\,(V_4)} = 0.3882\,.$$

Example: If $b_P = 0.48$, $b_W = 0.36$, then $b_{V4} = 0.2329$.

5 Case 5: Uncertain tax position, imperfect loss offset

In *case 5* the assumptions are as in case 4, except that there is uncertainty about the tax position in year 1.

One should observe that this may be relevant even if the probability of being out of tax position is very low. In an option valuation which is consistent with the CAPM, the probability should be adjusted up to reflect the market's pricing of the systematic risk of the tax base.

In order to simplify the analysis of case 5, we make the assumption that the factor price is perfectly correlated with the output price, or known with certainty. Thus the problem is reduced to uncertainty about the output price, and it is easier to arrive at an analytical formula for the option value of the tax claim.

Perfect correlation means that there are constants k_0 og k_1 such that

$$W = k_0 + k_1 P . \tag{18}$$

This also covers the special case $k_1 = 0$, which removes uncertainty from *W*.

While it is possible to derive the formulae below for each of these constants being positive, zero or negative, one should be careful to maintain consistency with the option valuation model. We need $PQ - WL$ to be distributed as a lognormal minus (or plus) a constant, so that the tax base is a lognormal minus a positive constant.

In particular we have

$$b_W = \frac{R_f}{\left[\frac{k_0}{k_1} + E(P)\right]\frac{\text{var}(R_m)}{\text{cov}(P, R_m)} - \left[E(R_m) - R_f\right]} . \tag{19}$$

(If $k_1 \to 0$, then $b_W \to 0$ as well.) Comparing with (1), we see that a positive b_P and positive constants k_0 and k_1 imply that $b_W < b_P$.

We will use the following relations, which are easily shown from the definitions above,

$$b_W = \frac{k_1 j(P)}{j(W)} b_P, \tag{20}$$

$$j(W) = \frac{k_0}{R_f} + k_1 j(P). \tag{21}$$

We assume that there are no more loss carry-forwards after year 1. This is of course an extreme assumption, since there will often be a possibility of being in tax position in a later year, and loss carry-forward to that year.

The cash flow to equity in year 1 will be

$$\begin{aligned} V_5 = {} & (PQ - WL) - B(1 + r_f) \\ & - t \cdot \max(0, PQ - WL - r_f B - aI - cI) \\ & - t \cdot \max(0, PQ - WL - r_f B - bI - dI). \end{aligned} \tag{22}$$

Using $W = k_0 + k_1 P$, this can be rewritten as

$$\begin{aligned} V_5 = {} & P(Q - k_1 L) - k_0 L - B(1 + r_f) \\ & - t(Q - k_1 L) \cdot \max\left[0, P - \frac{k_0 L + r_f B + (a + c)I}{Q - k_1 L}\right] \\ & - t(Q - k_1 L) \cdot \max\left[0, P - \frac{k_0 L + r_f B + (b + d)I}{Q - k_1 L}\right]. \end{aligned} \tag{23}$$

In order to find year-0 market values we use Black and Scholes's option value formula, as modified by McDonald and Siegel (1984) to allow for rate-of-return shortfall. The two max expressions are seen as the values at expiration of two European call options on one unit of output, with exercise prices equal to the fractions which are deducted from *P*.

The year-0 market value of this is

$$
\begin{aligned}
j\,(V_5) = {} & (Q - k_1 L) j\,(P) - k_0 \frac{L}{R_f} - B \\
& - t(Q - k_1 L)\left[j\,(P) N(x_1) - \frac{k_0 L + r_f B + (a + c) I}{R_f (Q - k_1 L)} N(x_2) \right] \\
& - t\,(Q - k_1 L)\left[j\,(P) N(x_3) - \frac{k_0 L + r_f B + (b + d) I}{R_f (Q - k_1 L)} N(x_4) \right],
\end{aligned}
\tag{24}
$$

with *N* being the cumulative standard normal distribution function.

$N(x_1)$ and $N(x_2)$ can be interpreted as probabilities that the corporation will be in position to pay the tax at the rate *t*. The two probabilities are adjusted, however, to reflect the market valuation of the systematic risk of the tax base. With a positive systematic risk $N(x_2)$ is adjusted downwards, and so is $N(x_1)$, but not that much.

$N(x_3)$ and $N(x_4)$ can be interpreted similarly for the tax at the rate t. These will be somewhat lower than $N(x_1)$ and $N(x_2)$, respectively, since the base for the special tax is strictly lower, so that the probability is strictly lower.

If all four probabilities approach unity, we get to the situation in case 4 above. The market value approaches the expression we had in case 4. To see this, introduce $W = k_0 + k_1 P$ in equation (13) for $j\,(V_4)$. This gives the limiting case,

$$
\begin{aligned}
\lim j\,(V_5) = j\,(V_4) = {} & (Q - k_1 L)(1 - t - t\,) j\,(P) - \frac{k_0 L (1 - t - t\,)}{R_f} \\
& - B + \frac{r_f (t + t\,) B}{R_f} + \frac{t(a + c) + t\,(b + d)}{R_f} I.
\end{aligned}
\tag{25}
$$

Equation (24) shows $j\,(V_5)$ as a function of $j\,(P)$, without using $j\,(W)$. In order to express the elements of the value as proportional to *I*, as in the previous cases, we need to reintroduce $Qj\,(P) = 2gI$ og $Lj\,(W) = gI$.

Then we find, from (20) and (21),

$$
\begin{aligned}
j\,(V_5) = {} & 2g_5 I\left[1 - tN(x_1) - tN(x_3)\right] \\
& - g_5 I\left\{\frac{b_W}{b_P}\left[1 - tN(x_1) - tN(x_3)\right] + \left(1 - \frac{b_W}{b_P}\right)\left[1 - tN(x_2) - tN(x_4)\right]\right\} \\
& - B + \frac{r_f B}{R_f}\left[tN(x_2) + tN(x_4)\right] \\
& + \frac{I}{R_f}\left[t(a + c)N(x_2) + t\,(b + d)N(x_4)\right].
\end{aligned}
\tag{26}
$$

Again we assume that financing is divided equally between equity and debt, and find

$$
g_5 = \frac{1 - \dfrac{1}{R_f}\left[\left(\dfrac{r_f}{2} + a + c\right)tN(x_2) + \left(\dfrac{r_f}{2} + b + d\right)tN(x_4)\right]}{\left(2 - \dfrac{b_W}{b_P}\right)\left[1 - tN(x_1) - tN(x_3)\right] - \left(1 - \dfrac{b_W}{b_P}\right)\left[1 - tN(x_2) - tN(x_4)\right]}.
\tag{27}
$$

Then it is possible to solve for the beta value of equity,

$$
b_{V5} = \frac{j\,(P)(Q - k_1 L)\left[1 - tN(x_1) - tN(x_3)\right]}{j\,(V_5)}\, b_P .
\tag{28}
$$

The value of the fraction is found from the first line and the first term in the second line of (26):

$$
\frac{j\,(P)(Q - k_1 L)\left[1 - tN(x_1) - tN(x_3)\right]}{j\,(V_5)} = \frac{g_5 I\left(2 - \dfrac{b_W}{b_P}\right)\left[1 - tN(x_1) - tN(x_3)\right]}{I/2}.
\tag{29}
$$

Example: Assume that $N(x_1) = 0.95$, $N(x_2) = 0.9$, $N(x_3) = 0.9$ og $N(x_4) = 0.85$. These would correspond to somewhat higher actual probabilities, say 0.97 for paying tax at the rate *t*, and 0.93 for paying tax at the rate t . Using the numbers given above, the fraction is equal to 0.7839. With $b_P = 0.48$ and $b_W = 0.36$, one gets $b_{V5} = 0.3763$. In this case we find $g_5 = 1.1041$.

The example shows that even with relatively high values of the probabilities of being in tax position, the required productivity of the marginal investment increases significantly, and the beta value of equity is also considerably higher than in a situation with a certain tax position.

6 Discussion and conclusion

An overview of the numerical examples is given in the table:

i	g_i	b_{Vi}
1	1	0.60
2	1	1.20
3	0.85	0.27
4	0.88	0.23
5	1.10	0.38

Even if the technology is the same in all cases, and even if the beta values of the two prices are the same, we see significant variations in the required marginal productivity before tax and large variations in the beta value, which determines the required expected rate of return after tax. We also see that the relationship between these is non-monotonous.

The interplay between debt finance and imperfect loss offset makes the investment decision very sensitive to changes in tax position. This was already shown by Lund

(1987) using the same method. This implies that tax authorities would have to be very careful to avoid distorting investment decisions.

The examples also show that it is difficult, and can be misleading, to use beta values from one tax regime to make decisions under a different regime. Text books warn against using the same required expected rate of return for all projects in a corporation. The reason which is often given for the warning is that projects differ in terms of pre-tax systematic risk. But we have shown that even with the same underlying systematic risk in pre-tax cash flows, the after-tax systematic risk may vary a lot. In addition to the variation shown, there is variation due to different ratios between operating income and cost.

Traditionally one has analyzed how tax systems influence the corporations' required pre-tax rate of return from projects. This is expressed in g . But the analysis shows that tax systems also affect the required after-tax expected rate of return, expressed in b . This means that one cannot even use the same required after-tax required expected rate of return when analyzing different (hypothetical, suggested) tax systems applied to the same projects.

Required expected rates of return after tax in various tax regimes are typically observed as (averages of) realized rates of return after tax, or realized beta values. In addition to the problems we have shown so far in the use of these for other tax regimes, there is the problem that they may include realized rent. Unless all after-tax rents have already been capitalized in the market value of shares, this is typically the case in resource extraction in regimes where companies do not pay any up-front fee to reflect the resource value. In that case the realized rates of return are average rates of return, while the required rate of return is a marginal rate.

While we suggested above that the results were based on the Black and Scholes option valuation model, they are in fact more general. The underlying asset, P in our case, does not need to have a price process which is a geometric Brownian motion with drift. The result that a European call option's value can be written as an expected present value under an adjusted probability measure, holds very generally, see, e.g., Björk (1998). Then the exercise price and the underlying asset price can be valued separately, as is done here, but the probabilities will of course depend on the price process and the systematic risk.

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