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***The rate of capital retirement: How is it related to the form of the survival function and the investment growth path?***

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# THE RATE OF CAPITAL RETIREMENT: HOW IS IT RELATED TO THE FORM OF THE SURVIVAL FUNCTION AND THE INVESTMENT GROWTH PATH?*)

by

**ERIK BIØRN**

ABSTRACT

We discuss the relationship between the retirement process of the capital, as formalized by its survival function, and the average retirement rate, and how this relationship is affected by changes in the investment path. The effect of the survival function on the age distribution of the capital goods, both those existing and those retired in each period, is also considered. These issues are illustrated by means of parametric (convex and concave) functions and numerical examples. We find that the retirement rate is a declining function of the growth rate of investment (except in the exponential decay case) and quite sensitive to the value of this parameter over a reasonable interval. Approximating the retirement rate by the inverse of the capital's maximal life-time or twice this value ('double declining balance') will in many cases produce very inaccurate results. The response of the capital/investment ratio and the retirement/investment ratio to changes in the investment growth rate and in the curvature of the survival function is also investigated.

**Keywords**: Capital. Retirement. Survival function. Age distribution.
Exponential decay. Mortality rate
**JEL classification**: C51, D24, E22, O47

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# 1 Introduction

The retirement of capital goods is a variable of considerable interest in economics and econometrics, and retirement (mortality) rates are important for several research purposes. In most countries, replacement investment accounts for a substantial part of gross investment and GNP, according to national accounts calculations. The process describing how capital retires is a basic determinant of the age distribution of the capital stock, as is also the time path of the investment in new assets. Using demographic analogues, we may associate (gross) investment with the birth of capital and retirement with death. From basic demography we know that the pattern of mortality rates of a population of persons and the time profile of the births jointly determine the age distribution of the population (when adjustment is made for migrations). Similarly, changes in the retirement process and in the time path of new investments jointly determine the average age of the capital and other characteristics of its age distribution (when purchases and sales of used capital are adjusted for).

In this paper, we discuss the relationship between the retirement process of the capital, formalized by its survival function, and the average retirement rate, and how this relationship is affected by changes in the investment growth path. The average retirement rate of capital goods corresponds to the average mortality rate of a population of persons and the ratio between the gross investment and the capital stock corresponds to the birth rate. We will also discuss how changes in the survival function affects the age distribution of the capital goods, partly in general terms, partly by means of parametric (convex and concave) functions, and partly through numerical examples. An integration of our analysis into a model of the firm's investment demand is, however, beyond our scope. Neither do we present econometric illustrations of the relationships under discussion.

It is well known that the average rate of capital retirement will be a constant independent of the time path of gross investment in the particular case where the survival function is exponentially declining in the age of the capital (exponential decay, declining balance). Exponential decay is the *only* case in which changes in the investment profile do not affect the average retirement rate; see Feldstein and Rothschild (1974, section 2). Often exponential decay is considered a benchmark case of capital retirement, and it has been argued that it is an acceptable simplifying approximation for long-term analysis, for instance in modelling economic growth, cf. Jorgenson (1963, p. 251, 1974). Yet, average retirement rates are applied by researchers even if exponential decay is not warranted, for instance in the calculation of capital service prices. This practise may be criticized, see Biørn (1989, section 11.7).[1]

An examination of the relationship between the form of the retirement process and the average retirement rate of capital under non-exponential decay is interesting for several purposes. We may, for example, wish to interpret time series for the retirement rates calculated from investment data by the perpetual inventory method with a given retirement process on a 'scale' comparable to the retirement rate under exponential decay. Which is the appropriate 'translation' of the retirement rate, $\delta$, if the retirement pattern of the capital correponds to, say, simultaneous retirement at a fixed age $N$, or to linear retirement up to a fixed life-time, $N$? Often $\delta = 1/N$ is used as a 'rule of translation' for the former and $\delta = 2/N$ for the latter.[2] Under which conditions are these translation rules valid, or acceptable, approximations? If direct registrations of the capital stock are available, when will constancy of the implied retirement rate hold as a satisfactory approximation? A further, more general question is: How will the changes in the curvature of the survival function affect the average retirement rate for different growth profiles of gross investment?

The following sections are disposed as follows. In Section 2, we describe the model and define the basic concepts. Section 3 considers the age distribution of the capital and the related age distribution of retirement. We next derive the relationship between the overall retirement rates, the hazard rate of the mortality distribution of the capital, and its age distribution. The hazard rates can be interpreted as age specific retirement rates. In Section 4, we consider the special case with constant gross investment. We show that in this case, the overall retirement rate coincides with the inverse of the total service flow from one capital unit during its life-time.

Three parametric survival functions are defined in Section 5, characterized by, respectively, exponentially declining survival function with infinite maximal life-time (exponential decay), convex survival function with finite maximal life-time, and concave survival function with finite maximal life-time. We discuss the way in which the curvature of the survival function interferes with the growth rate of investment in determining the age distribution of the capital and its average retirement rate. Three special cases – linear, simultaneous, and immediate retirement – are considered in Section 6. The translation formulae $\delta = 1/N$ for the first and $\delta = 2/N$ for the second hold under stationarity of gross investment only and may be quite inaccurate otherwise. Numerical illustrations of

[1] Retirement, as the term is being used here, is distinctly different from *depreciation*, which is related to the decline in the *value* of the capital, rather than to its deterioration as an input in production. A recent survey of empirical studies of depreciation is Jorgenson (1996).

[2] The latter is often denoted as the 'double declining balance' pattern; cf., *e.g.*, Fraumeni and Jorgenson (1980, p. 24).

the sensitivity of the retirement rate and properties of the age distribution to changes in the curvature of the survival function and the growth pattern of investment are given in Section 7. A main conclusion is that the retirement rate is a declining function of the growth rate of investment and quite sensitive to the value of this growth rate over a reasonable interval. If, for example, the (continuous) growth rate of investment increases from 0 to 5 per cent and all capital retires simultaneously at age 20, the retirement rate decreases from 5 per cent to 2.9 per cent. With linear retirement over 20 years, the corresponding retirement rate decreases from 10 per cent to 8.6 per cent.

## 2 Model and basic concepts

Let $J(t)$ denote gross investment at time $t$, with $t$ continuous. The investment brings an increase in the productive capacity of the capital stock, which disappears gradually as the age of the capital increases. This is characterized by the *survival function, or survival curve*, $B(s)$, indicating the proportion of the capacity of a capital stock which survives at age $s$ ($\geq 0$).[3] It is assumed to be time invariant and satisfies

$$B(s) \in [0,1], \quad B'(s) \leq 0, \quad B(0) = 1, \quad B(\infty) = 0. \tag{1}$$

The capital volume of age $s$ at time $t$, *i.e.*, belonging to vintage $t-s$ at time $t$, is

$$K(t,s) = B(s)\,J(t-s). \tag{2}$$

We make the usual assumption that *one unit of capital produces one unit of capital services per unit of time*, so that $K(t,s)$ both has a stock interpretation and represents the instantaneous flow of capital services produced at time $t$ by the capital which is of age $s$. Neo-classical theory of production implies that capital is malleable, *i.e.*, capital units belonging to different vintages are perfect substitutes at any point in time, so that the capital input in a production function at time $t$ can be represented by the aggregate

$$K(t) = \int_0^\infty K(t,s)ds = \int_0^\infty B(s)J(t-s)ds. \tag{3}$$

This variable, the *gross capital stock*, has the joint interpretation as the total number of capital units at time $t$ and the instantaneous flow of services 'produced' by these units

[3]This function represents both the *loss of efficiency* of existing capital units and their *physical disappearance*. We may therefore, in principle, consider $B(s)$ as the product of two factors, one indicating the relative number of capital units surviving at age $s$ (the survival function), the other indicating the relative decline in efficiency of each remaining unit (the efficiency function). We will not focus on such a decomposition here. However, for the purpose of measuring capital stocks from market data, the distinction between the survival function and the efficiency function is important; see Biørn (1998).

at time $t$.[4]

The flow of capital goods retired at time $t$ is the difference between the gross investment, $J(t)$, and the increase in the gross capital stock, $\dot{K}(t)$, the "dot" indicating the time derivative. Using (1) and (3), the *retirement* at time $t$ can be expressed as

$$D(t) = J(t) - \dot{K}(t) = \int_0^\infty b(s)J(t-s)\,ds \tag{4}$$

where

$$b(s) = -B'(s), \qquad s \geq 0. \tag{5}$$

The function $b(s)$, denoted as the *(relative) retirement function, or retirement (mortality) curve*, indicates the structure of the retirement process: $b(s)ds$ is the share of an *initial investment* of one unit which disappears from $s$ to $s + ds$ years after installation. From (1) and (5) it follows that $b(s)$ is non-negative with $\int_0^\infty b(s)ds = 1$.

Let

$$\beta(s) = \frac{b(s)}{B(s)} = -\frac{d\ln B(s)}{ds}, \tag{6}$$

so that $\beta(s)ds$ is the share of the *remaining capital* at age $s$ which disappears from $s$ to $s + ds$ years after installation. The difference between $b(s)$ and $\beta(s)$ follows from their different normalization: $b(s)$ relates to normalization with respect to the *initial* investment, $\beta(s)$ relates to normalization against the capital *remaining*. The latter corresponds to age specific mortality rates in demography.[5]

The volume of retirement at time t can be decomposed by vintages by

$$D(t,s) = b(s)J(t-s) = \beta(s)K(t,s) \tag{7}$$

which denotes the retirement of capital of age $s$ at time $t$, *i.e.*, belonging to vintage $t-s$ at time $t$, so that (4) can be written as

$$D(t) = \int_0^\infty b(s)J(t-s)\,ds = \int_0^\infty D(t,s)ds. \tag{8}$$

[4]This contrasts with the putty-clay approach in which (i) each capital vintage is assumed to have its specific input coefficients *ex post*, determined at the time of investment [cf. Johansen (1959, 1972)], (ii) capital is not malleable, and (iii) (planned and realized) retirement of capital is determined from profitability conditions, *e.g.*, formalized by means of a quasi-rent criterion [cf. Biørn and Frenger (1992)].

[5]In a probabilistic setting, $B(s)$ can be interpreted as the survival probability, $1 - B(s)$ as the non-survival probability, $b(s)$ as the density function and $\beta(s)$ as the failure rate or hazard rate function; see Barlow and Proschan (1975, section 3.1) and Kalbfleisch and Prentice (1980, section 1.2).

## 3 The age distribution of capital and retirement

We next consider the age distributions of the capital and the retirement. Using (2) and (3), the *age distribution of the capital stock* at time $t$ can be characterized by the share of the capital which at time $t$ is of age $s$,

$$k(t,s) = \frac{K(t,s)}{K(t)} = \frac{B(s)J(t-s)}{\displaystyle\int_0^\infty B(s)J(t-s)ds}, \qquad s \geq 0. \tag{9}$$

From (7) and (8) it follows that the *age distribution of the retirement of capital* at time $t$ can be characterized by the share of the retirement which at time $t$ relates to capital of age $s$ as

$$d(t,s) = \frac{D(t,s)}{D(t)} = \frac{\beta(s)K(t,s)}{\displaystyle\int_0^\infty \beta(s)K(t,s)ds} = \frac{b(s)J(t-s)}{\displaystyle\int_0^\infty b(s)J(t-s)ds}, \qquad s \geq 0. \tag{10}$$

Obviously, $k(t,s)$ and $d(t,s)$ are non-negative with $\int_0^\infty k(t,s)ds = \int_0^\infty d(t,s)ds = 1$.

From (7) we find that the *age specific retirement rate* at time $t$ and age $s$ is

$$\delta(t,s) = \frac{D(t,s)}{K(t,s)} = \beta(s), \tag{11}$$

and from (3), (8), and (9) it follows that the *overall retirement rate* can be expressed as

$$\delta(t) = \frac{D(t)}{K(t)} = \frac{\displaystyle\int_0^\infty D(t,s)ds}{\displaystyle\int_0^\infty K(t,s)ds} = \frac{\displaystyle\int_0^\infty b(s)J(t-s)ds}{\displaystyle\int_0^\infty B(s)J(t-s)ds} = \int_0^\infty \beta(s)k(t,s)ds. \tag{12}$$

This rate is an average of the age specific retirement rates, weighted by the age distribution of gross capital at time $t$. It will therefore, in general, be *time dependent*. If, however, $\beta(s)$ is constant or the age distribution $k(t,s)$ is time invariant, $\delta(t)$ will be constant. These are the only two cases in which $\delta(t)$ will be time independent.

From (9) and (10) it follows that the relationship between the age distributions of retirement and capital can be expressed as

$$d(t,s) = \frac{\beta(s)k(t,s)}{\displaystyle\int_0^\infty \beta(s)k(t,s)ds}. \tag{13}$$

Using (12), this relationship can be rewritten as

$$\frac{d(t,s)}{k(t,s)} = \frac{\beta(s)}{\delta(t)}, \tag{14}$$

*i.e.*, the ratio between the age specific retirement and capital distributions at time $t$ and age $s$ equals the ratio between the age specific retirement rate at age $s$ and the overall retirement rate at time $t$.

Let now $\gamma(t,s) = J(t-s)/J(t),\ s \geq 0$, denote the *growth path of gross investment* up to time $t$. The rates $k(t,s)$, $d(t,s)$, and $\delta(t)$ then can be expressed in a simpler form. The age distribution of the capital can be expressed in terms of the survival function $B(s)$ and the growth path of investment $\gamma(t,s)$ as

$$k(t,s) = \frac{B(s)\gamma(t,s)}{\int_0^\infty B(s)\gamma(t,s)ds}, \tag{15}$$

the age distribution of the retirement can be expressed in terms of the retirement function $b(s)$ and the path $\gamma(t,s)$ as

$$d(t,s) = \frac{b(s)\gamma(t,s)}{\int_0^\infty b(s)\gamma(t,s)ds}, \tag{16}$$

and the overall retirement rate can be expressed in terms of the survival function $B(s)$, the retirement function $b(s)$, and the growth path $\gamma(t,s)$ as

$$\delta(t) = \frac{\int_0^\infty \beta(s)B(s)\gamma(t,s)ds}{\int_0^\infty B(s)\gamma(t,s)ds} = \frac{\int_0^\infty b(s)\gamma(t,s)ds}{\int_0^\infty B(s)\gamma(t,s)ds}. \tag{17}$$

It follows from (15) – (17) that *$k(t,s)$, $d(t,s)$, and $\delta(t)$ are all time invariant for any $B(s)$ if and only if $\gamma(t,s)$ is time invariant.* The corresponds to a *stable population* in demography [cf. Coale (1972) and Keyfitz (1977)]. For $\gamma(t,s) = J(t-s)/J(t)$ to be independent of $t$ for all $t$ and $s$, it must be an *exponential function* of the form $\gamma(t,s) = e^{-\alpha s}$.[6] The latter assumption will be specifically considered in Sections 5 – 7. If $\beta(s)$ is constant, which implies that $B(s)$ is an exponential function [cf. (6)], $\delta(t)$ will be time invariant and equal to this constant for any $\gamma(t,s)$. *Hence, constancy of $\delta(t)$ implies that at least one of the functions $B(s)$ or $\gamma(t,s)$ must be exponential functions in $s$.*

[6] Nickell comments on this result by saying that if $\delta(t)$ "is to settle down to a constant level in the long run in the absence of exponential decay, investment must settle to an exponential growth path in the long run. This is, in fact the result of renewal theory which has been quoted ..... as a justification for assuming that the potential replacement-capital ratio is constant. Now, if investment was indeed moving on a fixed exponential growth path, a proliferation of investment equations would hardly be required to explain this movement and consequently any serious theoretical justification of the assumption of a fixed potential replacement-capital ratio for empirical work must fall back on exponential decay" [Nickell (1978, p. 119)].

## 4 The constant investment case

*If gross investment is constant*, then $\gamma(t,s) = 1$ for all $t$ and $s$, and as a consequence, the age distributions of capital and retirement and the overall retirement rates will be constant and depend on the form of the survival function only. Let us take a look at this case.

Let $\Phi$ denote the total flow of services produced by one capital unit during its entire life-time, *i.e.*,

$$\Phi = \int_0^\infty B(s)ds. \tag{18}$$

*If gross investment is constant* and equal to $J$, it follows from from (3) and (4) that

$$K(t) = K = \Phi J, \tag{19}$$

$$D(t) = D = J. \tag{20}$$

Then the (constant) capital stock equals the (constant) gross investment times the per unit of capital service flow during the unit's life-time; net investment, $\dot{K}$, is zero. This corresponds to a *stationary population* in demography. Hence,

$$\delta(t) = \delta = \frac{D}{K} = \frac{1}{\Phi}, \tag{21}$$

*i.e.*, under constant gross investment, the overall retirement rate is constant and equal to the inverse of the total service flow per capital unit during its life-time.[7]

From (9) and (10) we find that under constant gross investment, the age distribution of capital and retirement are time invariant and equal to

$$k(t,s) = k(s) = \frac{B(s)}{\Phi}, \tag{22}$$

$$d(t,s) = d(s) = b(s), \tag{23}$$

for all $t$. Then we have

$$\frac{d(t,s)}{k(t,s)} = \frac{d(s)}{k(s)} = \beta(s)\Phi, \tag{24}$$

so that the two age distributions will coincide if and only if the hazard rate $\beta(s)$ is constant and equal to the inverse of the total service flow per capital unit during its life-time. This particular case is 'exponential decay', which will be discussed in Section 5.

[7] If the survival function declines so rapidly that $\Phi < 1$ – which is rather unlikely to occur in practice – the overall retirement rate, with our continuous time formulation of the retirement process will exceed 1.

## 5 Three parametric survival functions

In this section, we describe three examples with parametric survival functions. The examples are charaterized by, respectively, (i) exponentially declining survival function with infinite maximal life-time, (ii) convex survival function with finite maximal life-time, and (iii) concave survival function with finite maximal life-time. Example (i) is one-parametric, Examples (ii) and (iii) are two-parametric, with one parameter representing the maximal life-time and the other describing the curvature of the function.

In characterizing the convex and the concave survival functions and their properties we need the following function, defined by the integral

$$F(i,z) = \int_0^1 e^{z\theta}(1-\theta)^i d\theta = \int_0^1 e^{z(1-\lambda)}\lambda^i d\lambda = e^z \int_0^1 e^{-z\lambda}\lambda^i d\lambda, \quad i = 0,1,2,\dots, \tag{25}$$

where the second equality is obtained by substituting $\lambda = 1 - \theta$. Some properties of this function to be used below are proved in the Appendix.

**Exponential decay**

This, probably the most famous example of a survival function of capital in the literature, is the case where the function declines exponentially at the rate $\delta$, which is its only parameter, and we have

$$B(s) = e^{-\delta s}, \qquad \delta \geq 0, \ s \geq 0, \tag{26}$$

$$b(s) = \delta\, e^{-\delta s}, \tag{27}$$

$$\beta(s) = \delta, \tag{28}$$

$$\Phi = \frac{1}{\delta}. \tag{29}$$

This distribution thus has a *constant hazard rate, equal to* $\delta$. We also find

$$k(t,s) = d(t,s) = \frac{e^{-\delta s}\gamma(t,s)}{\int_0^\infty e^{-\delta s}\gamma(t,s)ds}, \tag{30}$$

$$\delta(t) = \delta, \tag{31}$$

so that this survival function is characterized by identical age distributions of capital and retirement and a constant overall retirement rate.

Assume now *constant rate of investment growth* at the rate $\alpha$, $J(t-s) = J(t)e^{-\alpha s}$, *i.e.*,

$$\gamma(t,s) = e^{-\alpha s}, \quad \text{for all} \ \ s,t. \tag{32}$$

Then the ratio between capital and gross investment, which may be interpreted as the *inverse birth rate* of capital, can, for an arbitrary survival function, be written as

$$\frac{K(t)}{J(t)} = \int_0^\infty B(s)e^{-\alpha s}ds. \tag{33}$$

Similarly, the ratio between retirement and gross investment, which can be interpreted as the *death to birth ratio* of capital, can be written as

$$\frac{D(t)}{J(t)} = \int_0^\infty b(s)e^{-\alpha s}ds. \tag{34}$$

Under exponential decay and exponential investment growth, we get the following particular expressions for the capital stock and retirement:

$$K(t) = \frac{J(t)}{\delta + \alpha}, \tag{35}$$

$$D(t) = \frac{J(t)\delta}{\delta + \alpha}. \tag{36}$$

It is easy to show that net investment then equals

$$\dot{K}(t) = \frac{J(t)\alpha}{\delta + \alpha},$$

which implies that the net investment/gross investment ratio is larger the larger is the growth rate of the investment. There is proportionality between $K(t)$, $D(t)$, $\dot{K}(t)$, and $J(t)$. If $\delta$ and $\alpha$ are positive, the net investment/retirement ratio equals $\alpha/\delta$. The capital/gross investment ratio and the retirement/gross investment ratio are smaller the larger is the past growth rate of the investment. If investment is declining, with $\delta > -\alpha$, retirement will exceed gross investment, *i.e.*, net investment will be negative. The two age distributions coincide in this case,

$$k(t,s) = d(t,s) = (\delta + \alpha)e^{-(\delta+\alpha)s}, \tag{37}$$

*i.e.*, they follow an exponential decay retirement function with the parameter $\delta$ replaced by $(\delta + \alpha)$ .

Under *constant investment* ($\alpha = 0$), we have in particular

$$K(t) = \frac{J(t)}{\delta} = \frac{D(t)}{\delta}$$
$$k(t,s) = d(t,s) = b(s) = \delta e^{-\delta s}.$$

Then the age distribution of capital and retirement both coincide with the retirement function.

**Convex survival function**

In this example, the survival function is two-parametric and convex. One parameter, $N$, is the maximal life-time, the other, $\tau$, characterizes the curvature of the survival function. It is given by[8]

$$B(s) = \left(1 - \frac{s}{N}\right)^{\tau}, \qquad N > 0,\ \tau \geq 1,\ 0 \leq s \leq N, \tag{38}$$

$$b(s) = \frac{\tau}{N}\left(1 - \frac{s}{N}\right)^{\tau - 1}, \tag{39}$$

$$\beta(s) = \frac{\tau}{N - s}, \tag{40}$$

$$\Phi = \frac{N}{\tau + 1}. \tag{41}$$

The retirement function and the survival function belong to the same class of functions, as was also the case for exponential decay. The hazard rate is a monotonically increasing function of the capital's age, increasing in $\tau$, and goes to infinity at the maximal age $N$. The total per unit service flow is proportional to the maximal life-time and monotonically *decreasing* in $\tau$. The survival function is strictly convex for $\tau > 1$ and linear for $\tau = 1$.

Inserting (38) and (39) into (15) – (17), we find the following expressions for the age distributions of capital and retirement and the (time dependent) overall retirement rate for an arbitrary growth path of investment, $\gamma(t, s)$:

$$k(t,s) = \frac{\left(1 - \frac{s}{N}\right)^{\tau} \gamma(t,s)}{\int_0^N \left(1 - \frac{s}{N}\right)^{\tau} \gamma(t,s) ds}, \tag{42}$$

$$d(t,s) = \frac{\left(1 - \frac{s}{N}\right)^{\tau-1} \gamma(t,s)}{\int_0^N \left(1 - \frac{s}{N}\right)^{\tau-1} \gamma(t,s) ds}, \tag{43}$$

$$\delta(t) = \frac{\tau}{N} \frac{\int_0^N \left(1 - \frac{s}{N}\right)^{\tau-1} \gamma(t,s) ds}{\int_0^N \left(1 - \frac{s}{N}\right)^{\tau} \gamma(t,s) ds}. \tag{44}$$

We again assume exponentially growing investment, (32). Since, from (A.1) and (A.5), by substituting $\theta = s/N$ and $i = \tau$,

$$\int_0^N \left(1 - \frac{s}{N}\right)^{\tau} e^{-\alpha s} ds = NF(\tau, -\alpha N) = \frac{N}{\tau+1}[1 - \alpha N F(\tau+1, -\alpha N)],$$

[8] This survival function is also considered in Biørn (1989, section 11.2.2).

we can write the relationship between capital, gross investment, and retirement at time $t$ as

(45) $$K(t) = J(t)NF(\tau, -\alpha N) = J(t)\frac{N}{\tau+1}[1 - \alpha NF(\tau+1, -\alpha N)],$$

(46) $$D(t) = J(t)\tau F(\tau-1, -\alpha N) = J(t)[1 - \alpha NF(\tau, -\alpha N)].$$

From (32), (42), and (43) it then follows that

(47) $$k(t,s) = \frac{e^{-\alpha s}}{NF(\tau, -\alpha N)}\left(1 - \frac{s}{N}\right)^{\tau},$$

(48) $$d(t,s) = \frac{e^{-\alpha s}}{NF(\tau-1, -\alpha N)}\left(1 - \frac{s}{N}\right)^{\tau-1}.$$

From (45) and (46) we find that *the overall retirement rate is time invariant* and can be written as the following function of $N$, $\tau$, and $\alpha$:

(49) $$\delta(t) = \delta = \frac{\tau}{N}\frac{F(\tau-1, -\alpha N)}{F(\tau, -\alpha N)} = \frac{\tau+1}{N}\frac{1 - \alpha NF(\tau, -\alpha N)}{1 - \alpha NF(\tau+1, -\alpha N)},$$

where we in the second equality have used (A.5). If we approximate $F(\tau, -\alpha N)$ and $F(\tau+1, -\alpha N)$ by the *first* term in their Taylor expansions, which are $1/(\tau+1)$ and $1/(\tau+2)$, respectively [cf. (A.6)], we find the following approximation formula for the retirement rate, provided that $\alpha N$ is not too large:

(50) $$\delta \approx \frac{\tau+1}{N}\frac{1 - \frac{\alpha N}{\tau+1}}{1 - \frac{\alpha N}{\tau+2}} \approx \frac{\tau+1}{N}\left[1 - \left(\frac{1}{\tau+1} - \frac{1}{\tau+2}\right)\alpha N\right] = \frac{\tau+1}{N} - \frac{\alpha}{\tau+2}.$$

Under *constant investment* ($\alpha = 0$), this survival function implies

$$K(t) = J(t)\frac{N}{\tau+1},$$
$$k(t,s) = \frac{\tau+1}{N}\left(1 - \frac{s}{N}\right)^{\tau},$$
$$d(t,s) = \frac{\tau}{N}\left(1 - \frac{s}{N}\right)^{\tau-1},$$
$$\delta = \frac{\tau+1}{N}.$$

Comparing the last expression with (49) and (50), we see that we can interpret the retirement rate when gross investment grows at the rate $\alpha$ as obtained by multipliying its value under constant investment by a correction factor which is one for $\alpha = 0$ and in general depends on $\alpha N$. Approximately, when $\alpha$ is not too large, this correction simply implies that we should deduct a share $1/(\tau + 2)$ of the growth rate of investment. This share declines from $1/3$ to zero as $\tau$ increases from 1 to infinity. Numerical illustrations are given in Section 7.

**Concave survival function**

In this example, the survival function is two-parametric and concave. Again, one parameter, $N$, is the maximal life-time, the other, $\sigma$, characterizes the curvature of the survival function. It is given by[9]

$$B(s) = 1 - \left(\frac{s}{N}\right)^{\sigma}, \qquad N > 0,\ \sigma \geq 1,\ 0 \leq s \leq N, \tag{51}$$

$$b(s) = \frac{\sigma}{N}\left(\frac{s}{N}\right)^{\sigma-1}, \tag{52}$$

$$\beta(s) = \frac{\frac{\sigma}{N}\left(\frac{s}{N}\right)^{\sigma-1}}{1-\left(\frac{s}{N}\right)^{\sigma}} = \frac{\sigma s^{\sigma-1}}{N^{\sigma}-s^{\sigma}}, \tag{53}$$

$$\Phi = \frac{N\sigma}{\sigma+1}. \tag{54}$$

In this case, the retirement function and the survival function do *not* belong to the same class of functions. The hazard rate is a monotonically increasing function of the capital's age and goes to infinity at the maximal age $N$. The total per unit service flow is proportional to the maximal life-time and monotonically *increasing* in $\sigma$. The survival function is strictly concave for $\sigma > 1$ and linear for $\sigma = 1$.

Inserting (51) and (52) into (15) – (17), we find the following expressions for the age distributions of capital and retirement and the (time dependent) overall retirement rate for an arbitrary growth path of investment, $\gamma(t,s)$:

$$k(t,s) = \frac{\left[1-\left(\frac{s}{N}\right)^{\sigma}\right]\gamma(t,s)}{\int_0^N \left[1-\left(\frac{s}{N}\right)^{\sigma}\right]\gamma(t,s)ds}, \tag{55}$$

$$d(t,s) = \frac{\left(\frac{s}{N}\right)^{\sigma-1}\gamma(t,s)}{\int_0^N \left(\frac{s}{N}\right)^{\sigma-1}\gamma(t,s)ds}, \tag{56}$$

$$\delta(t) = \frac{\sigma}{N}\frac{\int_0^N \left(\frac{s}{N}\right)^{\sigma-1}\gamma(t,s)ds}{\int_0^N \left[1-\left(\frac{s}{N}\right)^{\sigma}\right]\gamma(t,s)ds}. \tag{57}$$

We again assume exponentially growing investment, (32). Since, from (A.1) and (A.5), by substituting $\lambda = s/N$ and $i = \sigma$,

$$\int_0^N \left(\frac{s}{N}\right)^{\sigma} e^{-\alpha s}ds = Ne^{-\alpha N}F(\sigma,\alpha N) = \frac{N}{\sigma+1}e^{-\alpha N}[1+\alpha N F(\sigma+1,\alpha N)],$$

[9] This survival function is also considered in Biørn (1989, section 11.2.1).

so that

$$\int_0^N \left[1 - \left(\frac{s}{N}\right)^\sigma\right] e^{-\alpha s} ds = N e^{-\alpha N}[F(0, \alpha N) - F(\sigma, \alpha N)],$$

we can write the relationship between capital, gross investment, and retirement at time $t$ as

$$K(t) = J(t) N e^{-\alpha N}[F(0, \alpha N) - F(\sigma, \alpha N)], \tag{58}$$

$$D(t) = J(t)\sigma e^{-\alpha N} F(\sigma - 1, \alpha N) = J(t) e^{-\alpha N}[1 + \alpha N F(\sigma, \alpha N)]. \tag{59}$$

From (32), (55), and (56) it follows that

$$k(t, s) = \frac{e^{-\alpha s}}{N e^{-\alpha N}[F(0, \alpha N) - F(\sigma, \alpha N)]} \left[1 - \left(\frac{s}{N}\right)^\sigma\right], \tag{60}$$

$$d(t, s) = \frac{e^{-\alpha s}}{N e^{-\alpha N} F(\sigma - 1, \alpha N)} \left(\frac{s}{N}\right)^{\sigma - 1}. \tag{61}$$

From (58) and (59) we find that *the overall retirement rate is time invariant* and can be written as the following function of $N$, $\sigma$, and $\alpha$:

$$\delta(t) = \delta = \frac{\sigma}{N} \frac{F(\sigma - 1, \alpha N)}{F(0, \alpha N) - F(\sigma, \alpha N)}. \tag{62}$$

If we approximate $F(\sigma - 1, \alpha N)$, $F(0, \alpha N)$ and $F(\sigma, \alpha N)$ by the *first and second* term in their Taylor expansions, which are $1/\sigma + \alpha N/(\sigma(\sigma+1))$, $1 + \alpha N/2$ and $1/(\sigma+1) + \alpha N/((\sigma+1)(\sigma+2))$, respectively, [cf. (A.6)] we find the following approximation formula for the retirement rate provided that $\alpha N$ is not too large:

$$\begin{aligned} \delta &\approx \frac{\sigma}{N} \frac{\frac{1}{\sigma} + \frac{\alpha N}{\sigma(\sigma+1)}}{\left(1 + \frac{\alpha N}{2}\right) - \left(\frac{1}{\sigma+1} + \frac{\alpha N}{(\sigma+1)(\sigma+2)}\right)} \\ &= \frac{1}{N} \frac{1 + \frac{\alpha N}{\sigma+1}}{\frac{\sigma}{\sigma+1} + \left(\frac{1}{2} - \frac{1}{(\sigma+1)(\sigma+2)}\right)\alpha N} = \frac{\sigma+1}{\sigma N} \frac{1 + \frac{\alpha N}{\sigma+1}}{1 + \frac{\sigma+3}{2(\sigma+2)}\alpha N}. \\ &\approx \frac{\sigma+1}{\sigma N}\left[1 - \left(\frac{\sigma+3}{2(\sigma+2)} - \frac{1}{\sigma+1}\right)\alpha N\right] = \frac{\sigma+1}{\sigma N} - \left(\frac{(\sigma+1)(\sigma+3)}{2\sigma(\sigma+2)} - \frac{1}{\sigma}\right)\alpha. \end{aligned} \tag{63}$$

Under *constant investment* ($\alpha = 0$), when we use (A.2), this survival function implies

$$\begin{aligned} K(t) &= J(t)\frac{N\sigma}{\sigma+1}, \\ k(t, s) &= \frac{\sigma+1}{N\sigma}\left[1 - \left(\frac{s}{N}\right)^\sigma\right], \\ d(t, s) &= \frac{\sigma}{N}\left(\frac{s}{N}\right)^{\sigma-1}, \\ \delta &= \frac{\sigma+1}{N\sigma}. \end{aligned}$$

Comparing the last expression with (62) and using (A.2), we see that we can interpret the retirement rate when gross investment grows at the rate $\alpha$ as obtained by multiplying its value under constant investment by a correction factor which is one for $\alpha = 0$ and in general depends on $\alpha N$. Approximately, when $\alpha$ is not too large, this correction implies that we should deduct a share of the growth rate of investment which increases from 1/3 to 1/2 as $\sigma$ increases from 1 to infinity. Numerical illustrations are given in Section 7.

## 6 Linear, simultaneous, and immediate retirement

Let us briefly consider retirement processes characterized by linear retirement, simultaneous retirement at a finite positive age, and immediate retirement after the first service period. They are special cases of the second and third examples in Section 5. We still assume a constant rate of investment growth.

**Linear retirement**

Linear retirement, in which a constant proportion, $1/N$, of the initial investment vanishes in each period is the special case of both the convex survival function (38) with $\tau = 1$ and of the concave survival function (51) with $\sigma = 1$, and we have

$$B(s) = 1 - \frac{s}{N},$$

$$b(s) = \frac{1}{N}.$$

It follows from (45) and (46) with $\tau = 1$ that capital and retirement become

$$K(t) = J(t)NF(1, -\alpha N) = J(t)\frac{1}{\alpha}\left[1 - \frac{1}{\alpha N}(1 - e^{-\alpha N})\right],$$

$$D(t) = J(t)F(0, -\alpha N) = J(t)\frac{1}{\alpha N}(1 - e^{-\alpha N}).$$

The overall retirement rate, obtained from (49), is

$$\delta = \frac{1}{N}\frac{F(0, -\alpha N)}{F(1, -\alpha N)} = \frac{1}{N}\frac{1 - e^{-\alpha N}}{1 - \frac{1}{\alpha N}(1 - e^{-\alpha N})}.$$

**Simultaneous retirement**

Simultaneous retirement (often called "sudden death" or "one-horse shay"), in which all capital invested vanishes completely at age $N$, can be considered the special case of the concave survival function (51) with $\sigma \to \infty$,

$$B(s) = 1 - \lim_{\sigma \to \infty}\left(\frac{s}{N}\right)^{\sigma} = \begin{cases} 1, & s < N, \\ 0, & s = N. \end{cases}$$

It follows from (58) and (59), by inserting $\sigma \to \infty$ and using (A.3) and (A.4), that capital and retirement become

$$K(t) = J(t)Ne^{-\alpha N}F(0, \alpha N) = J(t)NF(0, -\alpha N),$$
$$D(t) = J(t-N) = J(t)e^{-\alpha N}.$$

The overall retirement rate, obtained from (62), is

$$\delta = \frac{1}{NF(0, \alpha N)} = \frac{e^{-\alpha N}}{NF(0, -\alpha N)}.$$

**Immediate retirement**

This case, in which all capital goods vanish immediately after investment, may be considered the special case of the convex survival function (38) with $\tau \to \infty$, so that

$$B(s) = \lim_{\tau \to \infty} \left(1 - \frac{s}{N}\right)^{\tau} = \begin{cases} 1, & s = 0, \\ 0, & s > 0. \end{cases}$$

Since $K(t)$ is a stock concept and $J(t)$ and $D(t)$ are flow concepts, this specification, however, is not convenient. Instead, we consider immediate retirement as the special case of simultaneous retirement where the life-time is $N = 1$. We then get

$$K(t) = J(t)e^{-\alpha}F(0, \alpha) = J(t)F(0, -\alpha),$$
$$D(t) = J(t)e^{-\alpha},$$

and the overall retirement rate becomes

$$\delta = \frac{1}{F(0, \alpha)} = \frac{e^{-\alpha}}{F(0, -\alpha)}.$$

If we let $\alpha$ go to zero – which has the interpretation that investment is constant over the age interval $(0, 1)$ – we simply get $K(t) = D(t) = J(t)$ and $\delta = 1$, which agree with our intuitive understanding of immediate retirement when the life-time is arbitrarily short.

# 7 Numerical illustrations

In this section, we present numerical illustrations, relating to the two-parametric convex and concave survival functions described in Section 5, under exponential investment growth. Using a demographic analogue, what we describe here are characteristics of a stable population of capital assets for different parameter values. Six values of the maximal life-time $N$ are considered, 6, 10, 20, 30, 50, and 100 years, and six values of the curvature parameters $\tau$ and $\sigma$ are specified for each class of functions. The six panels of the tables below, a – f, represent the six values of the curvature parameters. Seven values of the (continuous) growth rate of investment, $\alpha$, are considered, ranging from -5 to +15 per cent. Tables 1 – 3 illustrate the convex class, Tables 4 – 6 illustrate the concave class of survival functions.[10]

Tables 1 and 4 contain the retirement rates (in per cent). Tables 2, 3, 5, and 6 describe properties of the implied age distributions. Tables 2 and 5 give the capital/investment ratio ($K/J$) – which is analogous to the inverse birth rate of capital – and Tables 3 and 6 give the retirement/investment ratio ($D/J$) – which is analogous to the death to birth ratio.

**Retirement rates**

From the columns of Tables 1 and 4 we see that the retirement rate under exponential investment growth shows considerable *sensitivity to the growth rate*. The relative sensitivity is larger the higher is the maximal life-time. In the concave class with the curvature parameter $\sigma = 2$ and the maximal life time $N = 6$, for example, the retirement rate declines from $\delta = 27.3$ per cent for $\alpha = -0.05$ via $\delta = 25$ per cent for $\alpha = 0$ to $\delta = 19.2$ per cent for $\alpha = 0.15$. With $\sigma = 2, N = 50$, the retirement rate declines from $\delta = 5.97$ per cent for $\alpha = -0.05$ via $\delta = 3$ per cent for $\alpha = 0$ to $\delta = 0.55$ per cent for $\alpha = 0.15$. For each value of the maximal life time $N$, the sensitivity of $\delta$ is smaller the higher is the curvature parameter $\tau$ for the convex class of functions, and larger the higher is the curvature parameter $\sigma$ for the concave class.

The rows for $\alpha = 0$ in Tables 1 and 4, which represent the *constant investment case*, give the value $\delta = 1/\Phi = (\tau + 1)/N$ for the convex class of survival functions [cf. (41), (49), and (50)] and the value $\delta = (\sigma + 1)/(\sigma N)$ for the concave class [cf. (54), (62), and (63)], as can be easily checked. The 'double declining balance' hypothesis, corresponding to $\delta = 2/N$, is represented by the $\alpha = 0$ rows for $\tau = 1$ in Table 1 and for $\sigma = 1$ in Table 4 (linear retirement). The 'inverse life-time' hypothesis, corresponding to $\delta = 1/N$, is represented by the $\alpha = 0$ rows for $\tau = 0$[11] in Table 1 and (approximately) for $\sigma = 1000$ in Table 4 (simultaneous retirement)

The formulae (50) and (63) give very good approximations to the exact $\delta$ value for $\alpha \in (-0.01, +0.01)$, $N \leq 50$ and fairly good approximations for $\alpha \in (-0.05, -0.01)$,

[10] The function $F(i, z)$ is evaluated by means of (A.6) (by truncating the expansion after a suitable number of terms), and the calculations are performed by means of routines in the Gauss software code constructed by the author.

[11] Strictly, (38) is defined for $\tau \geq 1$ only. The case $\tau = 0$ gives, however, simultaneous retirement. even if (39) and (40) and expressions derived from them are undefined.

$N \leq 20$ and $\alpha \in (0.01, 0.05)$, $N \leq 20$. For $\alpha = 0.01, \tau = 1, N = 30$ in the *convex* class, the exact formula gives for example $\delta = 6.35$ per cent and the approximate formula $\delta = 6.33$ per cent. For $\alpha = 0.05, \tau = 1, N = 20$ the exact formula gives $\delta = 8.59$ per cent and the approximate formula $\delta = 8.33$ per cent. For $\alpha = -0.01, \sigma = 2, N = 30$ in the concave class, the exact formula gives $\delta = 5.46$ per cent and the approximate formula $\delta = 5.44$ per cent. For $\alpha = -0.05, \sigma = 2, N = 20$, the exact formula gives $\delta = 10.00$ per cent and the approximate formula $\delta = 9.68$ per cent.

**Capital/investment ratios**

By examining the columns of Tables 2 and 5, we see how changes in the investment growth rate, for each given maximal life-time and each value of the curvature parameter, affects an important property of the age distribution of the capital, the capital/investment ratio, or the inverse birth ratio of capital. As a benchmark case, we may consider the constant investment case, where $K/J = N/(\tau+1)$ for the convex class and $K/J = N\sigma/(\sigma+1)$ for the concave class. These constants also represent the total number of services per capital unit during its life time and the inverse retirement rates under constant investment. The latter can be confirmed by comparing the $\alpha = 0$ rows in Tables 2 and 5 with the corresponding rows in Tables 1 and 4.

Increasing $\alpha$ implies a growing and gradually younger stock of capital (decreasing $K/J$). A few examples illustrate this. In the convex class with $\tau = 2, N = 6$, the capital/investment ratio declines from $K/J = 2.16$ for $\alpha = -0.05$ via $K/J = 2.00$ for $\alpha = 0$ to $K/J = 1.62$ for $\alpha = 0.15$. With $\tau = 2, N = 50$, the ratio declines from $K/J = 35.6$ for $\alpha = -0.05$ via $K/J = 16.7$ for $\alpha = 0$ to $K/J = 5.1$ for $\alpha = 0.15$. In the concave class with $\sigma = 2, N = 6$, the capital/investment ratio declines from $K/J = 4.49$ for $\alpha = -0.05$ via $K/J = 4.00$ for $\alpha = 0$ to $K/J = 2.92$ for $\alpha = 0.15$. With $\sigma = 2, N = 50$, the ratio declines from $K/J = 103.4$ for $\alpha = -0.05$ via $K/J = 33.3$ for $\alpha = 0$ to $K/J = 6.4$ for $\alpha = 0.15$. For each $N$ and $\alpha$, increasing $\tau$ leads to a younger stock of capital for the convex class (decreasing $K/J$), and increasing $\sigma$ leads to an older stock of capital for the concave class (increasing $K/J$).

**Retirement/investment ratios**

By examining the columns of Tables 3 and 6 we see how changes in the investment growth rate, for each given maximal life-time and each value of the curvature parameter, affects another basic property of the age distribution of the capital, the retirement/investment ratio, or the death to birth ratio of the capital. As a benchmark case, we may consider the constant investment case, in which $D/J = 1$ for any survival functions.

Increasing $\alpha$ leads to a decline in the death to birth ratio. This is also an indication of a gradually younger capital stock. A few examples illustrate this. In the convex class with $\tau = 2, N = 6$, the retirement/investment ratio declines from $D/J = 1.11$ for $\alpha = -0.05$ to $D/J = 0.75$ for $\alpha = 0.15$, whereas with $\tau = 2, N = 50$, the ratio declines from $D/J = 2.78$ for $\alpha = -0.05$ to $D/J = 0.23$ for $\alpha = 0.15$. In the concave class with $\sigma = 2, N = 6$, the retirement/investment ratio declines from $D/J = 1.22$ for $\alpha = -0.05$ to $D/J = 0.56$ for $\alpha = 0.15$, whereas with $\sigma = 2, N = 50$, the ratio declines from $D/J = 6.17$ for $\alpha = -0.05$ to $D/J = 0.04$ for $\alpha = 0.15$. For each $N$ and $\alpha$, increasing $\tau$ leads to a decreasing/increasing $D/J$ ratio for the convex class if $\alpha$ is negative/positive, and increasing $\sigma$ leads to increasing/decreasing $D/J$ ratio for the concave class if $\alpha$ is negative/positive.

## 8 Concluding remarks

In this paper, we have investigated the relationship between the retirement process of the capital, formalized by its survival function, and the average retirement rate and how this relationship is affected by changes in the investment path. As is well known, the average retirement rate will be a constant independent of the investment path and equal for all capital vintages only in the particular case where the survival function is exponentially declining (exponential decay). This is often regarded as a benchmark case in the literature. Otherwise, the average retirement rate will depend on both the growth profile of gross investment and the parameters describing the survival function. An interesting question for practical purposes is: How sensitive is it? Can constant average retirement rate be defended as a useful practical simplification?

Our numerical illustrations of how the average retirement rate responds to changes in the curvature of survival functions and to the growth pattern of investment under non-exponential survival functions show considerable sensitivity to both. The survival functions we have considered are two-parametric and characterized by the maximal life-time and a curvature parameter indicating the degree of convexity or concavity. We find that the retirement rate is a declining function of the growth rate of investment and quite sensitive to its value over a reasonable interval. Furthermore, we have illustrated how two basic properties of the age distribution of the capital, the capital/investment ratio and the retirement/investment ratio, are affected by changes in the growth rate of investment and in the form of the survival function.

Simultaneous retirement (sudden death) at a fixed age $N$, and linear retirement up to a fixed life-time, $N$ are often applied in practical research or in national accounting. Which is the appropriate 'translation' of the retirement rate, $\delta$, in these cases? We have shown, as particular cases of our parametric functions, that the ‘translation’ rules $\delta = 1/N$ for the former and $\delta = 2/N$ for the latter hold under constant investment, but may be very inaccurate otherwise.

Some limitations of the paper should be mentioned: First, an integration of our analysis into a model of the firm growth process is beyond our scope. Second, although our formulae in the first part of the paper can be used to represent cases with cyclical variation in the retirement rates and in the age distribution of the capital following from cyclical variations in investment, we have not illustrated such situations numerically in the last part. Third, due to lack of reliable empirical information on survival functions, we have not presented econometric evidence on the parameters and relationships under discussion.

Table 1:

**Table 1. Retirement rate, per cent.**
**Convex survival functions.**

Curvature Parameter = $\tau$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*a.* $\tau = 0$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 19.29148 | 12.70747 | 7.90988 | 6.43608 | 5.44713 | 5.03392 |
| -1 | 17.17167 | 10.50833 | 5.51666 | 3.85830 | 2.54149 | 1.58198 |
| 0 | 16.66667 | 10.00000 | 5.00000 | 3.33333 | 2.00000 | 1.00000 |
| 1 | 16.17167 | 9.50833 | 4.51666 | 2.85830 | 1.54149 | 0.58198 |
| 5 | 14.29148 | 7.70747 | 2.90988 | 1.43608 | 0.44713 | 0.03392 |
| 10 | 12.16369 | 5.81977 | 1.56518 | 0.52396 | 0.06784 | 0.00045 |
| 15 | 10.27677 | 4.30825 | 0.78594 | 0.16851 | 0.00830 | 0.00000 |

*b.* $\tau = 1$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 35.08496 | 21.80997 | 11.96106 | 8.78465 | 6.43968 | 5.17555 |
| -1 | 33.67001 | 20.33893 | 10.34459 | 7.01699 | 4.36199 | 2.39221 |
| 0 | 33.33333 | 20.00000 | 10.00000 | 6.66667 | 4.00000 | 2.00000 |
| 1 | 33.00332 | 19.67218 | 9.67763 | 6.34966 | 3.69348 | 1.71828 |
| 5 | 31.74829 | 18.46742 | 8.59141 | 5.37158 | 2.90097 | 1.23949 |
| 10 | 30.31943 | 17.18282 | 7.61594 | 4.63567 | 2.47898 | 1.11106 |
| 15 | 29.03567 | 16.11473 | 6.95350 | 4.22470 | 2.30622 | 1.07143 |

*c.* $\tau = 2$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 51.30766 | 31.34766 | 16.45308 | 11.56598 | 7.81152 | 5.48109 |
| -1 | 50.25226 | 30.25378 | 15.25763 | 10.26153 | 6.26953 | 3.29062 |
| 0 | 50.00000 | 30.00000 | 15.00000 | 10.00000 | 6.00000 | 3.00000 |
| 1 | 49.75224 | 29.75372 | 14.75738 | 9.76097 | 5.76797 | 2.78442 |
| 5 | 48.80485 | 28.83987 | 13.92211 | 8.99707 | 5.12694 | 2.35877 |
| 10 | 47.71383 | 27.84422 | 13.13035 | 8.36575 | 4.71755 | 2.19514 |
| 15 | 46.71877 | 26.99121 | 12.54863 | 7.96303 | 4.50917 | 2.13198 |

Table 1:

TABLE 1 (CONT.). RETIREMENT RATE, PER CENT.
CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = $\tau$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*d.* $\tau = 3$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 67.70771 | 41.06957 | 21.14513 | 14.56018 | 9.40887 | 5.95496 |
| -1 | 66.86827 | 40.20269 | 20.20543 | 13.54154 | 8.21391 | 4.22903 |
| 0 | 66.66667 | 40.00000 | 20.00000 | 13.33333 | 8.00000 | 4.00000 |
| 1 | 66.46826 | 39.80264 | 19.80524 | 13.14113 | 7.81277 | 3.82447 |
| 5 | 65.70565 | 39.06386 | 19.12235 | 12.50917 | 7.26932 | 3.44133 |
| 10 | 64.81863 | 38.24470 | 18.44938 | 11.95350 | 6.88267 | 3.26259 |
| 15 | 63.99986 | 37.52750 | 17.93025 | 11.57137 | 6.66214 | 3.18427 |

*e.* $\tau = 4$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 84.19718 | 50.88505 | 25.94119 | 17.66868 | 11.13855 | 6.56805 |
| -1 | 83.50120 | 50.16867 | 25.17070 | 16.83944 | 10.17701 | 5.18824 |
| 0 | 83.33333 | 50.00000 | 25.00000 | 16.66667 | 10.00000 | 5.00000 |
| 1 | 83.16785 | 49.83530 | 24.83724 | 16.50581 | 9.84285 | 4.85160 |
| 5 | 82.52903 | 49.21425 | 24.25800 | 15.96488 | 9.36918 | 4.50315 |
| 10 | 81.77993 | 48.51601 | 23.67037 | 15.46761 | 9.00630 | 4.31844 |
| 15 | 81.08197 | 47.89465 | 23.20141 | 15.10799 | 8.78340 | 4.23004 |

*f.* $\tau = 5$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 100.73780 | 60.75411 | 30.79721 | 20.84382 | 12.94851 | 7.28450 |
| -1 | 100.14378 | 60.14440 | 30.14597 | 20.14756 | 12.15082 | 6.15944 |
| 0 | 100.00000 | 60.00000 | 30.00000 | 20.00000 | 12.00000 | 6.00000 |
| 1 | 99.85806 | 59.85866 | 29.86016 | 19.86163 | 11.86450 | 5.87130 |
| 5 | 99.30814 | 59.32250 | 29.35648 | 19.38789 | 11.44380 | 5.55167 |
| 10 | 98.65903 | 58.71297 | 28.83385 | 18.93737 | 11.10334 | 5.36570 |
| 15 | 98.04970 | 58.16366 | 28.40606 | 18.59937 | 10.88266 | 5.27054 |

Table 2:

Table 2. Capital/Investment ratio.

Convex survival functions.

Curvature Parameter = $\tau$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*a.* $\tau = 0$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 6.99718 | 12.97443 | 34.36564 | 69.63378 | 223.64988 | 2948.26318 |
| -1 | 6.18365 | 10.51709 | 22.14028 | 34.98588 | 64.87213 | 171.82818 |
| 0 | 6.00000 | 10.00000 | 20.00000 | 30.00000 | 50.00000 | 100.00000 |
| 1 | 5.82355 | 9.51626 | 18.12692 | 25.91818 | 39.34693 | 63.21206 |
| 5 | 5.18364 | 7.86939 | 12.64241 | 15.53740 | 18.35830 | 19.86524 |
| 10 | 4.51188 | 6.32121 | 8.64665 | 9.50213 | 9.93262 | 9.99955 |
| 15 | 3.95620 | 5.17913 | 6.33475 | 6.59261 | 6.66298 | 6.66666 |

*b.* $\tau = 1$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 3.32392 | 5.94885 | 14.36564 | 26.42252 | 69.45995 | 569.65264 |
| -1 | 3.06091 | 5.17092 | 10.70138 | 16.61960 | 29.74425 | 71.82818 |
| 0 | 3.00000 | 5.00000 | 10.00000 | 15.00000 | 25.00000 | 50.00000 |
| 1 | 2.94089 | 4.83742 | 9.36538 | 13.60607 | 21.30613 | 36.78794 |
| 5 | 2.72121 | 4.26123 | 7.35759 | 9.64174 | 12.65668 | 16.02695 |
| 10 | 2.48019 | 3.67879 | 5.67668 | 6.83262 | 8.01348 | 9.00005 |
| 15 | 2.27089 | 3.21391 | 4.55508 | 5.20164 | 5.77827 | 6.22222 |

*c.* $\tau = 2$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 2.15947 | 3.79540 | 8.73127 | 15.23003 | 35.56796 | 207.86105 |
| -1 | 2.03036 | 3.41836 | 7.01379 | 10.79735 | 18.97702 | 43.65637 |
| 0 | 2.00000 | 3.33333 | 6.66667 | 10.00000 | 16.66667 | 33.33333 |
| 1 | 1.97036 | 3.25164 | 6.34623 | 9.29284 | 14.77547 | 26.42411 |
| 5 | 1.85857 | 2.95509 | 5.28482 | 7.14435 | 9.87466 | 13.58922 |
| 10 | 1.73269 | 2.64241 | 4.32332 | 5.44492 | 6.79461 | 8.19999 |
| 15 | 1.62025 | 2.38145 | 3.62995 | 4.35483 | 5.12579 | 5.83704 |

Table 2:

Table 2 (cont.). Capital/Investment ratio.

Convex survival functions.

Curvature Parameter = $\tau$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*d.* $\tau = 3$

| $\alpha$ | $N = 6$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|---|---|
| -5 | 1.59470 | 2.77242 | 6.19382 | 10.46006 | 22.68155 | 104.71663 |
| -1 | 1.51818 | 2.55085 | 5.20686 | 7.97350 | 13.86210 | 30.96910 |
| 0 | 1.50000 | 2.50000 | 5.00000 | 7.50000 | 12.50000 | 25.00000 |
| 1 | 1.48218 | 2.45082 | 4.80648 | 7.07157 | 11.34717 | 20.72766 |
| 5 | 1.41431 | 2.26943 | 4.14553 | 5.71129 | 8.15041 | 11.84647 |
| 10 | 1.33657 | 2.07277 | 3.51501 | 4.55508 | 5.92323 | 7.54000 |
| 15 | 1.26582 | 1.90376 | 3.03672 | 3.76345 | 4.61635 | 5.49926 |

*e.* $\tau = 4$

| $\alpha$ | $N = 6$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|---|---|
| -5 | 1.26267 | 2.17936 | 4.77528 | 7.89348 | 6.29049 | 63.77331 |
| -1 | 1.21210 | 2.03382 | 4.13724 | 6.31336 | 10.89679 | 23.87639 |
| 0 | 1.20000 | 2.00000 | 4.00000 | 6.00000 | 10.00000 | 20.00000 |
| 1 | 1.18810 | 1.96714 | 3.87038 | 5.71239 | 9.22267 | 17.08934 |
| 5 | 1.14248 | 1.84453 | 3.41787 | 4.76988 | 6.95934 | 10.52283 |
| 10 | 1.08956 | 1.70893 | 2.96997 | 3.92656 | 5.26141 | 6.98400 |
| 15 | 1.04078 | 1.58996 | 2.61770 | 3.32138 | 4.20461 | 5.20020 |

*f.* $\tau = 5$

| $\alpha$ | $N = 6$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|---|---|
| -5 | 1.04452 | 1.79359 | 3.87639 | 6.31161 | 12.58097 | 43.77331 |
| -1 | 1.00864 | 1.69078 | 3.43101 | 5.22260 | 8.96795 | 19.38194 |
| 0 | 1.00000 | 1.66667 | 3.33333 | 5.00000 | 8.33333 | 16.66667 |
| 1 | 0.99149 | 1.64315 | 3.24042 | 4.79349 | 7.77333 | 14.55329 |
| 5 | 0.95870 | 1.55467 | 2.91066 | 4.10040 | 6.08132 | 9.47717 |
| 10 | 0.92031 | 1.45533 | 2.57507 | 3.45574 | 4.73859 | 6.50800 |
| 15 | 0.88457 | 1.36680 | 2.30383 | 2.97625 | 3.86359 | 4.93327 |

Table 3:

TABLE 3. RETIREMENT/INVESTMENT RATIO.
CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = $\tau$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*a.* $\tau = 0$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.34986 | 1.64872 | 2.71828 | 4.48169 | 12.18249 | 148.41316 |
| -1 | 1.06184 | 1.10517 | 1.22140 | 1.34986 | 1.64872 | 2.71828 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.94176 | 0.90484 | 0.81873 | 0.74082 | 0.60653 | 0.36788 |
| 5 | 0.74082 | 0.60653 | 0.36788 | 0.22313 | 0.08208 | 0.00674 |
| 10 | 0.54881 | 0.36788 | 0.13534 | 0.04979 | 0.50337 | 0.00005 |
| 15 | 0.40657 | 0.22313 | 0.04979 | 0.01111 | 0.00055 | 0.00000 |

*b.* $\tau = 1$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.16620 | 1.29744 | 1.71828 | 2.32113 | 4.47300 | 29.48263 |
| -1 | 1.03061 | 1.05171 | 1.10701 | 1.16620 | 1.29744 | 1.71828 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.97059 | 0.95163 | 0.90635 | 0.86394 | 0.78694 | 0.63212 |
| 5 | 0.86394 | 0.78694 | 0.63212 | 0.51791 | 0.36717 | 0.19865 |
| 10 | 0.75198 | 0.63212 | 0.43233 | 0.31674 | 0.59933 | 0.10000 |
| 15 | 0.65937 | 0.51791 | 0.31674 | 0.21975 | 0.13326 | 0.06667 |

*c.* $\tau = 2$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.10797 | 1.18977 | 1.43656 | 1.76150 | 2.77840 | 11.39305 |
| -1 | 1.02030 | 1.03418 | 1.07014 | 1.10797 | 1.18977 | 1.43656 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.98030 | 0.96748 | 0.93654 | 0.90707 | 0.85225 | 0.73576 |
| 5 | 0.90707 | 0.85225 | 0.73576 | 0.64278 | 0.50627 | 0.32054 |
| 10 | 0.82673 | 0.73576 | 0.56767 | 0.45551 | 0.66027 | 0.18000 |
| 15 | 0.75696 | 0.64278 | 0.45551 | 0.34678 | 0.23113 | 0.12444 |

Table 3:

TABLE 3 (CONT.). RETIREMENT/INVESTMENT RATIO.

CONVEX SURVIVAL FUNCTIONS.

Curvature Parameter = $\tau$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*d.* $\tau = 3$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.07974 | 1.13862 | 1.30969 | 1.52300 | 2.13408 | 6.23583 |
| -1 | 1.01518 | 1.02551 | 1.05207 | 1.07974 | 1.13862 | 1.30969 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.98518 | 0.97549 | 0.95194 | 0.92928 | 0.88653 | 0.79272 |
| 5 | 0.92928 | 0.88653 | 0.79272 | 0.71444 | 0.59248 | 0.40768 |
| 10 | 0.86634 | 0.79272 | 0.64850 | 0.54449 | 0.70384 | 0.24600 |
| 15 | 0.81013 | 0.71444 | 0.54449 | 0.43548 | 0.30755 | 0.17511 |

*e.* $\tau = 4$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.06313 | 1.10897 | 1.23876 | 1.39467 | 1.81452 | 4.18867 |
| -1 | 1.01212 | 1.02034 | 1.04137 | 1.06313 | 1.10897 | 1.23876 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.98812 | 0.98033 | 0.96130 | 0.94288 | 0.90777 | 0.82911 |
| 5 | 0.94288 | 0.90777 | 0.82911 | 0.76151 | 0.65203 | 0.47386 |
| 10 | 0.89104 | 0.82911 | 0.70300 | 0.60734 | 0.73693 | 0.30160 |
| 15 | 0.84388 | 0.76151 | 0.60734 | 0.50179 | 0.36931 | 0.21997 |

*f.* $\tau = 5$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.05223 | 1.08968 | 1.19382 | 1.31558 | 1.62905 | 3.18867 |
| -1 | 1.01009 | 1.01691 | 1.03431 | 1.05223 | 1.08968 | 1.19382 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.99009 | 0.98357 | 0.96760 | 0.95207 | 0.92227 | 0.85447 |
| 5 | 0.95207 | 0.92227 | 0.85447 | 0.79498 | 0.69593 | 0.52614 |
| 10 | 0.90797 | 0.85447 | 0.74249 | 0.65443 | 0.76307 | 0.34920 |
| 15 | 0.86731 | 0.79498 | 0.65443 | 0.55356 | 0.42046 | 0.26001 |

Table 4:

Table 4. Retirement rate, per cent.

Concave survival functions.

Curvature Parameter = $\sigma$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*a.* $\sigma = 1$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 35.08496 | 21.80997 | 11.96106 | 8.78465 | 6.43968 | 5.17555 |
| -1 | 33.67001 | 20.33893 | 10.34459 | 7.01699 | 4.36199 | 2.39221 |
| 0 | 33.33333 | 20.00000 | 10.00000 | 6.66667 | 4.00000 | 2.00000 |
| 1 | 33.00332 | 19.67218 | 9.67763 | 6.34966 | 3.69348 | 1.71828 |
| 5 | 31.74829 | 18.46742 | 8.59141 | 5.37158 | 2.90097 | 1.23949 |
| 10 | 30.31943 | 17.18282 | 7.61594 | 4.63567 | 2.47898 | 1.11106 |
| 15 | 29.03567 | 16.11473 | 6.95350 | 4.22470 | 2.30622 | 1.07143 |

*b.* $\sigma = 2$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 27.27980 | 17.34218 | 10.00000 | 7.65851 | 5.96757 | 5.10736 |
| -1 | 25.44116 | 15.44361 | 7.94977 | 5.45596 | 3.46844 | 2.00000 |
| 0 | 25.00000 | 15.00000 | 7.50000 | 5.00000 | 3.00000 | 1.50000 |
| 1 | 24.56615 | 14.56857 | 7.07460 | 4.58057 | 2.59237 | 1.12081 |
| 5 | 22.90287 | 12.96184 | 5.60405 | 3.23783 | 1.47723 | 0.41574 |
| 10 | 20.98180 | 11.20811 | 4.22469 | 2.16496 | 0.83149 | 0.20398 |
| 15 | 19.22876 | 9.71349 | 3.24744 | 1.53313 | 0.55030 | 0.13453 |

*c.* $\sigma = 3$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 24.65389 | 15.83099 | 9.32958 | 7.27079 | 5.80413 | 5.08403 |
| -1 | 22.69281 | 13.80653 | 7.14642 | 4.93078 | 3.16620 | 1.86592 |
| 0 | 22.22222 | 13.33333 | 6.66667 | 4.44444 | 2.66667 | 1.33333 |
| 1 | 21.75946 | 12.87317 | 6.21297 | 3.99719 | 2.23215 | 0.92979 |
| 5 | 19.98593 | 11.16076 | 4.64894 | 2.57378 | 1.06190 | 0.21930 |
| 10 | 17.94009 | 9.29788 | 3.20119 | 1.47026 | 0.43860 | 0.06019 |
| 15 | 16.07784 | 7.72135 | 2.20539 | 0.86319 | 0.21197 | 0.02671 |

Table 4:

TABLE 4 (CONT.). RETIREMENT RATE, PER CENT.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = $\sigma$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*d.* $\sigma = 4$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 23.33183 | 15.06714 | 8.98805 | 7.07225 | 5.72011 | 5.07212 |
| -1 | 21.31660 | 12.98600 | 6.74284 | 4.66637 | 3.01343 | 1.79761 |
| 0 | 20.83333 | 12.50000 | 6.25000 | 4.16667 | 2.50000 | 1.25000 |
| 1 | 20.35826 | 12.02764 | 5.78443 | 3.70785 | 2.05458 | 0.83728 |
| 5 | 18.53924 | 10.27291 | 4.18638 | 2.25894 | 0.87506 | 0.14521 |
| 10 | 16.44555 | 8.37275 | 2.72774 | 1.16724 | 0.29043 | 0.02381 |
| 15 | 14.54601 | 6.77681 | 1.75087 | 0.60052 | 0.10782 | 0.00711 |

*e.* $\sigma = 5$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 22.53442 | 14.60503 | 8.78020 | 6.95095 | 5.66861 | 5.06486 |
| -1 | 20.48994 | 12.49276 | 6.49982 | 4.50688 | 2.92101 | 1.75604 |
| 0 | 20.00000 | 12.00000 | 6.00000 | 4.00000 | 2.40000 | 1.20000 |
| 1 | 19.51851 | 11.52132 | 5.52833 | 3.53532 | 1.94919 | 0.78311 |
| 5 | 17.67658 | 9.74596 | 3.91554 | 2.07760 | 0.77181 | 0.10978 |
| 10 | 15.56072 | 7.83108 | 2.46026 | 1.00386 | 0.21957 | 0.01166 |
| 15 | 13.64646 | 6.23279 | 1.50579 | 0.47074 | 0.06612 | 0.00237 |

*f.* $\sigma = 1000$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 19.30802 | 12.71726 | 7.91448 | 6.43886 | 5.44834 | 5.03409 |
| -1 | 17.18833 | 10.51832 | 5.52164 | 3.86160 | 2.54345 | 1.58290 |
| 0 | 16.68333 | 10.01000 | 5.00500 | 3.33667 | 2.00200 | 1.00100 |
| 1 | 16.18833 | 9.51832 | 4.52164 | 2.86160 | 1.54345 | 0.58290 |
| 5 | 14.30802 | 7.71727 | 2.91449 | 1.43886 | 0.44835 | 0.03409 |
| 10 | 12.17987 | 5.82898 | 1.56880 | 0.52561 | 0.06818 | 0.00046 |
| 15 | 10.29236 | 4.31658 | 0.78842 | 0.16928 | 0.00836 | 0.00000 |

Table 5:

Table 5. Capital/Investment ratio.

Concave survival functions.

Curvature Parameter = $\sigma$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*a.* $\sigma = 1$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 3.32392 | 5.94885 | 14.36564 | 26.42252 | 69.45995 | 569.65264 |
| -1 | 3.06091 | 5.17092 | 10.70138 | 16.61960 | 29.74425 | 71.82818 |
| 0 | 3.00000 | 5.00000 | 10.00000 | 15.00000 | 25.00000 | 50.00000 |
| 1 | 2.94089 | 4.83742 | 9.36538 | 13.60607 | 21.30613 | 36.78794 |
| 5 | 2.72121 | 4.26123 | 7.35759 | 9.64174 | 12.65668 | 16.02695 |
| 10 | 2.48019 | 3.67879 | 5.67668 | 6.83262 | 8.01348 | 9.00005 |
| 15 | 2.27089 | 3.21391 | 4.55508 | 5.20164 | 5.77827 | 6.22222 |

*b.* $\sigma = 2$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 4.48837 | 8.10230 | 20.00000 | 37.61501 | 103.35194 | 931.44422 |
| -1 | 4.09146 | 6.92347 | 14.38897 | 22.44185 | 40.51149 | 100.00000 |
| 0 | 4.00000 | 6.66667 | 13.33333 | 20.00000 | 33.33333 | 66.66667 |
| 1 | 3.91142 | 6.42320 | 12.38452 | 17.91930 | 27.83679 | 47.15178 |
| 5 | 3.58386 | 5.56736 | 9.43036 | 12.13912 | 15.43870 | 18.46468 |
| 10 | 3.22770 | 4.71518 | 7.03003 | 8.22033 | 9.23234 | 9.80010 |
| 15 | 2.92152 | 4.04637 | 5.48022 | 6.04846 | 6.43074 | 6.60741 |

*c.* $\sigma = 3$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 5.08805 | 9.23276 | 23.09691 | 44.03753 | 124.35752 | 1190.09138 |
| -1 | 4.60982 | 7.80852 | 16.26963 | 25.44026 | 46.16381 | 115.48455 |
| 0 | 4.50000 | 7.50000 | 15.00000 | 22.50000 | 37.50000 | 75.00000 |
| 1 | 4.39378 | 7.20816 | 13.86391 | 20.01126 | 30.93915 | 51.81916 |
| 5 | 4.00225 | 6.18783 | 10.36383 | 13.20344 | 16.49648 | 19.15967 |
| 10 | 3.57909 | 5.18192 | 7.57507 | 8.71820 | 9.57983 | 9.94017 |
| 15 | 3.21773 | 4.40115 | 5.81213 | 6.30390 | 6.57377 | 6.65482 |

Table 5:

TABLE 5 (CONT.). CAPITAL/INVESTMENT RATIO.
CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = $\sigma$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*d.* $\sigma = 4$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 5.45499 | 9.93330 | 25.07491 | 48.25666 | 138.86777 | 1386.53744 |
| -1 | 4.92208 | 8.34307 | 17.41298 | 27.27496 | 49.66652 | 125.37454 |
| 0 | 4.80000 | 8.00000 | 16.00000 | 24.00000 | 40.00000 | 80.00000 |
| 1 | 4.68203 | 7.67599 | 14.73964 | 21.24113 | 32.73769 | 54.42842 |
| 5 | 4.24823 | 6.54754 | 10.88568 | 13.77612 | 17.02110 | 19.43554 |
| 10 | 3.78135 | 5.44284 | 7.85685 | 8.95476 | 9.71777 | 9.97625 |
| 15 | 3.38455 | 4.59204 | 5.96984 | 6.41004 | 6.61909 | 6.66351 |

*e.* $\sigma = 5$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 5.70307 | 10.41121 | 26.45365 | 51.25708 | 149.56422 | 1541.72574 |
| -1 | 5.13085 | 8.70113 | 18.18241 | 28.51535 | 52.05607 | 132.26823 |
| 0 | 5.00000 | 8.33333 | 16.66667 | 25.00000 | 41.66667 | 83.33333 |
| 1 | 4.87365 | 7.98638 | 15.31785 | 22.04918 | 33.90760 | 56.08184 |
| 5 | 4.40984 | 6.78152 | 11.21637 | 14.12909 | 17.32559 | 19.57030 |
| 10 | 3.91225 | 5.60818 | 8.02551 | 9.08772 | 9.78515 | 9.98835 |
| 15 | 3.49083 | 4.70970 | 6.05848 | 6.46382 | 6.63741 | 6.66561 |

*f.* $\sigma = 1000$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 6.98909 | 12.95796 | 34.31138 | 69.49967 | 223.04288 | 2933.51031 |
| -1 | 6.17729 | 10.50605 | 22.11588 | 34.94544 | 64.78981 | 171.55690 |
| 0 | 5.99401 | 9.99001 | 19.98002 | 29.97003 | 49.95005 | 99.90010 |
| 1 | 5.81790 | 9.50722 | 18.11056 | 25.89597 | 39.31662 | 63.17527 |
| 5 | 5.17919 | 7.86332 | 12.63505 | 15.53070 | 18.35419 | 19.86456 |
| 10 | 4.50859 | 6.31753 | 8.64394 | 9.50063 | 9.93228 | 9.99954 |
| 15 | 3.95376 | 5.17690 | 6.33376 | 6.59227 | 6.66295 | 6.66666 |

Table 6:

TABLE 6. RETIREMENT/INVESTMENT RATIO.

CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = $\sigma$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*a.* $\sigma = 1$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.16620 | 1.29744 | 1.71828 | 2.32113 | 4.47300 | 29.48263 |
| -1 | 1.03061 | 1.05171 | 1.10701 | 1.16620 | 1.29744 | 1.71828 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.97059 | 0.95163 | 0.90635 | 0.86394 | 0.78694 | 0.63212 |
| 5 | 0.86394 | 0.78694 | 0.63212 | 0.51791 | 0.36717 | 0.19865 |
| 10 | 0.75198 | 0.63212 | 0.43233 | 0.31674 | 0.19865 | 0.10000 |
| 15 | 0.65937 | 0.51791 | 0.31674 | 0.21975 | 0.13326 | 0.06667 |

*b.* $\sigma = 2$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.22442 | 1.40511 | 2.00000 | 2.88075 | 6.16760 | 47.57221 |
| -1 | 1.04091 | 1.06923 | 1.14389 | 1.22442 | 1.40511 | 2.00000 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.96089 | 0.93577 | 0.87615 | 0.82081 | 0.72163 | 0.52848 |
| 5 | 0.82081 | 0.72163 | 0.52848 | 0.39304 | 0.22806 | 0.07677 |
| 10 | 0.67723 | 0.52848 | 0.29700 | 0.17797 | 0.07677 | 0.01999 |
| 15 | 0.56177 | 0.39304 | 0.17797 | 0.09273 | 0.03539 | 0.00889 |

*c.* $\sigma = 3$

| $\alpha$ | $N=6$ | $N=10$ | $N=20$ | $N=30$ | $N=50$ | $N=100$ |
|---|---|---|---|---|---|---|
| -5 | 1.25440 | 1.46164 | 2.15485 | 3.20188 | 7.21788 | 60.50457 |
| -1 | 1.04610 | 1.07809 | 1.16270 | 1.25440 | 1.46164 | 2.15485 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.95606 | 0.92792 | 0.86136 | 0.79989 | 0.69061 | 0.48181 |
| 5 | 0.79989 | 0.69061 | 0.48181 | 0.33983 | 0.17518 | 0.04202 |
| 10 | 0.64209 | 0.48181 | 0.24249 | 0.12818 | 0.04202 | 0.00598 |
| 15 | 0.51734 | 0.33983 | 0.12818 | 0.05441 | 0.01393 | 0.00178 |

Table 6:

TABLE 6 (CONT.). RETIREMENT/INVESTMENT RATIO.
CONCAVE SURVIVAL FUNCTIONS.

Curvature Parameter = $\sigma$.
Maximal life time = $N$.
Rate of investment growth, per cent = $\alpha$

*d.* $\sigma = 4$

| $\alpha$ | $N = 6$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|---|---|
| -5 | 1.27275 | 1.49667 | 2.25375 | 3.41283 | 7.94339 | 70.32687 |
| -1 | 1.04922 | 1.08343 | 1.17413 | 1.27275 | 1.49667 | 2.25375 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.95318 | 0.92324 | 0.85260 | 0.78759 | 0.67262 | 0.45572 |
| 5 | 0.78759 | 0.67262 | 0.45572 | 0.31119 | 0.14895 | 0.02822 |
| 10 | 0.62186 | 0.45572 | 0.21431 | 0.10452 | 0.02822 | 0.00238 |
| 15 | 0.49232 | 0.31119 | 0.10452 | 0.03849 | 0.00714 | 0.00047 |

*e.* $\sigma = 5$

| $\alpha$ | $N = 6$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|---|---|
| -5 | 1.28515 | 1.52056 | 2.32268 | 3.56285 | 8.47821 | 78.08629 |
| -1 | 1.05131 | 1.08701 | 1.18182 | 1.28515 | 1.52056 | 2.32268 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.95126 | 0.92014 | 0.84682 | 0.77951 | 0.66092 | 0.43918 |
| 5 | 0.77951 | 0.66092 | 0.43918 | 0.29355 | 0.13372 | 0.02149 |
| 10 | 0.60877 | 0.43918 | 0.19745 | 0.09123 | 0.02149 | 0.00116 |
| 15 | 0.47638 | 0.29355 | 0.09123 | 0.03043 | 0.00439 | 0.00016 |

*f.* $\sigma = 1000$

| $\alpha$ | $N = 6$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 50$ | $N = 100$ |
|---|---|---|---|---|---|---|
| -5 | 1.34945 | 1.64790 | 2.71557 | 4.47498 | 12.15214 | 147.67552 |
| -1 | 1.06177 | 1.10506 | 1.22116 | 1.34945 | 1.64790 | 2.71557 |
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 0.94182 | 0.90493 | 0.81889 | 0.74104 | 0.60683 | 0.36825 |
| 5 | 0.74104 | 0.60683 | 0.36825 | 0.22347 | 0.08229 | 0.00677 |
| 10 | 0.54914 | 0.36825 | 0.13561 | 0.04994 | 0.00677 | 0.00005 |
| 15 | 0.40694 | 0.22347 | 0.04994 | 0.01116 | 0.00056 | 0.00000 |

# Appendix: Properties of the function (25)

In this appendix, we prove some useful properties of the auxiliary function (25), defined by the intergral

$$\text{(A.1)}\quad F(i,z) = \int_0^1 e^{z\theta}(1-\theta)^i d\theta = \int_0^1 e^{z(1-\lambda)}\lambda^i d\lambda = e^z \int_0^1 e^{-z\lambda}\lambda^i d\lambda, \quad i = 0,1,2,\dots .$$

It is straightforward to show that

$$\text{(A.2)}\qquad F(i,0) = \frac{1}{i+1},$$

$$\text{(A.3)}\qquad \lim_{i\to\infty} F(i,z) = 0 \;\text{ for all } z,$$

$$\text{(A.4)}\qquad \lim_{i\to\infty} (i+1)F(i,z) = 1 \;\text{ for all } z.$$

If $z \neq 0$, the function satisfies the recursion

$$\text{(A.5)}\qquad \begin{cases} F(0,z) = \frac{1}{z}(e^z - 1), \\ F(i,z) = \frac{1}{z}[iF(i-1,z) - 1], \qquad i = 1,2,\dots , \end{cases}$$

*i.e.*,

$$F(0,-z) = e^{-z}F(0,z) = \tfrac{1}{z}(1 - e^{-z}),$$
$$F(i,-z) = \tfrac{1}{z}[1 - iF(i-1,-z)], \qquad i = 1,2,\dots ,$$

which can be shown by integration by parts. For $i = 1$ and $i = 2$, we have in particular (when $z \neq 0$)

$$F(1,z) = \tfrac{1}{z}\left[\tfrac{1}{z}(e^z - 1) - 1\right],$$
$$F(2,z) = \tfrac{1}{z}\left[\tfrac{2}{z}\left(\tfrac{1}{z}(e^z - 1) - 1\right) - 1\right],$$

*i.e.*,

$$F(1,-z) = \tfrac{1}{z}\left[1 - \tfrac{1}{z}(1 - e^{-z})\right],$$
$$F(2,-z) = \tfrac{1}{z}\left[1 - \tfrac{2}{z}\left(1 - \tfrac{1}{z}(1 - e^{-z})\right)\right].$$

If $z$ is small and $i$ is large, the above recursion may give inaccurate results in numerical calculations. A better way of keeping the accuracy under control is to represent $F(i,z)$ by its Taylor expansion. Expanding $e^z$ by Taylor's formula, we get, for $i = 0$,

$$F(0,z) = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1\right) = 1 + \frac{z}{1\cdot 2} + \frac{z^2}{1\cdot 2\cdot 3} + \frac{z^3}{1\cdot 2\cdot 3\cdot 4} + \cdots .$$

Substituting this expression in (A.5), it follows that

$$F(1,z) = \frac{1}{z}[F(0,z) - 1] = \frac{1}{2} + \frac{z}{2\cdot 3} + \frac{z^2}{2\cdot 3\cdot 4} + \frac{z^3}{2\cdot 3\cdot 4\cdot 5} + \cdots ,$$

$$F(2,z) = \frac{1}{z}[2F(0,z) - 1] = \frac{1}{3} + \frac{z}{3\cdot 4} + \frac{z^2}{3\cdot 4\cdot 5} + \frac{z^3}{3\cdot 4\cdot 5\cdot 6} + \cdots ,$$

and in general,

$$F(i,z) = \frac{1}{i+1} + \frac{z}{(i+1)(i+2)} + \frac{z^2}{(i+1)(i+2)(i+3)} + \cdots = \sum_{j=1}^{\infty} \frac{i!\, z^{j-1}}{(i+j)!}, \quad i = 0, 1, 2, \dots\ , \tag{A.6}$$

which can be verified by induction.

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