A NOTE ON FINITE DIFFERENCES AND ADDITIVE SEPARABILITY

Aanund Hylland

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1. Introduction

In this note, we prove that certain conditions concerning finite differences of functions of two variables, imply that the functions are additively separable. If the functions are twice continuously differentiable, the result follows immediately from well-known facts about such functions. The purpose of the note, therefore, is to generalize the result to functions which do not satisfy this differentiability condition.

The result is useful in the study of strategy-proof or incentive-compatible decision procedures. But it is a purely mathematical result and makes no reference to specific features of the situations to which it is applied. This suggests that it can have applications in other areas and that it possibly may have been stated and proved elsewhere in the literature. I am, however, aware of no earlier statement or proof.

The proof below uses some well-known results from real analysis. Specific references are not made; any textbook on the subject contains the necessary theory.

When f is a function of two variables, we define:

$$\Delta_1^f(x_1,x_2;y) = f(x_2,y) - f(x_1,y)$$

$$\Delta_2^f(x;y_1,y_2) = f(x,y_2) - f(x,y_1)$$

$$\begin{aligned}\Delta_{12}^f(x_1,x_2;y_1,y_2) &= \Delta_1^f(x_1,x_2;y_2) - \Delta_1^f(x_1,x_2;y_1) \\ &= \Delta_2^f(x_2;y_1,y_2) - \Delta_2^f(x_1;y_1,y_2) \\ &= f(x_2,y_2) - f(x_2,y_1) - f(x_1,y_2) + f(x_1,y_1) .\end{aligned}$$

The definition shows that Δ_{12}^f is symmetrical in the two variable, therefore, it could also have been denoted Δ_{21}^f. The function Δ_{12}^f is additive in x in the sense that

$$\Delta_{12}^f(x_1,x_3;y_1,y_2) = \Delta_{12}^f(x_1,x_2;y_1,y_2) + \Delta_{12}^f(x_2,x_3;y_1,y_2) \quad .$$

Similarly, Δ_{12}^f is additive in y.

2. The Result

Proposition: Let X and Y be non-degenerate, finite, closed intervals on the real line. That is, $X = [\underline{x},\bar{x}]$ and $Y = [\underline{y},\bar{y}]$ for real numbers $\underline{x}$, $\bar{x}$, $\underline{y}$ and $\bar{y}$ with $\underline{x} < \bar{x}$ and $\underline{y} < \bar{y}$. Let ϕ and ψ be non-decreasing, continuously differentiable functions of one variable; ϕ is defined on X and ψ on Y. (At the end points of X and Y, one-sided derivatives must exist and satisfy the continuity assumption.) Assume that $\phi(x) \neq \psi(y)$ for all $x \in X$, $y \in Y$.

Let g and h be functions of two variables, defined on $X \times Y$. Suppose that g(x,y) and h(x,y) are non-decreasing in each variable. Finally, assume that (1) and (2) hold:

$$(1)\begin{cases} \text{For all } x_1,x_2 \in X \text{ with } x_1 \leq x_2 \text{ and all } y \in Y, \text{ there} \\ \text{exists an } x' \text{ such that } x_1 \leq x' \leq x_2 \text{ and} \\ \Delta_1^h(x_1,x_2;y) = \phi(x')\Delta_1^g(x_1,x_2;y) \quad . \end{cases}$$

$$(2)\begin{cases} \text{For all } x \in X \text{ and all } y_1,y_2 \in Y \text{ with } y_1 \leq y_2, \text{ there} \\ \text{exists a } y' \text{ such that } y_1 \leq y' \leq y_2 \text{ and} \\ \Delta_2^h(x;y_1,y_2) = \psi(y')\Delta_2^g(x;y_1,y_2). \end{cases}$$

Then g and h are additively separable. That is, there exist functions g_1 and h_1 defined on X and g_2 and h_2 defined on Y such that

$$g(x,y) = g_1(x) + g_2(y)$$

and

$$h(x,y) = h_1(x) + h_2(y),$$

for all $x \in X$, $y \in Y$.

Note that we have not assumed that g and h are continuous. Since monotonicity is assumed, all one-sided limits will exist. We need not assume that $\phi(x)$ and $\psi(y)$ are non-negative; if g is strictly increasing this will, however, follow from (1) and (2). Equations similar to (1) and (2) for the cases $x_1 \geq x_2$ and $y_1 \geq y_2$ follow from (1) and (2), since interchanging x_1 and x_2 (y_1 and y_2) simply changes the sign of Δ_1^g and Δ_1^h (Δ_2^g and Δ_2^h).

Let us now prove that the Proposition is true if g and h are twice continuously differentiable. (In fact, we need only assume differentiability for g; then it will follow that h is also differentiable.) From (1), we get

$$\frac{\Delta_1^h(x_1,x_2;y)}{x_2 - x_1} = \phi(x') \frac{\Delta_1^g(x_1,x_2;y)}{x_2 - x_1} .$$

For any x, let $x_1 = x$ and let x_2 tend to x from above. Then x' will also tend to x. Since we know that the derivatives of g and h exist, we get

$$h_1(x,y) = \phi(x)g_1(x,y) ,$$

where g_1 and h_1 denote derivatives with respect to the first variable.

Differentiating this expression with respect to y gives

$$h_{12}(x,y) = \phi(x)g_{12}(x,y) .$$

From (2), we can similarly conclude

$$h_{21}(x,y) = \psi(y)g_{12}(x,y) .$$

Since $\phi(x) \neq \psi(y)$ and since the cross derivatives are equal for the twice continuously differentiable functions g and h, this implies $g_{12}(x,y) = h_{12}(x,y) = 0$. Hence g and h must be additively separable.

Let (x_0,y_0) be an arbitrary but fixed reference point. Then the definition of Δ^g_{12} gives, for any x and y

$$g(x,y) = g(x,y_0) + g(x_0,y) - g(x_0,y_0) + \Delta^g_{12}(x_0,x;y_0,y) .$$

If Δ^g_{12} is identically equal to 0, we can conclude that g is additively separable. For example, we can define $g_1(x) = g(x,y_0)$ and $g_2(y) = g(x_0,y) - g(x_0,y_0)$. Hence we want to prove $\Delta^g_{12} = \Delta^h_{12} = 0$.

A possible strategy might be to use (1) to prove

$$(3) \qquad \Delta^h_{12}(x_1,x_2;y_1,y_2) = \phi(x')\Delta^g_{12}(x_1,x_2;y_1,y_2) ,$$

for some x' with $x_1 \leq x' \leq x_2$, while (2) would be used to prove a similar formula with $\psi(y')$ instead of $\phi(x')$. Since $\phi(x') \neq \psi(y')$, the result would then follow. But (3) does not, in general, follow from (1). In fact, (3) is not always true. To see what the problem is, consider the two equations obtained by setting $y = y_1$ and $y = y_2$ in (1). Subtracting the former from the latter gives (we omit some of the arguments to simplify notation):

$$\Delta_{12}^{h} = \phi(x'')\Delta_1^g(x_1,x_2;y_2) - \phi(x')\Delta_1^g(x_1,x_2;y_1)$$

$$= \phi(x'')\Delta_{12}^{g} + (\phi(x'') - \phi(x'))\Delta_1^g(x_1,x_2;y_1) \;,$$

where x' and x" lie between x_1 and x_2. If x" is close to x_2 and x' is close to x_1, a possibility which cannot be ruled out, the right-hand side of this expression may be greater than $\phi(x_2)\Delta_{12}^g$ and (3) may be wrong.

We shall eventually prove that (3) holds when g is continuous and $\Delta_{12}^g(x_1,x_2;y_1,y_2)$ is "almost monotone" in x_2; see Lemma 3 in Section 5 below. Then the Proposition can be proved. Before reaching that stage, we consider what is essentially a one-dimensional version of the Proposition; see Section 3. Both in that section and in the subsequent discussion of the two-dimensional case, we first consider the points at which the functions g and h are discontinuous.

Since ϕ is a continuous function defined on the compact interval X, its range $\phi(X)$ is a compact interval. Similarly, $\psi(Y)$ is a compact interval. We have assumed that $\phi(x) \neq \psi(y)$ for all $x \in X$ and $y \in Y$; hence these intervals do not overlap. Therefore, $\phi(X)$ lies either entirely above or entirely below $\psi(Y)$. The two cases can be treated similarly; without loss of generality we assume the former. Let $d_1 = \inf_{x\in X} \phi(x) - \sup_{y\in Y} \psi(y)$. Then $d_1 > 0$, and $\phi(x) \geq \psi(y) + d_1$ for all $x \in X$ and $y \in Y$. The derivatives ϕ' and ψ' are continuous and defined on compact sets; hence they are bounded. We can find a positive number d_2 such that $|\phi'(x)| \leq d_2$ and $|\psi'(y)| \leq d_2$ for all $x \in X$ and $y \in Y$.

3. The One-dimensional Case

In this section, the symbols u, u_1, v etc. will denote non-decreasing functions of one variable, defined on X. One might think of u as being the function obtained by keeping y fixed and varying x in $g(x,y)$, while v is derived from h in the same way. When f is a function of one variable, we define

$$\Delta^f(x_1,x_2) = f(x_2) - f(x_1) .$$

Let X and ϕ be as in the Proposition, let u and v be non-decreasing functions on X, and assume that the following equivalent of (1) holds:

$$(4)\begin{cases} \text{For all } x_1,x_2 \in X \text{ with } x_1 \leq x_2, \text{ there exists an } x' \text{ such that} \\ x_1 \leq x' \leq x_2 \text{ and} \\ \qquad \Delta^v(x_1,x_2) = \phi(x')\Delta^u(x_1,x_2) . \end{cases}$$

The function u is monotone; hence $\lim_{z\to x^-} u(z)$ exists for all $x \in (\underline{x},\bar{x}]$ and $\lim_{z\to x^+} u(z)$ exists for all $x \in [\underline{x},\bar{x})$. Now we consider the points of discontinuity of u. We count left and right discontinuities separately; that is, if $\lim_{z\to x^-} u(z) < u(x) < \lim_{z\to x^+} u(z)$, then u has two discontinuities at x. Since u is monotone, it has at most a countable number of discontinuities. Let an enumeration of the discontinuities be given. Suppose that the first item in the enumeration is a left discontinuity of size a at x. That is, assume

$$\lim_{z\to x^-} \Delta^u(z,x) = a ,$$

where $a > 0$. Let $x_2 = x$ in (4), and let x_1 tend to x from below. Then x' tends to x, and we get

$$\lim_{z \to x_-} \Delta^v(z,x) = a\phi(x) .$$

We want to eliminate the left discontinuity at x. To that end, we define functions u_1 and v_1 by

$$(5) \qquad u_1(z) = \begin{cases} u(z) & \text{for} \quad \underline{x} \leq z < x \\ u(z) - a & \text{for} \quad x \leq z \leq \bar{x} \end{cases} ;$$

$$(6) \qquad v_1(z) = \begin{cases} v(z) & \text{for} \quad \underline{x} \leq z < x \\ v(z) - a\phi(x) & \text{for} \quad x \leq z \leq \bar{x}. \end{cases}$$

Then u_1 and v_1 are non-decreasing, and they are continuous to the left at x. We shall show that (4) holds if u_1 and v_1 are substituted for u and v. If $x_2 < x$ or $x_1 \geq x$, this is obvious since $\Delta^{u_1} = \Delta^u$ and $\Delta^{v_1} = \Delta^v$. Then assume $x_1 < x \leq x_2$. From (4), we get

$$(7) \qquad \Delta^v(x_1,z) = \phi(x')\Delta^u(x_1,z)$$

for all z with $x_1 \leq z < x$, and

$$(8) \qquad \Delta^v(x,x_2) = \phi(x'')\Delta^u(x,x_2) ,$$

where x' and x'' are appropriately chosen numbers, satisfying $x_1 \leq x' \leq z$ and $x \leq x'' \leq x_2$. Let z in (7) take on a sequence of values converging to x from below. Then x' will vary inside the compact set $[x_1,x]$. Hence a sub-sequence can be chosen for which x' converges. This implies that we have

$$\lim_{z \to x^-} \Delta^v(x_1, z) = \phi(x') \lim_{z \to x^-} \Delta^u(x_1, z) , \tag{9}$$

for some x' with $x_1 \leq x' \leq x$. The definition of u_1 gives

$$\Delta^{u_1}(x_1, x_2) = \lim_{z \to x^-} \Delta^u(x_1, z) + \Delta^u(x, x_2) , \tag{10}$$

and a similar formula holds for v and v_1. Adding (8) and (9), we get

$$\Delta^{v_1}(x_1, x_2) = \phi(x') \lim_{z \to x^-} \Delta^u(x_1, z) + \phi(x'')\Delta^u(x, x_2) .$$

The right-hand side of this expression lies between $\phi(x')\Delta^{u_1}(x_1, x_2)$ and $\phi(x'')\Delta^{u_1}(x_1, x_2)$, because both terms in the right-hand side of (10) are non-negative. By continuity of ϕ, there must then exist an x''' such that $x' \leq x''' \leq x''$ and

$$\Delta^{v_1}(x_1, x_2) = \phi(x''')\Delta^{u_1}(x_1, x_2) ,$$

which gives the desired conclusion, namely that (4) holds for u_1 and v_1.

The functions u_1 and v_1 are non-decreasing. Except for the left discontinuity at x, they have the same discontinuities as u and v, respectively. Next we find the second item in the given ordering of the discontinuities of u. Then we construct functions u_2 and v_2 from u_1 and v_1 by eliminating that discontinuity, in the same way as u_1 and v_1 were constructed from u and v. (The construction is a little different if we are dealing with a right discontinuity, but the modifications are easily made.) We go on constructing

u_3 and v_3, u_4 and v_4, etc. If u has infinitely many discontinuities, this gives two infinite sequences of functions. If there are n discontinuities, we let $u_i = u_n$ and $v_i = v_n$ for all $i > n$, and again we have two infinite sequences. For any positive integer i, we can now prove the following:

(i) u_i and v_i are non-decreasing;

(ii) $u_i(\underline{x}) = u(\underline{x})$ and $v_i(\underline{x}) = v(\underline{x})$;

(iii) equation (4) holds if u_i and v_i are substituted for u and v.

For any $x \in X$, $u_i(x)$ and $v_i(x)$ are non-increasing sequences in i. By (i) and (ii), the sequences are bounded from below by $u(\underline{x})$ and $v(\underline{x})$, respectively. Hence they converge, and we define

$$u_0(x) = \lim_{i\to\infty} u_i(x)$$

$$v_0(x) = \lim_{i\to\infty} v_i(x) .$$

It is clear that (i) and (ii) hold for i = 0. Concerning (iii), as i increases, the number x' in (4) will vary. But since x' always belongs to the compact set $[x_1,x_2]$, we can find an infinite increasing sequence of numbers i such that the corresponding numbers x' will converge. Then it follows that (iii) is true for i = 0. Finally, it is not difficult to prove that u_0 is continuous.

We are now able to prove that for a given u, there is essentially only one function v which satisfies (4).

Lemma 1: Let u, v and v* be non-decreasing functions defined on X, assume that (4) holds for u and v, and assume that (4) also holds if v* is substituted for v. Then there exists a number a such that

$$v^*(x) \;=\; v(x) + a \qquad \text{for all } x \in X.$$

<u>Proof</u>: Let u, v and v* satisfy the premise of the lemma, and define $a = v^*(\underline{x}) - v(\underline{x})$. If the conclusion does not hold, there exists an x with $\underline{x} < x \leq \bar{x}$ and $v^*(x) - v(x) \neq a$. Let $b = v^*(x) - v(x) - a$. There is no loss of generality in assuming $b > 0$. We construct the sequences of functions $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ as described above. Simultaneously, we construct functions $v_1^*, v_2^*, \ldots$ by making modifications on v* similar to the ones made on v according to (6). Finally, we define the limit functions u_0, v_0 and v_0^*. By the argument above, these functions satisfy the premise of the lemma. From (6) and the definitions of a and b, we get $v_0^*(\underline{x}) - v_0(\underline{x}) = a$ and $v_0^*(x) - v_0(x) - a = b$. Moreover, u_0 is continuous.

This proves that if a counterexample to the lemma exists, there exists one in which u is continuous. Hence we can assume that u is continuous in the first place. Then u is uniformly continuous on the compact set X, and there exists a number δ such that $z, z' \in X$ and $|z-z'| \leq \delta$ imply $|u(z) - u(z')| \leq \frac{b}{2d_2(x-\underline{x})}$. If one positive number δ satisfies this, a smaller one will also do; hence we can assume that $\delta = \frac{x - \underline{x}}{n}$ for some integer n. Let $x_i = \underline{x} + i\delta$, for $i = 0, \ldots, n$. By the choice of δ, $0 \leq \Delta^u(x_{i-1}, x_i) \leq \frac{b}{2d_2(x-\underline{x})}$, for $i = 1, \ldots, n$. If x' and x" both lie between x_{i-1} and x_i, the definition of d_2 gives $|\phi(x'') - \phi(x')| \leq d_2|x''-x'| \leq d_2\delta$. Hence $|\phi(x'')\Delta^u(x_{i-1}, x_i) - \phi(x')\Delta^u(x_{i-1}, x_i)| \leq \frac{b}{2n}$. The assumptions of the lemma then imply that the difference between $\Delta^v(x_{i-1}, x_i)$ and $\Delta^{v^*}(x_{i-1}, x_i)$ is at most $\frac{b}{2n}$. Moreover, $\Delta^v(\underline{x},x) = \sum_{i=1}^{n} \Delta^v(x_{i-1},x_i)$, and similarly for v*. Hence $\Delta^v(\underline{x},x)$ and $\Delta^{v^*}(\underline{x},x)$ can differ by at most $\frac{b}{2}$. But $\Delta^{v^*}(\underline{x},x) - \Delta^v(\underline{x},x) = v^*(x) - v^*(\underline{x}) - v(x) + v(\underline{x}) = b$. This contradiction completes the proof of Lemma 1. ||

The next lemma essentially proves the Proposition when ψ is a constant function.

<u>Lemma 2</u>: Let u, u*, v and v* be non-decreasing functions defined on X, assume that (4) holds for u and v, and assume that (4) also holds if u* and v* are substituted for u and v. Let the number ψ_0 satisfy $\psi_0 = \psi(y)$ for some $y \varepsilon Y$, and assume

(11) $$v^*(x) - v(x) = \psi_0(u^*(x) - u(x)) \qquad \text{for all } x \varepsilon X.$$

Then there exists a number a such that

$$u^*(x) = u(x) + a \qquad \text{for all } x \varepsilon X.$$

<u>Proof</u>: First we shall prove that the discontinuities of u and u* are at the same places and of the same size. We consider the (possible) left discontinuity at x for some $x \varepsilon (\underline{x},\bar{x}]$; the treatment of right discontinuities is similar. To simplify notation, define $u_-(x) = \lim_{z \to x^-} u(z)$, and similarly for the functions u*, v and v*. Let $x_2 = x$ in (4) and let x_1 tend to x from below. Then x' will converge to x, and we get

$$v(x) - v_-(x) = \phi(x)(u(x) - u_-(x)) .$$

A similar formula holds for u^* and v^*, and subtracting the former equation from the latter gives

$$(12)\qquad v^*(x) - v_-^*(x) - v(x) + v_-(x) = \phi(x)(u^*(x) - u_-^*(x) - u(x) + u_-(x)).$$

On the other hand, by applying (11) to z where z < x and letting z converge to x, we get

$$v_-^*(x) - v_-(x) = \psi_0(u_-^*(x) - u_-(x)) ,$$

which, together with (11), gives

$$(13)\qquad v^*(x) - v_-^*(x) - v(x) + v_-(x) = \psi_0(u^*(x) - u_-^*(x) - u(x) + u_-(x)) .$$

Since $\psi_0 = \psi(y) \neq \phi(x)$, this is only possible if the last term in the right-hand side of (12) and (13) is 0. Hence we get

$$u^*(x) - u_-^*(x) = u(x) - u_-(x) ,$$

which proves that u and u^* have equal left discontinuities at x.

The next step is to enumerate the discontinuities of u, and construct u_1 and v_1 by (5) and (6). Simultaneously, we construct functions u_1^* and v_1^* from u^* and v^*, in the same way and by eliminating the same discontinuity. We go on constructing u_i, v_i, u_i^* and v_i^* for i = 2,3,..., at each stage considering the same discontinuity of u and u^*. Finally, we construct the limit functions u_0, v_0, u_0^* and v_0^*. Since u and u^* have equal sets of discontinuities, which we have eliminated in the same order, $u_i^*(x) - u_i(x) = u^*(x) - u(x)$ for all $x \in X$ and all i = 0,1,...,. A similar statements holds for the v-functions. Hence, u_0, u_0^*, v_0 and v_0^* satisfy (11). Moreover, u_0 and u_0^*

satisfy the conclusion of Lemma 2 if and only if u and u* do. The rest of the premise is preserved when u_0 is substituted for u, etc., by earlier arguments. Earlier arguments also imply that u_0 is continuous.

This shows that if a counterexample to Lemma 2 exists, we can construct one in which u is continuous, and there is no loss of generality in assuming continuity of u from the beginning. Define $a = u^*(\underline{x}) - u(\underline{x})$. If the lemma does not hold, we can find $x \in (\underline{x},\bar{x}]$ and $b \neq 0$ such that $u^*(x) = u(x) + a + b$. Assume $b > 0$; the case $b < 0$ is similar. Since u is uniformly continuous, there exists $\delta > 0$ such that $z,z' \in X$ and $|z-z'| < \delta$ imply $|u(z) - u(z')| \leq \frac{bd_1}{2d_2(x-\underline{x})}$. Choose δ such that $x - \underline{x} = n\delta$ for some integer n, and define $x_i = \underline{x} + i\delta$ for $i = 0,\ldots,n$. Define

$$c_i = u^*(x_i) - u(x_i) - u^*(x_{i-1}) + u(x_{i-1})$$

and

$$\bar{c}_i = v^*(x_i) - v(x_i) - v^*(x_{i-1}) + v(x_{i-1}) ,$$

for $i = 1,\ldots,n$. Then

$$\sum_{i=1}^{n} c_i = u^*(x) - u(x) - u^*(\underline{x}) + u(\underline{x}) = b ,$$

and there must exist an i for which $c_i \geq \frac{b}{n}$. Fix such an i. By applying (11) with $x = x_i$ and with $x = x_{i-1}$ and subtracting, we get

$$\bar{c}_i = \psi_0 c_i .$$

For any $x'' \in X$, $\phi(x'') - \psi_0 \geq d_1$. Since $c_i \geq \frac{b}{n} > 0$, this implies

$$(14) \qquad \bar{c}_i \leq (\phi(x'') - d_1)c_i \leq \phi(x'')c_i - \frac{bd_1}{n} .$$

On the other hand, (4) and the corresponding equation for u^* and v^* can be used to prove the existence of x', $x'' \in [x_{i-1}, x_i]$ such that

$$(15) \qquad \begin{aligned} \bar{c}_i &= \phi(x'')\Delta^{u^*}(x_{i-1},x_i) - \phi(x')\Delta^{u}(x_{i-1},x_i) \\ &= \phi(x'')c_i + (\phi(x'') - \phi(x'))\Delta^{u}(x_{i-1},x_i) . \end{aligned}$$

Since $|\phi(x'') - \phi(x')| \leq d_2|x''-x'| \leq d_2\delta$ and $0 \leq \Delta^u(x_{i-1},x_i) \leq \frac{bd_1}{2d_2(x-\underline{x})}$, the last term in (15) has absolute value at most $\frac{bd_1}{2n}$. But then (14) and (15) provide a contradiction, and the proof of Lemma 2 is complete. ||

Let us justify the claim that Lemma 2 proves the Proposition when the function ψ is constant. Fix y_1 and y_2, and define u and u^* by $u(x) = g(x,y_1)$ and $u^*(x) = g(x,y_2)$ for all $x \in X$. Define v and v^* from h in the same way. Now (1) implies both equation (4) and the corresponding formula for u^* and v^*. Let ψ_0 be the constant value of ψ; then (2) implies (11). The conclusion of Lemma 2 gives $\Delta^g_{12}(x_1,x_2;y_1,y_2) = 0$ for all x_1 and x_2, from which additive separability of g follows. Lemma 1 then implies that h is additively separable; see the end of Section 6 below.

4. Eliminating Discontinuities in Two Dimensions

Returning to the general two-dimensional case, we are now able to eliminate all discontinuities of the function g without affecting its satisfying the premise or conclusion of the Proposition.

Let g and h satisfy the premise of the Proposition. For any $y \in Y$, consider $g(x,y)$ as a function of x, and look at its discontinuities. We shall prove that both the location and the size of these discontinuities are independent of the chosen y. Fix $x > \underline{x}$ and define, for $y \in Y$,

$$u^*(y) = g(x,y)$$

$$v^*(y) = h(x,y)$$

$$u(y) = \lim_{z \to x^-} g(z,y)$$

$$v(y) = \lim_{z \to x^-} h(z,y) .$$

Choose $x_2 = x$ and let x_1 tend to x from below. Then (1) gives

$$v^*(y) - v(y) = \phi(x)(u^*(y) - u(y)) \qquad \text{for all } y \in Y . \tag{16}$$

For any $y_1, y_2 \in Y$ with $y_1 < y_2$ and any $z < x$, (2) gives

$$\Delta_2^h(z;y_1,y_2) = \psi(y')\Delta_2^g(z;y_1,y_2) \quad ,$$

for some $y' \in [y_1,y_2]$. It is possible to let z converge to x from below in such a way that y' converges. (If necessary, we choose a sub-sequence of the original converging sequence of z-values.) This gives

$$\Delta^v(y_1,y_2) = \psi(y')\Delta^u(y_1,y_2) \ , \tag{17}$$

for some $y' \in [y_1,y_2]$. A formula similar to (17) with u* and v* substituted for u and v follows directly from (2). Since x is a fixed number, $\phi(x)$ is fixed. Now we have established the premise of Lemma 2, but with the roles of x and y interchanged; (17) corresponds to (4) and (16) corresponds to (11).

Clearly, Lemma 2 holds even if x and y are interchanged; we could have reversed the roles of the variables in the proof above. Hence there exists a number a such that

$$u^*(y) \;=\; u(y) + a \qquad \text{for all } y \in Y .$$

This is exactly what we wanted to prove. If $a = 0$, g is, for any y, continuous to the left at x. If $a > 0$, g has a left discontinuity at x of size a for any y. (Since g is non-decreasing in the first variable, $a < 0$ is impossible.)

Discontinuities to the right can be treated in exactly the same way.

Now we enumerate the discontinuities of $g(x,y)$ viewed as a function of x, where $y \in Y$ is arbitrary. We construct functions g_1, h_1, g_2, h_2, etc., in the same way as u_1, v_1, etc., were constructed in Section 3. That is, if the first item in the ordering of the discontinuities is a left discontinuity of size a at x, we define, for all $y \in Y$,

$$g_1(z,y) \;=\; \begin{cases} g(z,y) & \text{for} \quad \underline{x} \leq z < x \\ g(z,y) - a & \text{for} \quad x \leq z \leq \bar{x} \;; \end{cases}$$

$$h_1(z,y) \;=\; \begin{cases} h(z,y) & \text{for} \quad \underline{x} \leq z < x \\ h(z,y) - a\phi(x) & \text{for} \quad x \leq z \leq \bar{x} \;. \end{cases}$$

Then we define $g_0(x,y)$ and $h_0(x,y)$ as the limits of the sequences $g_i(x,y)$ and $h_i(x,y)$; the limits exist since the sequences are non-increasing and bounded from below by $g(\underline{x},\underline{y})$ and $h(\underline{x},\underline{y})$. The arguments of Section 3 show that (1) holds if g_0 and h_0 are substituted for g and h, that g_0 and h_0

are non-decreasing in x for each y, and that g_0 is continuous in x. It is clear from the construction of g_i and h_i that $\Delta_2^{g_0} = \Delta_2^g$ and $\Delta_2^{h_0} = \Delta_2^h$. Hence (2) holds for g_0 and h_0, and these functions are non-decreasing in y for each x. Moreover, $\Delta_{12}^{g_0} = \Delta_{12}^g$.

The argument above can now be applied to g_0 and h_0, with x and y interchanged. By using Lemma 2, we can prove that the discontinuities of $g_0(x,y)$, viewed as a function of y, are the same for all x. Then we eliminate these discontinuities. Let $\bar{g}$ and $\bar{h}$ be the resulting functions. They will be non-decreasing in both variables, they will satisfy (1) and (2), and we have $\Delta_{12}^{\bar{g}} = \Delta_{12}^{g_0} = \Delta_{12}^g$. Moreover, $\bar{g}$ is continuous in y. The construction of $\bar{g}$ from g_0 cannot destroy the property of being continuous in x. Hence $\bar{g}$ is continuous in both variables. Since $\bar{g}$ is also monotone in both variables, this implies that $\bar{g}$ is continuous as a function of two variables.

5. Continuous and "Almost Monotone" Δ_{12}^g

As was noted in Section 2, the Proposition is proved if we can establish equations (3) and a similar statement with ψ substituted for ϕ. Lemma 3 proves this in a special case, namely the case where g is continuous and $\Delta_{12}^g(x_1,x;y_1,y_2)$ is almost monotone in x, in a sense to be made precise by equation (18) below. This turns out to be sufficient to prove the Proposition.

<u>Lemma 3</u>: Let g and h satisfy the premise of the Proposition and assume that g is continuous. Let numbers $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ be given, and assume

(18) $0 \leq \Delta^g_{12}(x_1,x;y_1,y_2) \leq \Delta^g_{12}(x_1,x_2;y_1,y_2)$ for all $x \in [x_1,x_2]$.

Then there exists an $x' \in [x_1,x_2]$ such that

(19) $\Delta^h_{12}(x_1,x_2;y_1,y_2) = \phi(x')\Delta^g_{12}(x_1,x_2;y_1,y_2)$.

Proof: If Δ^h_{12} lies between $\phi(x_1)\Delta^g_{12}$ and $\phi(x_2)\Delta^g_{12}$, the result follows by continuity of ϕ. Assume that $\Delta^h_{12} > \phi(x_2)\Delta^g_{12}$ and define $a = \Delta^h_{12} - \phi(x_2)\Delta^g_{12}$; the case $\Delta^h_{12} < \phi(x_1)\Delta^g_{12}$ is treated similarly.

Let $z_0,z_1,\ldots,z_n$ and $z_1',\ldots,z_n'$ be sequences of numbers satisfying $z_0 = x_1$, $z_n = x_2$, and $z_{i-1} \leq z_i' \leq z_i$ for $i = 1,\ldots,n$. Then we shall prove that

(20) $$\sum_{i=1}^{n} \phi(z_i')\Delta^g_{12}(z_{i-1},z_i;y_1,y_2) \leq \phi(x_2)\Delta^g_{12}(x_1,x_2;y_1,y_2) .$$

To simplify notation, let $\Delta_i = \Delta^g_{12}(z_{i-1},z_i;x_1,x_2)$. If Δ_{i-1} and Δ_i have the same sign, we can conclude that $\phi(z_{i-1}')\Delta_{i-1} + \phi(z_i')\Delta_i$ lies between $\phi(z_{i-1}')(\Delta_{i-1} + \Delta_i)$ and $\phi(z_i')(\Delta_{i-1} + \Delta_i)$. Continuity of ϕ then implies the existence of z_i'' such that $z_{i-1}' \leq z_i'' \leq z_i'$ and

$$\begin{aligned} \phi(z_{i-1}')\Delta_{i-1} + \phi(z_i')\Delta_i &= \phi(z_i'')(\Delta_{i-1} + \Delta_i) \\ &= \phi(z_i'')\Delta^g_{12}(z_{i-2},z_i;y_1,y_2) . \end{aligned}$$

Hence we can eliminate z_{i-1} and z'_{i-1} and substitute z''_i for z'_i, without changing the left-hand side of (20). By repeating this argument, we eventually reach a situation where the terms $\Delta_1, \Delta_2, \ldots$ alternate in sign, and there is no loss of generality in assuming that this is the case from the beginning. (To be precise, the logic of the argument is the following: If there exists a case in which (20) fails, then there exists an example in which (20) fails and $\Delta_1, \Delta_2, \ldots$ alternate in sign. Hence it is sufficient to prove (20) for the latter type of cases.) We can also assume that n is even; if n is originally odd, we can add a $z_{n+1} = z_n$, and neither side of (20) changes. For any integer m with $1 \leq m \leq n$, (18) gives

$$(21) \qquad \sum_{i=1}^{m} \Delta_i = \Delta_{12}^{g}(x_1, z_m; y_1, y_2) \geq 0 \quad .$$

In particular, $\Delta_1 \geq 0$; hence $\Delta_i \geq 0$ for i odd and $\Delta_i \leq 0$ for i even. Now we shall make a series of changes in the left-hand side of (20), each time increasing its value or at least not decreasing it. We will end up with something which is less than or equal to the right-hand side. Then (20) is proved.

For i odd, replace the term $\phi(z'_i)\Delta_i$ by $\phi(z_i)\Delta_i$; since $\phi(z'_i) \leq \phi(z_i)$ and $\Delta_i \geq 0$, this cannot reduce the sum. For i even, replace $\phi(z'_i)\Delta_i$ by $\phi(z_{i-1})\Delta_i$; here the sum moves in the right direction because $\phi(z'_i) \geq \phi(z_{i-1})$ and $\Delta_i \leq 0$. The first two terms are now equal to

$$\phi(z_1)(\Delta_1 + \Delta_2) \quad .$$

We replace $\phi(z_1)$ by $\phi(z_3)$; since $\Delta_1 + \Delta_2 \geq 0$ by (21), this represents a change in the right direction. Now the four first terms are equal to

$$\phi(z_3)\sum_{i=1}^{4}\Delta_i \quad .$$

The sum is non-negative by (21) and we can replace $\phi(z_3)$ by $\phi(z_5)$. After having repeated this step $\frac{n}{2}$ - 1 times, the entire expression has become

$$\phi(z_{n-1})\sum_{i=1}^{n}\Delta_i \;=\; \phi(z_{n-1})\Delta_{12}^g(x_1,x_2;y_1,y_2) \quad .$$

This is less than or equal to the right-hand side of (20). This completes the proof of that equation.

Note that only the first inequality in (18) has been explicitly used so far. But if we had started by assuming $\Delta_{12}^h < \phi(x_1)\Delta_{12}^g$, we would have needed a version of (20) with the inequality reversed and $\phi(x_1)$ substituted for $\phi(x_2)$. In order to prove that formula, the second half of (18) must be used.

Now we choose $\delta > 0$ such that $z,z' \in X$ and $|z-z'| \leq \delta$ imply $|g(z,y_1) - g(z',y_1)| \leq \frac{a}{2d_2(x_2-x_1)}$, and such that $x_2 - x_1 = n\delta$ for a positive integer n. This is possible by (uniform) continuity of g. Define $z_i = x_1 + i\delta$. For any i, (1) implies the existence of $z_i', z_i'' \in [z_{i-1},z_i]$ such that

$$\begin{aligned}
\Delta_{12}^h(z_{i-1},z_i;y_1,y_2) &= \phi(z_i'')\Delta_1^g(z_{i-1},z_i;y_2) - \phi(z_i')\Delta_1^g(z_{i-1},z_i;y_1) \\
&= \phi(z_i'')\Delta_{12}^g(z_{i-1},z_i;y_1,y_2) + (\phi(z_i'') - \phi(z_i'))\Delta_1^g(z_{i-1},z_i;y_1) \\
&\leq \phi(z_i'')\Delta_{12}^g(z_{i-1},z_i;y_1,y_2) + d_2|z_i'' - z_i'|\,\frac{a}{2d_2(x_2-x_1)} \\
&\leq \phi(z_i'')\Delta_{12}^g(z_{i-1},z_i;y_1,y_2) + \frac{a}{2n} \;.
\end{aligned}$$

(We use the definition of d_2 and the fact that $|z_i'' - z_i'| \leq z_i - z_{i-1} = \delta$.) Summing these equations for $i = 1,\ldots,n$, and using (20), we get

$$\begin{aligned}\Delta_{12}^h(x_1,x_2;y_1,y_2) &= \sum_{i=1}^n \Delta_{12}^h(z_{i-1},z_i;y_1,y_2) \\ &\leq \sum_{i=1}^n (\phi(z_i'')\Delta_{12}^g(z_{i-1},z_i;y_1,y_2) + \frac{a}{2n}) \\ &\leq \phi(x_2)\Delta_{12}^g(x_1,x_2;y_1,y_2) + \frac{a}{2} .\end{aligned}$$

This contradicts the definition of a and completes the proof of Lemma 3. ||

The lemma remains true, and the proof is similar, if $\Delta_{12}^g(x_1,x_2;y_1,y_2)$ is non-positive, and the assumption (18) is replaced by

$$0 \geq \Delta_{12}^g(x_1,x;y_1,y_2) \geq \Delta_{12}^g(x_1,x_2;y_1,y_2) \quad \text{for all } x \in [x_1,x_2] .$$

Moreover, a result similar to Lemma 3 can be obtained in which the roles of x and y are interchanged and ψ is substituted for ϕ.

6. Completing the Proof of the Proposition

Now we are able to complete the proof. Assume that g and h satisfy the premise of the Proposition, and assume that Δ_{12}^g is not identically equal to 0. By the construction of Section 4, we can modify g and h such that the premise is still satisfied, Δ_{12}^g is not changed, and the resulting g is continuous. Therefore, there is no loss of generality in assuming that g is continuous.

Find numbers $x_1, x_1' \in X$ and $y_1, y_1' \in Y$ such that $x_1 \leq x_1'$, $y_1 \leq y_1'$, and $\Delta_{12}^g(x_1,x_1';y_1,y_1') \neq 0$. Let $a = \Delta_{12}^g(x_1,x_1';y_1,y_1')$ and assume $a > 0$; the case $a < 0$ is similar.

Consider $\Delta_{12}^g(x_1,x;y_1,y_1')$ as a function of x. The function is continuous, and the value is 0 for $x = x_1$ and a for $x = x_1'$. But the function is not necessarily monotone or "almost monotone"; the value may go below 0 and above a for x between x_1 and x_1'. Let x_2 be the largest value of x in this interval for which the function has value 0; this is well defined by continuity. Moreover, let x_2' be the smallest value of x in the interval $[x_2,x_1']$ for which the function has value a. Then the function will be almost monotone, in the sense of Lemma 3, on $[x_2,x_2']$. Formally, define

$$x_2 := \sup\{x \mid x \leq x_1' \text{ and } \Delta_{12}^g(x_1,x;y_1,y_1') \leq 0\}$$

and

$$x_2' = \inf\{x \mid x \geq x_2 \text{ and } \Delta_{12}^g(x_1,x;y_1,y_1') \geq a\} \quad .$$

These definitions and additivity of the function Δ_{12}^g give

$$x_1 \leq x_2 \leq x_2' \leq x_1' \tag{22}$$

and

$$0 \leq \Delta_{12}^g(x_2,x;y_1,y_1') \leq \Delta_{12}^g(x_2,x_2';y_1,y_1') = a \quad , \text{ for all } x \in [x_2,x_2'] \; . \tag{23}$$

Now we consider $\Delta^g_{12}(x_2,x_2';y_1,y)$ as a function of y and go through the same argument. That is, we define

$$y_2 \quad = \quad \sup\{y \mid y \leq y_1' \text{ and } \Delta^g_{12}(x_2,x_2';y_1,y) \leq 0\}$$

and

$$y_2' \quad = \quad \inf\{y \mid y \geq y_2 \text{ and } \Delta^g_{12}(x_2,x_2';y_1,y) \geq a\} \quad .$$

These definitions imply

$$y_1 \leq y_2 \leq y_2' \leq y_1'$$

and

$$(24) \qquad 0 \leq \Delta^g_{12}(x_2,x_2';y_2,y) \leq \Delta^g_{12}(x_2,x_2';y_2,y_2') = a \qquad \text{for all } y \in [y_2,y_2'].$$

We construct x_3 and x_3', y_3 and y_3', x_4 and x_4', etc., in the same way. Formulas similar to (22) will hold, and they show that x_i is a non-decreasing sequence which is bounded from above. Hence it converges; let the limit be x_0. The sequence x_i' converges for similar reasons, and its limit x_0' satisfies $x_0 \leq x_0'$. In the same way, we define $y_0 = \lim\limits_{i\to\infty} y_i$ and $y_0' = \lim\limits_{i\to\infty} y_i'$ and get $y_0 \leq y_0'$. The function Δ^g_{12} is continuous in its four arguments. Hence (23) and corresponding formulas imply

$$(25) \qquad \Delta^g_{12}(x_0,x_0';y_0,y_0') = a > 0 \quad .$$

(This immediately rules out the possibility $x_0 = x_0'$ or $y_0 = y_0'$.) The same formulas also give

$$0 \leq \Delta_{12}^{g}(x_0,x;y_0,y_0') \leq \Delta_{12}^{g}(x_0,x_0';y_0,y_0') \qquad \text{for all } x \in [x_0,x_0'].$$

By Lemma 3, there must exist an $x' \in [x_0,x_0']$ such that

$$(26) \qquad \Delta_{12}^{h}(x_0,x_0';y_0,y_0') = \phi(x')\Delta_{12}^{g}(x_0,x_0';y_0,y_0') \quad .$$

From (24) and similar formulas, and by using Lemma 3 with x and y interchanged, we can conclude that there exists a $y' \in [y_0,y_0']$ such that

$$(27) \qquad \Delta_{12}^{h}(x_0,x_0';y_0,y_0') = \psi(y')\Delta_{12}^{g}(x_0,x_0';y_0,y_0') \quad .$$

Equations (25) - (27) provide a contradiction, since $\phi(x') \neq \psi(y')$.

So far we have proved that if the premise of the Proposition holds, then Δ_{12}^{g} is identically 0. We have earlier seen that this implies additive separability of g. Concerning h, fix any numbers $y_1,y_2 \in Y$ and define

$$u(x) = g(x,y_1)$$

$$u^*(x) = g(x,y_2)$$

$$v(x) = h(x,y_1)$$

$$v^*(x) = h(x,y_2) \quad .$$

Then (1) implies that (4) holds for u and v, and that (4) also holds if u* and v* are substituted for u and v. By additive separability of g, $\Delta^{u} = \Delta^{u^*}$. Hence (4) holds if u is kept unchanged and v* is substituted for v. Lemma 1 can be applied. The conclusion of that lemma implies $\Delta^{v} = \Delta^{v^*}$, which gives $\Delta_{12}^{h}(x_1,x_2;y_1,y_2) = 0$ for any x_1 and x_2. Additive separability of h follows. The proof of the Proposition is complete.

7. Relaxing the Monotonicity Assumptions

Suppose that g and h satisfy the premise of the Proposition, except that $g(x,y)$ and $h(x,y)$ are non-increasing (instead of non-decreasing) in x for every y. Then the proof of the Proposition can be carried through with insignificant changes. The same is true if $g(x,y)$ is non-decreasing in x for every y while $h(x,y)$ is non-increasing in x for every y. (In the latter case, the assumption (1) will require that ϕ have non-positive values, provided that $g(x,y)$ is strictly increasing in x.)

Similar remarks will of course apply with x and y interchanged.

Hence it is sufficient to assume that g and h are monotone in each variable, and that $g(x,y)$, as a function of x, moves in the same direction for all y, and similarly for the other variable and for the function h. One could ask whether this condition can be weakened further. The answer is yes. For one thing, if g is continuous, no monotonicity assumption is required at all. (This is clear from the relevant parts of the proof; monotonicity is only used in Sections 3 and 4 to eliminate discontinuities.) In order to apply the constructions of the first part of Section 3, we must at least assume that $g(x,y)$, as a function of x, is of bounded variation. In fact, this assumption is almost sufficient. Suppose that it is satisfied, and suppose that ϕ is strictly increasing. Except for certain special cases, if g is not monotone in x, (1) cannot be satisfied by any function h. Hence the Proposition holds vacuously. (If ϕ is constant on an interval, the Proposition is easily proved from Lemma 2.) We will not go into detail concerning the "special cases"; the Proposition extended to monotone g and h, as described above, is sufficiently general for our purpose.

Finally, suppose that ϕ or ψ is non-increasing. Again, the proof applies essentially without change. (Monotonicity of ϕ is only used in the proof of Lemma 3.)

8. Functions Defined on More General Domains

The Proposition assumes that the functions g and h are defined on a rectangle $X \times Y$. This is unnecessarily restrictive.

Let S be an open and convex subset of the Euclidean plane. Let ϕ and ψ be non-decreasing and continuously differentiable functions defined on intervals of the real line, such that $\phi(x)$ is defined whenever there exists a y with $(x,y) \in S$, while $\psi(y)$ is defined whenever there exists an x with $(x,y) \in S$. Assume that $(x,y) \in S$ implies $\phi(x) \neq \psi(y)$. (We need not assume $\phi(x) \neq \psi(y)$ for all values of x and y for which these numbers are defined, but only for pairs which belong to S.) Let g and h be defined on S and satisfy the premise of the Proposition. It should be clear what this means. For example, for any y, g(x,y) is non-decreasing in x on the set $\{x|(x,y) \in S\}$, and (1) holds whenever $(x_1,y) \in S$ and $(x_2,y) \in S$. Then the conclusion of the Proposition will hold, that is, h and g are additively separable.

To prove this, let $(x_0,y_0) \in S$ be an arbitrary but fixed reference point. For any $(x,y) \in S$, find a "path" in S from (x_0,y_0) to (x,y) consisting of a finite number of horizontal and vertical line segments. That is, find numbers $x_1,x_2,\ldots,x_n$ and $y_1,\ldots,y_n$ such that $(x_n,y_n) = (x,y)$, and $(x_i,y_{i-1}) \in S$ and $(x_i,y_i) \in S$ for $i = 1,\ldots,n$. Since S is open and convex, it is always possible to find such a path, and it is also possible to choose the path such that the sequences $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ are monotone. (Let $\varepsilon > 0$ be such that the ε-neighborhoods around (x_0, y_0) and (x,y) are contained in S. For any point on the straight line from (x_0, y_0) to (x,y), the ε-neighborhood around that point will also be contained in S. A path can clearly be found which never is farther than ε away from this line.) Define

$$\delta_x = \sum_{i=1}^{n} \Delta_1^g(x_{i-1}, x_i; y_{i-1})$$

$$\delta_y = \sum_{i=1}^{n} \Delta_2^g(x_i; y_{i-1}, y_i) \quad .$$

As the expressions are given here, δ_x and δ_y may depend not only on the point (x,y) but also on the path. We shall prove, however, that δ_x is a function only of x, not of y and not of the path. By a similar argument, δ_y depends only on y. The definitions give

$$g(x,y) = g(x_0, y_0) + \delta_x + \delta_y \quad ,$$

which proves that g is additively separable. The argument is similar for the function h.

The proof that δ_x depends only on x remains. Let $(x,y') \in S$ be another point, and let $x_1', \ldots, x_{n'}'$ and $y_1', \ldots, y_{n'}'$ represent a path from (x_0, y_0) to (x,y'). Both the possibilities $y' = y$ and $y' \neq y$ are covered here. Now define

$$\delta_x' = \sum_{i=1}^{n'} \Delta_1^g(x_{i-1}', x_i'; y_{i-1}') \quad .$$

We must prove that $\delta_x = \delta_x'$. Let z satisfy $x_{i-1} \leq z \leq x_i$. We add new segments, given by the number z and y_{i-1}, to the path from (x_0,y_0) to (x,y), such that the sequences defining the path become $x_1,\ldots,x_{i-1},z,x_i,\ldots,x_n$ and $y_1,\ldots,y_{i-1},y_{i-1},y_i,\ldots,y_n$. This amounts to cutting the horizontal line segment from (x_{i-1},y_{i-1}) to (x_i,y_{i-1}) in two pieces, and adding a vertical segment of length 0 at (z,y_{i-1}). This operation will not change the sum which defines δ_x. By adding points in this way in both the paths, we can assume that $n = n'$ and $x_i = x_i'$ for $i = 1,\ldots,n$. For any i, let X be the interval with endpoints x_{i-1} and x_i, and let Y be the interval with endpoints y_{i-1} and y_{i-1}'. The corners of the rectangle X×Y all belong to S, hence X×Y $\subseteq$ S by convexity. On the set X×Y, all the assumptions of the Proposition are satisfied. Therefore, g is additively separable on that set, which implies

$$\Delta_1^g(x_{i-1},x_i;y_{i-1}) \quad = \quad \Delta_1^g(x_{i-1},x_i;y_{i-1}') \quad .$$

Since this holds for all i, we get $\delta_x = \delta_x'$. The proof is complete.

The condition that S is convex is necessary. We need not, however, assume that S is open. The proof that δ_x only depends on x never uses this condition. But we did use it to prove that there always exists a path of the required type from (x_0,y_0) to (x,y). In order to remove the condition, define

$$S_0 \quad = \quad \{(x,y) \mid (x,y) \in S \text{ and there exists an } x' \neq x \text{ with } (x',y) \in S\}.$$

That is, if a straight line parallel to the x-axis intersects S in exactly one point, we eliminate that point from S. S_0 is convex, and in S_0 paths can be constructed and the argument above can be applied. Hence g and h are additively separable on S_0. For any point $(x,y) \in S \setminus S_0$, we can define $g_2(y)$ and $h_2(y)$ such that the conclusion of the Proposition becomes true, since there is no point $(x',y) \in S$ with $x' \neq x$ which can cause problems. (Note that S_0 may be empty, but that does not create difficulties.)