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19 On a Problem in Pure Economics*

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Intermediate between mathematics, statistics, and economics, we find a new discipline which, for lack of a better name, may be called *econometrics.*

Econometrics has as its aim to subject abstract laws of theoretical political economy or "pure" economics to experimental and numerical verification, and thus to turn pure economics, as far as is possible, into a science in the strict sense of the word.

The econometric study that I shall present is an attempt to realize the dream of Jevons[1]: to measure the variation in the marginal utility of economic goods. I shall give special attention to the variation in the marginal utility of money.

The study was made during a stay in Paris, 1923. I initially had the intention of extending the application of my method to more extensive statistical material, but other preoccupations have prevented it. The very lively interest in econometric research which has been manifested among mathematical economists in recent

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[1] *Theory of Political Economy* (1871). See in particular the paragraph, Numerical Determination of the Laws of Utility, p. 146 of the 4th edition (1911).

years[2] leads me to believe that it might perhaps be of interest to publish separately the results already obtained.

1. UTILITY AS A QUANTITY

The real advances in a science of the outside world begin on the day that it is realized that vague common sense notions must be replaced by notions capable of objective definition. It is thus that in mechanics the notion of force based on the feeling induced by muscular exertion has been substituted by the abstract notion of force employed in rational mechanics. It is in the same way that utility in the ordinary sense of the word has been replaced by the abstract notion of utility used in pure economics. Although the search for an objective definition of utility has been the object of works of scholars such as Messrs. Edgeworth, Fisher, and Pareto, and still others, it does not seem to me that definitive results have been obtained. In particular it seems to me that the axioms on which one must base oneself to establish the definition of utility as a quantity have not been displayed.

To save writing I employ vector notation. It would perhaps be possible to accomplish the research without this notation. But then it would be necessary—at least if rigor is to be maintained—to employ such an extensive verbal apparatus that reading would become not only very tedious, but very arduous as well, not to speak of the risk of logical errors.

Let us consider a *homo œconomicus* who possesses quantities $x_1, x_2, \ldots, x_M$ of M economic goods. We represent his economic position by the vector $\mathbf{x}$ which can be drawn in M-dimensional space from the origin of a system of rectangular coordinates to the point $(x_1, x_2, \ldots, x_M)$. We shall say that the individual possesses the resources $\mathbf{x}$. In the sequel $\mathbf{x}$ will be employed indifferently to designate both the point $(x_1, x_2, \ldots, x_M)$ and the vector leading to this point.

If the individual, finding himself initially at position $\mathbf{x}$, acquires the quantities $p_1, p_2, \ldots, p_M$ of goods $1, 2, \ldots, M$ (for instance by exchange, some of the p's being negative), we represent the displacement in his position by the vector $\mathbf{p}$ with components $p_1, p_2, \ldots, p_M$. We call the vector $\mathbf{p}$ the increment of his resources. The final position will obviously be represented by the vector $\mathbf{x} + \mathbf{p}$. If the individual undergoes a series of successive displacements $\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \ldots$ we shall say that he traverses an *acquisition path*, which we can henceforth assume to be piecewise continuous. Likewise the individual's consumption can be represented by a *consumption path*, the latter not being necessarily the same as an acquisition path. If the individual possesses resources $\mathbf{x}$ and if he can choose at will the consumption path from the origin to $\mathbf{x}$, we shall say that the consumption path is free. In the case in which the consumption path is partly or wholly prescribed we shall say that a constraint is imposed on the consumption path.

[2] For English language writings the bibliography in the *Quarterly Journal of Economics* (1925) may be consulted.

That being the case, to establish an objective definition of utility in the economic sense one may base one's self on the two categories of axioms that follow:

I. Axioms of the First Kind (Axioms Relating to a Given Position)

a. *Axiom of choice.* If the individual, finding himself at point **x** has a choice between two displacements **p** and **q**, we assume that his choice is always well-defined both in the case of free paths and in the contrary case. That is to say, one of the three following cases always holds:

1. The individual prefers **p** to **q**;
2. He prefers **q** to **p**;
3. The choice between **p** and **q** is indifferent to him. For brevity we use the notation

$$(\mathbf{xp}) \gtreqqless (\mathbf{xq})$$

to designate these three cases.

We conceive that it is in principle possible, by means of "experiments by interrogation" carried out on *homo œconomicus*, to determine objectively which of the three cases occurs. Likewise, it is in principle possible by "experiments by trial and error" to determine a displacement **p** such that $(\mathbf{xp}) = (\mathbf{xq})$, **q** being some given displacement, for example $\mathbf{q} = 0$. We shall say that the displacement $\mathbf{q} = 0$ is of zero choice, since the individual will obviously be indifferent as to whether he undergoes this displacement or not. A displacement **p** will be said to be of positive, zero, or negative choice according as $(\mathbf{xp}) \gtreqqless (\mathbf{x}, 0)$.

Two displacements **p** and **q** such that $(\mathbf{xp}) = (\mathbf{xq})$ will be called equivalent.

b. *Axiom of coordination* (transitivity). If

$$(\mathbf{xp}) > (\mathbf{xq}) \quad \text{and} \quad (\mathbf{xq}) > (\mathbf{xr})$$

then the choice will be

$$(\mathbf{xp}) > (\mathbf{xr})$$

and similarly for the symbols "=", "<", and "not >".

Since this last axiom assures the transitivity of the relation (a), we can take this relation to be the definition of equality and inequality of two utilities. This equality or inequality is therefore a fact that can be objectively established, provided the increments **p** and **q** used for comparing utility refer to the same position **x**.

c. *Axiom of addition.* If

$$(\mathbf{xp}) > (\mathbf{xq}) \quad \text{and} \quad (\mathbf{xr}) > (\mathbf{xs}),$$

p, **q**, **r**, and **s** *being infinitesimals,* the choice will be

$$(\mathbf{x}, \mathbf{p} + \mathbf{r}) > (\mathbf{x}, \mathbf{q} + \mathbf{s})$$

and similarly for the other symbols.

It should be noted that we do not assume the axiom of addition in the case when **p**, **q**, **r**, and **s** are not infinitesimals. One consequence of the axiom of addition is that a form in rational coefficients, linear and homogeneous in infinitesimal displacements of zero choice, is itself a displacement of zero choice, and it follows by a passage to the limit that this is also the case for a form in any real coefficients.

II. Axioms of the Second Kind (Axioms Relating to Different Positions)

a. *Axiom of choice.* If the individual knows that he will find himself on two different occasions in positions **x** and **y**, respectively, and if he has a choice between the displacement **p** in position **x** and the displacement **q** in position **y**, we assume that his choice is always well-defined, both in the case of free paths and in the contrary case. That is to say, one of the three cases

$$(\mathbf{xp}) \gtreqqless (\mathbf{yq})$$

always holds.

b. *Axiom of coordination* (transitivity). If

$$(\mathbf{xp}) \gtrless (\mathbf{yq}) \quad \text{and} \quad (\mathbf{yq}) > (\mathbf{zr})$$

then the choice will be

$$(\mathbf{xp}) > (\mathbf{zr})$$

and similarly for the other symbols.

The axioms of the second kind having been set down, the remarks under (Ib) concerning the objective definition of equality between two utilities apply again to increments **p** and **q** which refer to different positions **x** and **y**.

c. *Axiom of addition.* If

$$(\mathbf{xp}) > (\mathbf{yq}) \quad \text{and} \quad (\mathbf{xr}) > (\mathbf{ys}),$$

p, **q**, **r**, and **s** being infinitesimals,

the choice will be

$$(\mathbf{x}, \mathbf{p} + \mathbf{r}) > (\mathbf{y}, \mathbf{q} + \mathbf{s})$$

and similarly for the other symbols.

One consequence of this axiom is that two linear and homogeneous forms in infinitesimal displacements which are equivalent to each other are themselves equivalent displacements.

In all that follows, we assume that the consumption path is free. The study of the case in which the path is not free is infinitely more complex and requires among other things the introduction of two different concepts of utility. The study of this case would carry us too far from the objective we are pursuing.

To begin, we base ourselves on the axioms of the first kind. We assume that the position **x** does not belong to an indifferent part of the space. By this it is meant that the individual's choice is not such that every infinitesimal displacement around **x** is of zero choice.

Let L_μ denote a μ-dimensional linear variety imbedded in M-dimensional space ($\mu < M$). Assume moreover that **x** is not a singular point in the space,[3] that is to say a point such that every infinitesimal displacement of zero choice around **x** is situated on an L_{M-2} passing through **x**.

That being the case, every infinitesimal displacement of zero choice around **x** is situated in an L_{M-1} passing through **x**, and conversely every infinitesimal displacement situated on this L_{M-1} is of zero choice. In fact, consider $M - 1$ displacements of zero choice which are not located on an L_{M-2} passing through **x**. Let L be the L_{M-1} passing through **x** and through the end points of the $M - 1$ displacements under consideration. Then every infinitesimal displacement around **x** ending at a point on L is of zero choice since it can be expressed as a linear form in the $M - 1$ displacements under consideration which are all of zero choice. On the other hand, an infinitesimal displacement **p** which does not end at a point on L cannot be of zero choice since in that case any displacement around **x**, being expressible as a linear form in **p** and in the $M - 1$ displacements defining L, would have to be of zero choice, contrary to the hypothesis that **x** does not belong to an indifferent part of the space.

Let L' be an L_{M-1} parallel to L at an infinitesimal distance **q**$'$ from it. All displacements ending on L', and only those, are equivalent, because such a displacement is the sum of **q**$'$ and of a displacement on L (which is of zero choice). The displacements which end on L' will be of positive or negative choice according as **q**$'$ is of positive or negative choice. Moreover, two planes L' and L'' parallel to L at infinitesimal distances **q**$'$ and **q**$''$ from L enjoy the property that the displacements which end on L' will be preferred to those which end on L'' if **q**$'$ is preferred to **q**$''$. Finally, since the **q**'s are directed along the normal at L, we see that one can define the positive direction of this normal in such a way that **q**$'$ will be preferred to **q**$''$ or conversely according as **q**$'$ ends at a point on the normal whose abscissa is larger or smaller than the abscissa of the corresponding point for **q**$''$.

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[3] Readers familiar with the theory of indifference surfaces—which we do not require in the present work—will easily recognize that this is the case in which **x** is a singular point of the indifference surface passing through **x**.

Such are the patterns of choice to which the axioms of the first kind lead.

Consider now a vector $\mathbf{u}$ proceeding along the normal to L in the positive direction and of arbitrary length. Let $\delta\mathbf{x}$ be any infinitesimal displacement around $\mathbf{x}$. Then one can see that the possible values of the inner product $\mathbf{u}\,\delta\mathbf{x}$ are distributed around $\mathbf{x}$ in analogous fashion to the preferences of the individual. More precisely: if $\delta^{(1)}\mathbf{x}$ and $\delta^{(2)}\mathbf{x}$ are any two infinitesimal displacements around $\mathbf{x}$, the individual's choice will be

$$(\mathbf{x}\ \delta^{(1)}\mathbf{x}) \gtreqless (\mathbf{x}\ \delta^{(2)}\mathbf{x})$$

according as[4]

$$\mathbf{u}\,\delta^{(1)}\mathbf{x} \gtreqless \mathbf{u}\,\delta^{(2)}\mathbf{x}.$$

This justifies the definition that we shall adopt.

The inner product $\mathbf{u}\,\delta\mathbf{x}$ will be by definition the utility of the displacement $\delta\mathbf{x}$ around the fixed point $\mathbf{x}$. The ratios of the components $u_1, u_2, \ldots, u_M$ of $\mathbf{u}$ will be the ratios of the marginal utilities of goods $1, 2, \ldots, M$ at the point $\mathbf{x}$.

Since one can in principle determine "experimentally" $M - 1$ displacements of zero choice not all on the same L_{M-2}, the orientation of the plane L is a fact that can be objectively determined, and the same applies to the positive direction of its normal. As this "experiment" can be carried out for all points $\mathbf{x}$ of the space, we see that the vector $\mathbf{u}(\mathbf{x})$ is defined save for a factor which is a function of $\mathbf{x}$. Otherwise expressed, the axioms of the first kind allow for an objective definition of the direction of $\mathbf{u}$ but not its magnitude. The definition of the direction of $\mathbf{u}$—which we can call the *maximum direction* for obvious reasons—suffices to elaborate the principal part of the theory of the static equilibrium of exchange. In fact, the equation of the plane L

$$\mathbf{u}\,\delta\mathbf{x} = 0,$$

which can also be written

$$u_1\,\delta x_1 + u_2\,\delta x_2 + \cdots + u_M\,\delta x_M = 0$$

is the fundamental equation of exchange, and to know it one does not need to know the length of $\mathbf{u}$. It is by basing themselves on the notion of maximum direction and on the equation $\mathbf{u}\,\delta\mathbf{x} = 0$ that Mr. Fisher and Mr. Pareto after him have studied the static equilibrium of exchange.

For the goal we pursue, this definition does not suffice. We also need an objective definition of the length of $\mathbf{u}$. To obtain it we base ourselves on the axioms of the second kind.

Let $\delta\mathbf{x}$ and $\delta\mathbf{y}$ be two infinitesimal displacements starting from the points $\mathbf{x}$ and $\mathbf{y}$, respectively. Let $L(\mathbf{x})$ and $L(\mathbf{y})$ be the two L_{M-1}'s passing through $\mathbf{x}$ and $\mathbf{y}$,

[4] [Editor's note: in the first of these expressions, $(\mathbf{x}\ \delta^{(1)}\mathbf{x})$ denotes the ordered pair $(\mathbf{x}, \delta^{(1)}\mathbf{x})$, whereas in the second $\mathbf{u}\,\delta^{(1)}\mathbf{x}$ denotes the inner product $\mathbf{u}\cdot\delta^{(1)}\mathbf{x}$.]

respectively, and whose normals proceed in the maximum direction from **x** and **y**. Considering δ**x** as a given displacement and δ**y** as a variable displacement, it is an immediate consequence of axioms (IIa,b) and of the properties of the distribution of preferences around a fixed point, that all displacements δ**y** which are equivalent to δ**x**, and only those, end on one and the same L_{M-1} parallel to $L(\mathbf{y})$. Denote this L_{M-1} by $L'(\mathbf{y})$.

On the other hand, let $L'(\mathbf{x})$ be the L_{M-1} parallel to $L(\mathbf{x})$ which passes through the end point of δ**x**. The proposition we have just set forth with respect to δ**x** obviously remains true for all displacements ending on $L'(\mathbf{x})$ and for those only. There is therefore a one-to-one correspondence between $L'(\mathbf{x})$ and $L'(\mathbf{y})$ in the sense that an arbitrary displacement δ**y** ending on $L'(\mathbf{y})$ is equivalent to an arbitrary displacement δ**x** ending on $L'(\mathbf{x})$ and conversely, it being impossible for a displacement ending on one of these two L_{M-1} to be equivalent to a displacement that does not end on the other. For this reason we shall say that $L'(\mathbf{x})$ and $L'(\mathbf{y})$ are *two equivalent planes*. To compare the individual's choice between a displacement around **x** and a displacement around **y** it suffices therefore to study the ordering of pairs of equivalent planes such as $L'(\mathbf{x})$ and $L'(\mathbf{y})$.

Consider two pairs $L'(\mathbf{x})L'(\mathbf{y})$ and $L''(\mathbf{x})L''(\mathbf{y})$, and let $p'q'$ and $p''q''$ be the algebraic values of the infinitesimal distances of the planes from the points **x** and **y** respectively, these distances being taken with the appropriate sign defined by the maximum direction at **x** and **y**. If $p' < p''$, $L''(\mathbf{x})$ will be preferred to $L'(\mathbf{x})$, but then according to axioms (IIa,b) $L''(\mathbf{y})$ will be preferred to $L'(\mathbf{y})$, and consequently we will have $q' < q''$. Conversely if $q' < q''$, one will obtain $p' < p''$. From this we conclude that to a sequence of consecutive planes $L^{(k)}(\mathbf{x})$—in the sense that the distances $p^{(k)}$ from the $L^{(k)}(\mathbf{x})$ to the point **x** ($k = 1, 2, \ldots$) form a sequence of (algebraically) increasing numbers—there corresponds a sequence of equivalent planes $L^{(k)}(\mathbf{y})$ which are consecutive also. But so far nothing permits one to conclude anything relative to the distances of the planes $L^{k}(\mathbf{y})$ in relation to the distances of the planes $L^{(k)}(\mathbf{x})$.

In order to draw such a conclusion we base ourselves on axiom (IIc). If (**x** δ**x**) and (**y** δ**y**) are two equivalent displacements, the displacements (**x**, δ**x** $-$ δ**x**) and (**y**, δ**y** $-$ δ**y**), that is to say the displacements (**x**, 0) and (**y**, 0), will also be equivalent. The plane $L^{(k)}(\mathbf{y})$ will therefore be of zero, positive, or negative choice according as the equivalent plane $L^{(k)}(\mathbf{x})$ is of zero, positive, or negative choice. On the other hand, if (**x** δ**x**) and (**y** δ**y**) are equivalent, we conclude that (**x**, $k\delta$**x**) and (**y**, $k\delta$**y**) will also be equivalent, k being a finite factor, which shows that the distances of the planes $L^{(k)}(\mathbf{y})$ of positive choice are proportional to the distances of the planes $L^{(k)}(\mathbf{x})$ of positive choice, and analogously for planes of negative choice. To give the complete description of choices between displacements around **x** and displacements around **y**, it suffices therefore to add to the data on the maximum direction at **x** and **y** the proportionality coefficient c which expresses the density of the planes $L^{(k)}(\mathbf{y})$ relative to the density of the planes $L^{(k)}(\mathbf{x})$. This coefficient being given, if $L^{(k)}(\mathbf{x})$ is a given plane parallel to $L(\mathbf{x})$ at an infinitesimal (positive or negative) distance $p^{(k)}$, the displacements $\delta\mathbf{y}$ which are equivalent to the displacements $\delta\mathbf{x}$ ending on $L^{(k)}(\mathbf{x})$, will be the displacements ending on the plane $L^{(k)}(\mathbf{y})$ whose distance from $\mathbf{y}$ is $q^{(k)} = cp^{(k)}$.

Consider now a third point $\mathbf{z}$ and let $c_{\mathbf{xy}}$ be the proportionality coefficient of $\mathbf{y}$ relative to $\mathbf{x}$, $c_{\mathbf{xz}}$ the coefficient of $\mathbf{z}$ relative to $\mathbf{x}$, and $c_{\mathbf{yz}}$ the coefficient of $\mathbf{z}$ relative to $\mathbf{y}$.

Consider the three planes $L^{(k)}(\mathbf{x})$, $L^{(k)}(\mathbf{y})$, $L^{(k)}(\mathbf{z})$ at distances p, q, and r from $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ respectively. If on the one hand the first two planes are equivalent, and on the other the last two, the first and the last will also be equivalent by virtue of Axiom (IIb). We have

$$c_{\mathbf{xy}} = \frac{q}{p}, \qquad c_{\mathbf{xz}} = \frac{r}{p}, \qquad c_{\mathbf{yz}} = \frac{r}{q},$$

whence

$$c_{\mathbf{yz}} = \frac{c_{\mathbf{xz}}}{c_{\mathbf{xy}}}.$$

Thus, to know the coefficient of an arbitrary point $\mathbf{z}$ relative to another arbitrary point $\mathbf{y}$, it suffices to know the coefficients of $\mathbf{z}$ and of $\mathbf{y}$ relative to a fixed point $\mathbf{x}$.

To give the complete description of individual choices throughout the entire space, and for arbitrary infinitesimal displacements, it suffices then to adjoin to the datum on the maximum direction, for each point in the space, the datum of the coefficient c relative to a fixed point.

Let us alter the notation slightly. Let $\mathbf{a}$ be the fixed point, $\mathbf{x}$ and $\mathbf{y}$ two arbitrary points, $c_{\mathbf{x}}$ and $c_{\mathbf{y}}$ the coefficients of $\mathbf{x}$ and $\mathbf{y}$ relative to the fixed point $\mathbf{a}$. Let p be an arbitrary finite factor, positive or negative. Then the displacements $(\mathbf{y}\ \delta\mathbf{y})$ which end on the $L^{(k)}(\mathbf{y})$ whose distance from $\mathbf{y}$ is $pc_{\mathbf{y}}$—and those only—are equivalent to the displacements $(\mathbf{x}\ \delta\mathbf{x})$ which end on the $L^{(k)}(\mathbf{x})$ whose distance from $\mathbf{x}$ is $pc_{\mathbf{x}}$—and to those only.

Such are the patterns of choice to which the axioms of the second kind lead.

Consider now the vector $\mathbf{u}(\mathbf{x})$ defined at every point in the space in the following manner. The positive direction of $\mathbf{u}(\mathbf{x})$ will be the maximum direction at the point $\mathbf{x}$, its length will be equal to the quotient [length of $\mathbf{u}(\mathbf{a})]/c_{\mathbf{x}}$; $\mathbf{a}$ is the fixed point relative to which $c_{\mathbf{x}}$ is defined. The length of $\mathbf{u}(\mathbf{a})$ is chosen arbitrarily. We see that the values of the inner product $\mathbf{u}(\mathbf{x})\ \delta\mathbf{x}$ are distributed *throughout the entire space* in a manner analogous to individual preferences. More precisely, if $(\mathbf{x}\ \delta\mathbf{x})$ and $(\mathbf{y}\ \delta\mathbf{y})$ are two arbitrary infinitesimal displacements, the choice will be

$$(\mathbf{x}\ \delta\mathbf{x}) \gtreqless (\mathbf{y}\ \delta\mathbf{y})$$

according as

$$\mathbf{u}(\mathbf{x})\ \delta\mathbf{x} \gtreqless \mathbf{u}(\mathbf{y})\ \delta\mathbf{y}.$$

This justifies the following definition.

The inner product $\mathbf{u}(\mathbf{x})\,\delta\mathbf{x}$ will be the utility of the displacement $(\mathbf{x}, \delta\mathbf{x})$. The vector $\mathbf{u}(\mathbf{x})$ will be the marginal utility of the resources $\mathbf{x}$, and the components $u_1, u_2, \ldots, u_M$ of $\mathbf{u}(\mathbf{x})$ will be the marginal utilities of the goods $1, 2, \ldots, M$. The vector field so defined will be called the *choice field* of the individual considered.

Since it is in principle possible to determine "experimentally" for every point $\mathbf{x}$ of the space a plane $L^{(k)}(\mathbf{x})$ equivalent to a given plane $L^{(k)}(\mathbf{a})$, the constant $c_{\mathbf{x}}$ is capable of being objectively determined; the same therefore applies to the vector $\mathbf{u}(\mathbf{x})$.

We see the difference between the definition set forth here and the definition set forth above only with the aid of the axioms of the first kind. There we defined only the ratio of two infinitesimal [marginal[5]] utilities with reference to one and the same position, whereas here we have defined the ratio of two arbitrary infinitesimal [marginal[5]] utilities. It is only the latter definition that constitutes utility as a quantity. The fixing of the arbitrary factor which measures the length of $\mathbf{u}(\mathbf{a})$ is a question of fixing the unit of measurement for marginal utility. This factor is constant for all points of the field considered (which distinguishes it from the factor which figures in the first version of the definition). But it may vary from one individual's field to another's.

The definition adopted is therefore not universal in the sense that it permits the measurement of marginal utilities relating to different individuals. Each individual's choice field is affected by a proportionality coefficient that we have not defined, and which would probably be impossible to define in any objective manner. This fact is analogous for example to the comparison of the sense of color among different individuals. It is an objective fact that individuals who are not afflicted by color blindness are always in agreement on the question of knowing whether in a given situation the right vocabulary to employ is "red" or "green", and even on the question of knowing how to distinguish what is harmony from what is not in the world of colors. But there is no objective fact which permits one to exclude the hypothesis that the psychological state that with A is always associated with the word "red" is precisely the state that with B is associated with the word "green".

This lack of universality in the objective definition of marginal utility is nevertheless not essential to the aim we pursue. What matters for us is to have defined the field of choice of a specific individual.

According to the definition adopted, marginal utility is in short nothing else but a coefficient of contingent choice. If one offers an individual a quantity δx_i of good i in return for a quantity δx_j of good j, the individual will accept or not according as the relative price

$$\frac{\delta x_j}{\delta x_i} \lesseqgtr \frac{u_i}{u_j}.$$

It follows that at the moment when equilibrium actually occurs in the market our

[5] [Added in translation. Ed.]

individual forms part of, he will find himself at a point in his field of choice where the components of **u**(x) are proportional to the prices. Since marginal utility is nothing but a choice coefficient, one could just as well designate it by this name. This would provide a safeguard against sterile discussions about the "cause" of value and analogous questions. The usefulness of such a safeguard manifests itself quite often. We shall employ the term choice coefficient synonymously with marginal utility, while as a rule retaining the latter term. One might also envisage other terms such as Mr. Pareto's *ophelimity* or the *desirability* introduced by Mr. Gide and recently adopted by Mr. Fisher.

Starting from the definition of marginal utility one might seek a definition of total utility by considering the integral of **u**(x) taken along the consumption path. Nevertheless such a passage from marginal utility to total utility is not as simple as it might seem at first glance. In particular, one must not only distinguish two cases according as **u**(x) is or is not the derivative of a potential. One must also make more delicate distinctions. But since the question of total utility is not essential for what follows, we do not dwell on it further.

2. THE MARGINAL UTILITY CURVES

Consider more closely the variation in a single component u_m of **u**, that is to say, the variation in the marginal utility of good m. It is a function of the vector **x**. In the case in which u_m depends only on the component x_m of **x** we say that good m is *independent* of the other goods. In this case we can obviously represent the variation in u_m by a plane curve $u_m = f(x_m)$ whose ordinate is u_m and abscissa x_m. It is the marginal utility curve. Attempts have sometimes been made to provide a quantitative definition of marginal utility based solely on the notion of maximum direction, but with the supplementary hypothesis that all goods are independent.

It seems to me that this is a vicious circle, because one cannot provide an objective definition of what is to be understood by independent goods without first providing a quantitative definition of marginal utility. In fact, the single datum of maximum direction is not sufficient to discern whether or not we have a case of independent goods. True, the change in the maximum direction must satisfy a necessary condition, which is the integrability condition of the differential equation $\mathbf{u}\,\delta\mathbf{x} = 0$, or what amounts to the same thing: the condition for which the maximum direction be the direction of a vector which is the derivative of a potential. But this is a necessary condition, and not a sufficient one for the independence of goods.

On the other hand if goods are independent and **u**(x) is the corresponding vector, the vector $\psi(\mathbf{x})\mathbf{u}(\mathbf{x})$, where $\psi(\mathbf{x})$ is an arbitrary function of **x**, corresponds to nonindependent goods for which nevertheless the maximum direction is everywhere the same. This is why it has seemed to me necessary to seek a quantitative definition of marginal utility by taking the general point of view adopted in

Section 1. In what follows I abandon the general point of view to take up the case of independent goods.

Instead of $u_m = f(x_m)$ I shall employ the abbreviated notation $u = f(x)$ to indicate that the marginal utility of the good under consideration is a function of the quantity x of this good.

It is often necessary to characterize the increase (or decrease) of $f(x)$ relative to the increase of x, whether at a given point or in a prescribed interval. I assume that $f(x)$ has a well defined value and possesses a derivative in the interval under consideration. The derivative $f'(x)$ cannot by itself provide information about the nature of the increase (or decrease) of marginal utility, since it changes with a change in the unit of measurement whether of the quantity x or of the marginal utility. It is clearly possible to construct an infinity of functions derived from $f(x)$ characterizing the increase of $f(x)$ and whose value is unchanged by a change in the unit of measurement. Such for example are the functions

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{f(x)} \cdot \frac{x}{h} = \frac{f'(x)x}{f(x)}$$

and

$$\lim_{h\to 0} \left[\frac{f(x+h)}{f(x)}\right]^{x/h} = e^{f'(x)x/f(x)}$$

I consider in particular the function $f'(x)x/f(x)$ which will be called the *percentage increase* of $f(x)$. We designate it by $'f(x)$. Its negative value $-'f(x)$ will be called the *percentage decrease*. The reciprocal value $-1/'f(x)$ is sometimes called the elasticity coefficient. The function $'f(x)$ plays an important role in many problems of pure economics. It is obviously simply the logarithmic derivative $d \log f(x)/d \log x$. For positive x and $f(x)$ it is therefore represented geometrically by the slope of the tangent to the curve under consideration plotted in the logarithmic scale.

By analogy with successive derivatives one could define percentage increases of higher order. Assuming the existence of higher order derivatives one could for example consider the iterated percentage increases or else the functions $f^{(n)}(x)x^n/f(x)$ or again the functions ${}^{(n)}f(x) = d^n \log f(x)/d(\log x)^n$.

These functions are quite interesting from the point of view of applications since their values at a given point characterize the shape of the curve in the neighborhood of that point. Thus the values of the functions ${}^{(n)}f(x)/n!$ for $x = 1$ are clearly the coefficients of the development of $\log f(x)$ in powers of $\log x$.

In what follows I shall consider especially the marginal utility of the particular good, money.

For this it will be necessary to insist on an important distinction between two categories of economic goods: direct goods, which can be directly consumed, and indirect goods such as money whose sole object is to be exchanged for other goods.

The condition that would make it plausible to postulate the axioms of choice of Section 1 in the case in which some of the M goods are indirect goods, is obviously that the ratios in which individuals exchange indirect goods for direct ones be given.

Consider a list of exchange prices and define the *price vector* as the vector whose components are proportional to the prices appearing on the list, an arbitrary one of its components being chosen equal to unity. If some of the M goods do not appear on the list, we will for convenience define the corresponding components of the price vector to be equal to zero. We shall say that the price vector *encompasses* the goods appearing on the list.

That being the case, we can formulate the above remark as follows: In the case in which some of the M goods are indirect goods, the definition of the choice field implies the existence of a price vector that encompasses all the indirect goods and at least one direct good. To each price vector there corresponds a choice field. We can therefore consider the field as defined by the price vector under consideration, and we see that one can define the ratio of lengths of $\mathbf{u}$ for fields defined by different price vectors in the same way that one has defined the ratio of lengths of $\mathbf{u}$ for different positions in the same field.

Money is the indirect good *par excellence.* To define its marginal utility, one must therefore furnish not only the individual's position but also the price vector which defines the choice field. Let $\mathbf{r}$ be the individual's position, $\mathbf{z}$ the price vector. Then the marginal utility of money will be a function

$$u = \varphi(\mathbf{r}, \mathbf{z}).$$

Under this general form, the function φ depends on such a large number of variables that it is not manageable, at least if one proposes to arrive at precise conclusions that could be verified by concrete observation. One must therefore consider some hypotheses that will make it possible to simplify the form of φ. The two hypotheses that appear to me to be the most interesting from the economic point of view are the following.

First of all, if $\mathbf{r}$ remains constant and $\mathbf{z}$ encompasses all goods, I assume that the value of φ will be unchanged if the vector $\mathbf{z}$ is changed in such a way that the mean value of its components, which is the *price level*, stays constant. We know that there are several ways to define the price level. From the theoretical point of view the content of the hypothesis being considered will differ according to the index number construction adopted for the price level. We cannot dwell here on the various price index number constructions. We simply assume that the hypothesis we have adopted is the hypothesis that corresponds to the particular construction employed in the statistics that we will use later on. The diversity of price index number constructions does not have as great an importance from the practical as it does from the theoretical point of view.

The second hypothesis I adopt is the following. Given a constant price level, I assume that the value of φ is not altered if the position $\mathbf{r}$ is changed in such a way that the inner product $\mathbf{rz} = r_1 z_1 + r_2 z_2 + \cdots + r_M z_M$ stays constant. This expression is clearly nothing but the valuation in monetary units of the resources the individual disposes of during the unit of time, that is to say his *money income* earned per unit of time.

Granted these hypotheses, the marginal utility of money will be a function

$$u = \varphi(r, z)$$

of the money income r and the price level z.

By a simple transformation the function $\varphi(r, z)$ can be expressed as a function of a single variable.

In fact, according to the axioms adopted the individual's choice cannot be affected by a change in the monetary unit. He must be indifferent, for instance, as to whether his income is counted in dollars or cents. It follows that the function $\varphi(r, z)$ must satisfy the functional equation

$$k\varphi(kr, kz) = \varphi(r, z),$$

k being an arbitrary factor.

Assuming that the partial derivatives of φ exist, we obtain by differentiating with respect to k and then setting $k = 1$,

$$r\frac{\partial \varphi}{\partial r} + z\frac{\partial \varphi}{\partial z} + \varphi = 0.$$

The solution of this partial differential equation is, as is well-known,

$$\varphi(r, z) = \frac{1}{z}\,\Phi\left(\frac{r}{z}\right),$$

Φ being an arbitrary function. To determine it, set $z = z_0$, z_0 being a given price level, for instance the level serving as base in the index number construction. This gives

$$\frac{1}{z_0}\,\Phi\left(\frac{r}{z_0}\right) = \varphi(r, z_0) = g(r)$$

whence

$$\varphi(r, z) = \frac{1}{(z/z_0)}\,g\left(\frac{r}{(z/z_0)}\right),$$

$g(r)$ being the function that defines the variation in the marginal utility of money at the price level z_0.

Let us alter the notation slightly. Denote the *relative* price level z/z_0 by z, and let $\varphi(r, z)$ denote the marginal utility of money at the relative level z. Then we shall have

$$u = \varphi(r, z) = \frac{1}{z}\,g\left(\frac{r}{z}\right),$$

a formula which we can, moreover, also deduce directly without using the differential equation.

The marginal utility of money corresponding to any money income and price level is therefore known if one knows the variation in marginal utility with income at a fixed price level, that is if one knows the marginal utility curve at a constant price level. This property of the function $\varphi(r, z)$ is of fundamental importance in applications.

The formula $\varphi(r, z) = (1/z)g(r/z)$ gives rise to a rather interesting property of the function $\varphi(r, z)$. For a simple calculation yields

$$_z'\varphi = (-_r'\varphi) - 1,$$

where $_r'\varphi$ and $_z'\varphi$ denote the percentage increase of φ with respect to r and z respectively.

If money income r is held constant, an infinitesimal increment in the price level z therefore has the effect of bringing about a rise or fall in the marginal utility of money according as the percentage decrease of marginal utility with respect to r (z being held constant) is larger or smaller than unity for the value of r under consideration.

The relation $_z'\varphi = (-_r'\varphi) - 1$ can be explained roughly as follows. If the variation of marginal utility with income (the price level being held constant) is such that a small positive increment in income of $h\%$ brings about a $qh\%$ diminution in marginal utility, so that the percentage decrease is equal to q, then a small increment in the price level of $k\%$ (money income held constant) will bring about a (positive or negative) increment in marginal utility of $(q - 1)k\%$. This increment is therefore all the larger (algebraically) the larger is the percentage decrease q. It is positive if $q > 1$ and negative if $q < 1$.

Mr. Birck, in his treatise *Laeren om Graensevaerdien* (Copenhagen, 1918, p. 117), has put forward the following proposition.

If the "speed" of decrease is "large" or "small," an increment in the price level will lead to a diminution in marginal utility, whereas the latter will be increased if the "speed" of decrease has a "medium" value.

Mr. Birck has not defined what he means by "speed" of decrease, but his reasoning shows that he had in mind the slope of the curve representing the variation in the marginal utility of money at a constant price level, the curve being drawn on an ordinary scale. As we have seen, the problem in question has nothing to do with the value of this coefficient.

Mr. Birck's erroneous proposition is accounted for by the fact that he has reasoned on a numerical example while confusing two distinct properties of the figures he has chosen. One of them is essential for the results of his calculations, which he has moreover carried out correctly. The other is not, and yet it is the latter which has attracted Mr. Birck's attention. The first property is the behavior of the curve drawn on the logarithmic scale, the other is the behavior of the curve drawn on the ordinary scale. Mr. Birck's numerical example is by accident such that $\partial \log \varphi/\partial \log r$ is less than unity in the interval in which $\partial\varphi/\partial r$ is "large" as well as in the interval in which $\partial\varphi/\partial r$ is "small," whereas $\partial \log \varphi/\partial \log r$ is larger than unity in the interval in which $\partial\varphi/\partial r$ has a "medium" value. Had Mr. Birck drawn his curve on the logarithmic scale, he would doubtless have formulated the proposition correctly.

The behavior of the curve drawn on the logarithmic scale shows something else as well, namely that Mr. Birck's numerical example is not appropriate for illustrating the variation in the marginal utility of money. There is reason to believe that in actuality the percentage decrease is larger than unity for small incomes but less than unity for large incomes, which leads to the conclusion that for the poor the marginal utility of money will be increased as a consequence of an increase in the price level (money income remaining constant), whereas for the rich it will be diminished. This is precisely the opposite conclusion to that obtained by Mr. Birck.

The preceding remarks show to what extent conclusions drawn from numerical examples can be erroneous in economics. Even in cases as elementary as the one we have considered it is indispensable to make use of the notation of mathematical analysis. That obviously does not prevent numerical examples—even though they cannot serve as a means of research—from maintaining a high value as a means of popularizing a theory once its correctness has been assured by means of a rigorous mathematical study.

There are numerous analogies between rational mechanics and pure economics. Thus the vector **u** plays a role in pure economics analogous to universal attraction in rational mechanics. But there are also essential differences. Concrete economic phenomena are too complex for it to be possible on the basis of *a priori* considerations to determine precisely the forms of the functions given by the components of **u**. There is no universal law of economic attraction as there is a universal law of gravitation. In particular, it is not possible to assign *a priori* a well-determined form to the function $g(r)$. However, this does not mean that it must be considered to be a completely arbitrary function.

On the basis of economic considerations which we cannot dwell on here, we arrive at the conclusion that the function $g(r)$ must satisfy the following conditions.

(1) There exists a positive number a (the minimum subsistence at the given price level z_0) such that $g(r) > 0$ and possesses derivatives of first and second order for $a < r < \infty$.

(2A) $\lim_{r \to a} g(r) = \infty$. (2B) $\lim_{r \to \infty} g(r) = 0$.

(3) $\frac{d}{dr} g(r) = g'(r) < 0$ in the interval $a < r < \infty$.

(4) The percentage decrease $-'g(r) = -g'(r)r/g(r)$ is larger than unity for sufficiently small values of $r\ (> a)$.

(5) $\lim_{r \to \infty} 'g(r) = 0$.

These conditions restrict considerably the choice of empirical interpolation formulas that one might try out.

Formulas such as $'g(r) = O(r^{-k})$ where k is an arbitrary positive number must be excluded, for example. In fact we have

$$\log g(R) = \int_b^R 'g(r)r^k \frac{dr}{r^{1+k}} + \text{constant}$$

where b is an arbitrary number $> a$.

By virtue of condition (1), $|'g(r)r^k|$ is bounded in all finite intervals starting at $r = b$, and also from $r = b$ up to $r = \infty$ provided $'g(r) = O(r^{-k})$. In this case we therefore have from $R = b$ up to $R = \infty$

$$|\log g(r)| \lesseqgtr M\frac{(1/b)^k - (1/R)^k}{k} + \text{constant},$$

where M is a finite number.

This shows that $\text{Lim}_{R\to\infty} |\log g(r)|$ cannot be infinite and consequently that condition (2B) cannot be satisfied.

In order that this condition be satisfied it is necessary that

$$\lim_{r\to\infty} |'g(r)r^k| = \infty$$

and this must hold for arbitrarily small $k\,(>0)$.

We see in particular that a function $g(r)$ whose percentage increase has the form

$$'g(r) = Re^Q,$$

where R and Q are rational functions in r, cannot satisfy (2B) and (5) simultaneously; the same is true *a fortiori* of a function $g(r)$ which can itself be expressed in the form $g(r) = Re^Q$ since its percentage increase is a rational function.

We conclude that the function $g(r) = \text{constant}/(r - a)$ to which Daniel Bernoulli's[6] hypothesis leads must be rejected, and the same applies to the function $g(r) = \text{constant}/(r - a)^2$ recently introduced by Mr. Jordan.[7]

A function which satisfies all the assumed conditions provided the constants are appropriately chosen is the function

$$g(r) = Re^Q,$$

where R and Q are rational functions in $\log r$.

To see this we can for example set

$$Q = 0,$$

$$R = \frac{A}{B},$$

[6] "Specimen Theoriae novae de Mensura Sortis," *Commentarii academiae scientiarum imperialis Petropolitanae*. Tom. V (1738), p. 175. German translation by Pringsheim, Leipzig (1896). [English translation: "Exposition of a New Theory on the Measurement of Risk," *Econometrica*, 22, 1954, 23–36. Ed.]

[7] *The American Mathematical Monthly*. Vol. XXXI, No. 4 (1924).

where A and B are polynomials in $\log r$

$$A = \sum_{\nu=0}^{m} A_\nu(\log r)^\nu \qquad B = \sum_{\nu=0}^{n} B_\nu(\log r)^\nu \qquad (m < n)$$

whose coefficients are nonnegative except for B_0 which we set negative and $>$

$$-\sum_{\nu=1}^{n} B_\nu.$$

Since B is negative for $\log r = 0$ and positive for $\log r = 1$, the equation $B = 0$ has a root $\log r = \log a$ between 0 and 1. Moreover it is the only real and positive root of this equation, B being monotone increasing for positive $\log r$. Since a positive value of $\log r$ cannot make A vanish, we see that $\text{Lim}_{r \to a}\, g(r) = \infty$. Moreover $g(r)$ is always positive for $r > a$ and we have $\text{Lim}_{r \to \infty}\, g(r) = 0$, m being $< n$. Conditions (1) and (2) are therefore satisfied. On the other hand we have $g'(r) = -C/rB^2$, C being the polynomial in $\log r$,

$$C = \sum_{k=1}^{m+n} (\log r)^{k-1} \sum_{\nu=0}^{[(k-1)/2]} (k - 2\nu)(A_\nu B_{k-\nu} - A_{k-\nu}B_\nu).$$

This expression is certainly positive if all the A_ν's are chosen nonzero and also such that the sequence B_ν/A_ν ($\nu = 0, 1, \ldots$) forms a nondecreasing sequence.

The percentage decrease is equal to $-'g(r) = C/AB$. This expression can be made arbitrarily large by making the difference $\log r - \log a$ sufficiently small, because $\log a$ is a root of $B = 0$, and C, having all its coefficients nonnegative, is positive for $\log r = \log a$. Finally since C is of degree $m + n - 1$ whereas A and B are of degrees m and n respectively, we see that $\text{Lim}_{r \to \infty}\, 'g(r) = 0$. All the conditions (1)–(5) are therefore satisfied.

The simplest form that we could employ as interpolation formula for the marginal utility of money is therefore the formula

$$(1) \qquad g(r) = \frac{\text{constant}}{\log r - \log a}.$$

This is Bernoulli's formula with the exception that r has been replaced by $\log r$. The function (1) with the constant set equal to unity has among other things the interesting property of being equal to its percentage increase. I believe that one may arrive at interesting results if it is taken as the point of departure of a new theory of moral expectation.

3. THE METHOD OF ISOQUANTS

What makes it possible to establish a link between the abstract theory of pure economics and concrete economic phenomena is the fact which was mentioned in Section 1: The proportionality between prices and marginal utilities at the point of market equilibrium.

This proportionality is the cornerstone of all investigations whose aim is to reveal the properties of concrete fields of choice existing in the economic world.[8] Every point of equilibrium that we have a chance to observe provides some indication of the nature of the choice field in the neighborhood of the point corresponding to the observed equilibrium.

It is obviously an impossible task to study the choice fields of every individual participating in a given market. It would be a task comparable to studying the trajectories of individual molecules in a gaseous mass. But in the economic world as in the world of molecules, what is important is not to provide individual description of the elements, but to arrive at the knowledge of certain *average* properties that characterize the set of elements.

What matters in pure economics is not knowing the behavior of the personalities that form part of a given market, but knowing the behavior of the typical individual. To investigate the behavior of this typical individual, pure economics cannot, like the physical sciences, have recourse to laboratory experiments, but to make up for this it has at its disposal an enormous body of statistical observation.

The utilization of this material can be intensive or extensive. In the first case one seeks to restrict as far as possible the category of economic individuals entering into the analysis. One considers a well-defined economic milieu: a particular region, a certain social class, a sufficiently limited interval of time so that it may be assumed that the structure of the choice field has not changed, and so on. The averages that result are particular averages.

In the second case one seeks "second-order" means, that is averages with respect to all classes, an entire country or even averages characterizing the world market. The results that one can obtain by this latter method may perhaps be the most interesting from the point of view of the immediate use that public administration can make of it, but the results obtained by the first method are clearly more interesting from the theoretical point of view. The day that economic research has developed so far that one can think of reconstructing the global phenomenon with the partial phenomena studied by the intensive method, this latter will have proved its superiority even from the point of view of applications. The viewpoint we adopt in the present paragraph and the next is above all the viewpoint of the intensive method.

In economics as in medicine, it is often the extreme or even abnormal manifestations which can best serve to shed light on the law obeyed by the phenomenon studied. To determine the marginal utility curve it is important to know not only

[8] This is a problem in market statics. The study of the manner in which this equilibrium changes over time—whether it changes more or less rapidly, and so on—is a problem in market dynamics. This is obviously a much more complex problem. I employ the terms static and dynamic here in a sense analogous to that in which the terms are employed in rational mechanics. Certain economic authors use these terms in other senses which may easily lead to misunderstandings. Thus Mr. Schultz in his fine statistical works in the *Journal of Political Economy*, 33, 1925, 481–504, 577–637, uses (p. 501) the terms "static" and "dynamic" according as the number of variables introduced into the study of static equilibrium is small or large.

a number of points in the neighborhood of the point corresponding to "normal" economic equilibrium, but also some points further away from the "normal."

It is here that international politics have come to the aid of the economist in carrying out the vast economic experiment of the world war. Granted that this same upheaval that has given rise to these great fluctuations in statistical data needed by pure economics has also made analysis of the material very difficult. But this is a difficulty inherent in the problem, and the economic statistician has only to seek to overcome it by doubling precautions against error and by seeking as far as possible to check the results obtained by independent methods. I believe that the systematic utilization of this vast material to the benefit of pure economics —utilization which has hardly begun—will yield very important results.

In the present paragraph and the next we will seek to use a certain category of this material to study the marginal utility of money.

Let $f(x)$ be the function which expresses the marginal utility of an independent good as a function of the quantity x which the individual has at his disposal during a unit of time.

Let

$$\frac{1}{z} g\left(\frac{r}{z}\right) = \frac{1}{z} g(\rho)$$

be the function determining the marginal utility of money, where z denotes the relative price level, r the money income expressed as a sum of monetary units, and $\rho = r/z$ the real income. Finally let p be the price of the particular good considered expressed in monetary units, $v = p/z$ the relative price and $w = 1/v$ its reciprocal value.

Then we have at the point of market equilibrium

$$wf(x) = g(\rho).$$

This relation among the variables x, ρ, and w defines a surface which we call the *consumption surface.*

Consider x to be the dependent variable, and project the level curves of the surface on the (ρw) plane. We will call these curves *isoquants*, because each of them corresponds to a fixed quantity of the good under consideration.

From the relation $wf = g$ we obtain

$$wf'(x)\frac{\partial x}{\partial \rho} = g'(\rho)$$

$$w^2 f'(x)\frac{\partial x}{\partial w} = -g(\rho).$$

If $f'(x)$ is negative, corresponding to the hypothesis of diminishing marginal utility of the good under consideration, and if moreover $f(0)$ is finite and $f(x)$ zero

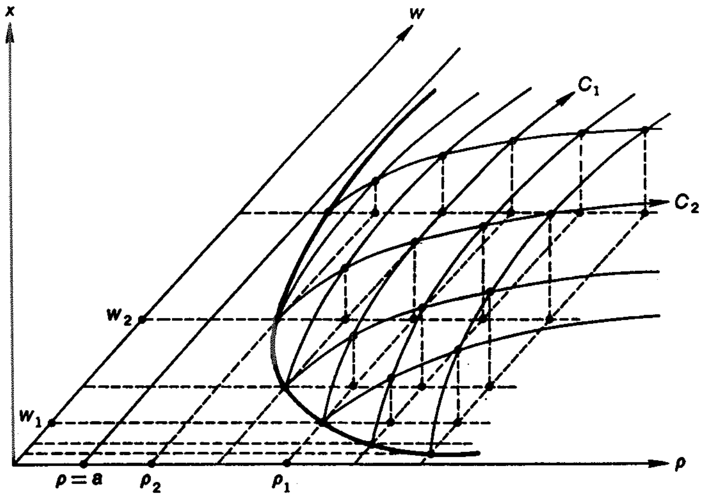

FIGURE 1

for a finite value x_0 of x (the point of satiety), the consumption surface will have the form indicated in Figure 1. Any curve of the system obtained by cutting the surface by planes parallel to the (wx) plane, is increasing with w, and any curve of the system parallel to the (ρx) plane is increasing with ρ. The asymptotes of the two systems are all situated in the plane $x = x_0$ in such a way that the portion of the surface corresponding to positive ρ and w are situated entirely between the planes $x = 0$ and $x = x_0$.

The economic interpretation of these properties is simple. Noting that x, ρ, and w are quantities that have reference to market equilibrium, we see that if real income ρ has a fixed value ρ_1, the individual will buy none of the good considered if w is very small, that is to say if the relative price is very large. He begins his purchases at $w = w_1$ and increases them successively following along the curve C_1 as the relative price falls (i.e., as w increases). On the other hand, if the relative price is a constant $1/w_2$, he buys nothing if his real income is smaller than ρ_2. Starting from ρ_2 he increases his purchases following along the curve C_2 as real income ρ increases.

Consider now the isoquants. Their equation is

$$w = \text{constant} \cdot g(\rho)$$

the constant being equal in turn to the value assumed by $1/f(x)$ when one sets $x =$ the constant quantity to which the isoquant in question corresponds.

Except for the constant and the notation, this is the equation of the curve

$$u = g(r)$$

which characterizes the variations in the marginal utility of money at the fixed price level z_0.

Abstracting from the unit of measurement of income and the unit of measurement of marginal utility (which can always be chosen arbitrarily) the curve sought is therefore given by any one of the isoquants. In Figure 1 we have drawn the isoquant corresponding to $x = 0$, as well as its asymptote $\rho = a$, a being the minimum subsistence level at the price level z_0.

The problem of determining the curve of marginal utility of money at the fixed price level z_0 is therefore reduced to the determination of the consumption surface, which is a problem in which the constancy of the price level does not enter.

Supposing that one has statistical data available from which one can draw n triples[9] of values $\rho_\nu w_\nu x_\nu$ ($\nu = 1, 2, \ldots, n$) corresponding to points on the consumption surface of the typical individual in the market under consideration. In order to infer the form of the surface from these n observations it is necessary as far as possible to guard against the two imperfections that always affect statistical data: their incompleteness and the irregularity in the covariance among the variables. This is the purpose of interpolation and curve fitting. In an analysis such as the present one it is often hard to distinguish what is interpolation and what is curve fitting. For this reason we distinguish the methods that we have to consider in terms of the character of the procedures rather than in terms of the aim we have in view.

The *direct method* consists in plotting the observed points $\rho_\nu w_\nu$ in a system of rectangular coordinates and assigning each point a height equal to the corresponding value of x_ν. If the irregularity of the data is not too great, the graph so obtained will already give some idea of the shape of the isoquants. As a first approximation one could draw them free-hand. In this case it would be preferable to employ the logarithmic scale along the w axis because the isoquants will then have the same form and will be distinguished only by a translation in the direction of the w axis. If necessary one could construct a stereometric model and perform a graphical fit with the help of a curve cut out from a loose sheet, the w axis carrying of course a logarithmic scale.

However, more precise results are obtained by employing the *time interpolation* method. First one traces the three polygonal lines representing the variation of ρ, w, and x with time. The scale must be large enough to permit graphical linear interpolation. If ρ, w, and x are given monthly, one could if necessary make a mechanical fit by replacing the observation for each month by the mean for the trimester that the month is in the middle of. Then one draws a straight line $x =$ constant $= x_1$ and notes the pairs of values ρ, w corresponding to the intersection of the straight line $x = x_1$ with the polygonal line which represents the variation of x with time. In this way one obtains a sequence of values ρ, w characterizing the isoquant corresponding to $x = x_1$.

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[9] [Translator's note: The word "triple" has been substituted for the word "couple" of the original.]

In this manner it is possible to determine different isoquants independently from one another and to check whether the results are in agreement in the sense that the values of w given by one of the isoquants are proportional to the values of w given by another.

The best method is perhaps *analytic fitting*. It consists in adopting certain forms for the functions $f(x)$ and $g(r)$ and determining the parameters that enter into them by means for instance of the method of least squares.

In the present case where we have certain *a priori* information on the nature of the function $g(r)$, which must satisfy the conditions listed in Section 2, it seems that the analytic method must be able to lead to interesting results. In the following section we shall employ this method and shall check the results obtained by showing that the points (ρw) determined by the time interpolation method group themselves distinctly around the isoquants determined by the method of least squares.

4. APPLICATIONS TO STATISTICAL DATA

The great cooperative association "l'Union des Coopérateurs" of Paris has for several years organized a statistical service which records in a continuous fashion the outflows of various articles sold at retail in the many branches of the company, as well as the prices of these articles and other statistical data which interest the Administrative Council.

This very detailed material has a high value as a means of studying the concrete properties of the fields of choice among the company's clientele. The details of this material are not published, but the management was kind enough to furnish me all the data I needed. I express herewith my hearty thanks. The data utilized in the present work are the following:

Quantity of sugar sold,
Price of sugar,
Sales turnover,
Size of the membership,
Cost of living index calculated by the company's statistical service (arithmetic mean with fixed weights).

All these data are monthly. I shall use them only starting from June 1920. The principal reason is that beginning on June 1, 1920, the "Union des Coopérateurs Parisiens" merged with the company, considerably increasing the turnover. I do not believe that sugar rationing has appreciably hindered the free play of the market during the period in question. Starting on August 17, 1920, rationing was limited to a very restricted category of consumers (children, etc.) but already by the decrees of February 1 and October 10, 1919, it was rendered far less severe than during the war.

As a check I have used a good deal of data furnished by the Statistique Générale de France. I wish to thank Mr. de Bernonville for imparting some unpublished data relating to the Paris duty[10] and workers' family budgets.

I cannot dwell here on the preliminary work (comparison of different statistical series that were available to me, critical examination of the data, etc.) which led me to choose the data listed below for a more thorough analysis. Nor can I go into the details concerning the relatively lengthy calculations that had to be carried out. I will limit myself to summarizing the principal results.

The first reduction of raw data consisted in the following. First I eliminated the annual periodic fluctuations by Persons' method[11] which can be described as follows. If one has at one's disposal a series of monthly data, the observations are replaced by their logarithms. Let y_ν be the logarithm of the νth observation, the data being numbered in such a way that the index ν of the observations corresponding to the months of January, February, and so on, are respectively $\equiv 1, 2, \ldots$ (mod 12). The sequence of differences $\Delta y_\nu = y_{\nu+1} - y_\nu$ is formed and one calculates the arithmetic mean G_1 of the Δy_ν ($\nu \equiv 1$), the mean G_2 of the Δy_ν ($\nu \equiv 2$), and so on. One then forms the sequence $\bar{G}_\mu = G_\mu - \sum_{i=1}^{12} G_i$ ($\mu = 1, 2, \ldots, 12$) and calculates the sums $K_\mu = \sum_{i=1}^{\mu} \bar{G}_i$. As a check one has $K_{12} = 0$. Finally one sets

$$Y_1 = -\frac{1}{12}\sum_{\mu=1}^{12} K_\mu, \quad Y_\mu = K_{\mu-1} + Y_1 \qquad (\mu = 2, 3, \ldots, 12).$$

Then the sequence $Y_1, Y_2, \ldots, Y_{12}$ is the sequence of logarithms in the indexes of periodic fluctuations. If $\eta_1, \eta_2, \ldots, \eta_{12}$ are the corresponding numbers, the elimination of periodic fluctuations consists in dividing the observations for January, February, etc., by $\eta_1, \eta_2, \ldots$ etc.

Since the raw data on membership size refer to the last day of the month, a linear interpolation was made in order to ascertain the size corresponding to the middle of the month.

After carrying out these reductions, I took the ratio of sales turnover to membership size as an index of variation in the monthly money income r of the typical individual in the market. The quotient (r/cost of living index) then becomes proportional to the variable ρ arising in the isoquant method. For the cost of living index I used the one calculated by the Union des Coopérateurs. The variable w of the isoquant method becomes proportional to the quotient (cost of living index/price of sugar), and the variable x proportional to the quotient (sugar outflow/membership). Choosing appropriate units of measurement for ρ, w, and x, the variables may be considered to be not only proportionate but equal to the calculated quotients. Table 1 shows the results of these calculations.

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[10] [Translator's note: The French word is *octroi*. Professor Frisch explains: "Octroi simply means duties on import of goods (primarily consumed goods) *into a city* from the surrounding country. In other words it is simply import duties in the usual sense, except that the frontier is the city limits, not the national border."]

[11] "Indices of General Business Conditions," *Review of Economic Statistics*, **1**, 1919, 5–205.

Table 1

VARIATION IN SUGAR CONSUMPTION WITH VARIATIONS IN PRICE OF SUGAR AND REAL INCOME

		Real income	Reciprocal of relative price of sugar	Quantity of sugar consumed
		p	w	x
1920	June	376	756	2,710
	July	374	749	1,750
	August	385	742	2,976
	September	307	794	2,015
	October	273	837	1,115
	November	252	1,018	1,164
	December	226	1,266	928
1921	January	232	1,223	940
	February	226	1,162	710
	March	240	1,100	850
	April	253	978	1,223
	May	277	975	1,181
	June	262	1,081	1,510
	July	269	953	1,971
	August	254	944	1,602
	September	237	1,040	1,266
	October	245	1,228	1,832
	November	254	1,208	1,905
	December	258	1,196	2,000
1922	January	255	1,221	1,652
	February	268	1,200	1,728
	March	281	1,130	1,870
	April	273	1,125	1,710
	May	259	1,218	1,780
	June	243	1,178	1,962
	July	252	1,117	1,749
	August	252	1,070	1,823
	September	263	1,096	1,778
	October	253	1,197	1,900
	November	259	1,160	1,945
	December	273	1,092	1,955

SOURCE: *L'Union des Coopérateurs Parisiens* and *Statistique Générale de France.*

I at first thought that perhaps the response of quantity sold to a change in price would not manifest itself in the same month as the price change but in the month following. Nevertheless the calculation of (zero and first order) correlation coefficients for the different lags between the x and w series showed that the maximum correlation was obtained when there was no lag. This shows moreover that it is plausible to consider the quantities sold as quantities effectively consumed.

For a first orientation I then carried out a time interpolation for $x = 1250$,

$x = 1500$, and $x = 1750$ by the method indicated in Section 3. In each of the three series of ρ, w so obtained I fitted an isoquant by the method of least squares, using for the marginal utility of money the function $g(\rho) = \text{constant}/(\log \rho - \log a)$ of Section 2. Using this function the constants to be determined are the constants $\log a$ and c which enter linearly in the relation

$$\log \rho = c\left(\frac{1}{w}\right) + \log a.$$

Theoretically one should expect to obtain the same values of $\log a$ for the three isoquants considered (the values of the constant c would of course be different). The results were also rather close. Nevertheless there was a small difference. To account for it I compared the results obtained by treating $\log \rho$ and $1/w$ in turn as independent variable, that is by minimizing in the first case the sum of deviations $\sum (\log \rho - (c(1/w) + \log a))^2$ and in the second case the sum of deviations $\sum (1/w - (1/c \log \rho - (\log a)/c))^2$. It turned out that the discrepancy in the results obtained for different isoquants was not as great as the discrepancy in results obtained by treating $\log \rho$ and $1/w$ in turn as dependent variable. Therefore in comparing the errors introduced on the one hand by the arbitrariness in the choice of independent variable and on the other by the lack of proportionality of the three isoquants we find that the first error is the larger. This fact pointed out to me the necessity of employing a mean regression line with the variables ρ, w, and x treated symmetrically. I tried the Reed–Pearson[12] mean regression line but the result was not satisfactory. I was therefore led to consider the mean regression problem in a more general fashion. I settled on the following method.

Consider m variables $x_1, x_2, \ldots, x_m$ for which $n(>m)$ observations $x_1^{(\nu)}, x_2^{(\nu)}, \ldots, x_m^{(\nu)}$ $(\nu = 1, 2, \ldots, n)$ are available. Assume that all the variables are reckoned from their means as origin. Then

$$\sum_{\nu=1}^{n} x_j^{(\nu)} = 0 \qquad (j = 1, 2, \ldots, m).$$

Let

(2) $$\sum_{j=1}^{m} b_{ij} x_j = 0 \qquad (i = 1, 2, \ldots, m); \qquad b_{ii} = -1$$

be the m linear regressions determined by the method of least squares with the variable x_i $(i = 1, 2, \ldots, m)$ taken as dependent variable. For efficient calculation one could use Mr. Yule's[13] calculating scheme or Gauss' method of normal equations. Explicit expressions for the coefficients will be given below.

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[12] See Czuber, "Lineare Ausgleichung und Korrelation," *Archiv für die gesamte Psychologie*, 44, 1923, where one will find bibliographical references.

[13] *Proc. Roy. Soc.* Series A., 79, 1907, 182. See also Chapter XII of the same author's treatise *Theory of Statistics*.

The necessary and sufficient condition for the m equations obtained to be identical is clearly that $b_{ij} = \alpha_i \beta_j$ where the α_i and β_j $(i, j = 1, 2, \ldots, m)$ are two sequences of given nonzero numbers. Since $b_{ii} = -1$ we have $\alpha_i = -1/\beta_i$ whence $b_{ij} = -\beta_j/\beta_i$. From this we obtain the necessary condition

$$b_{ij} b_{jk} b_{ki} = -1 \qquad (i, j, k = 1, 2, \ldots, m) \tag{3}$$

whence in particular for $k = j$

$$b_{ij} b_{ji} = +1.$$

Condition (3) is also sufficient since if the coefficients of one of the equations, say the b_{hj}'s $(j = 1, 2, \ldots, m)$ of the hth equation, are given then

$$b_{ij} = -\frac{b_{hj}}{b_{hi}}$$

whence upon setting $b_{hj} = \beta_j$,

$$b_{ij} = -\frac{\beta_j}{\beta_i}.$$

Let $\varepsilon_{ij} = b_{ij}/|b_{ij}|$ be the sign of the coefficient b_{ij}. We see that if condition (3) is satisfied, the coefficients of any equation, say the b_{hj}'s of the hth equation, are proportional to $\varepsilon_{hj}\sqrt[m]{|b_{1j}, b_{2j}, \ldots, b_{mj}|}$. We can therefore replace all m equations by a single one, namely

$$\sum_{j=1}^{m} B_j x_j = 0, \tag{4}$$

where

$$B_j = \varepsilon_{hj}\sqrt[m]{|b_{1j}, b_{2j}, \ldots, b_{mj}|},$$

h being an arbitrary index whose choice is immaterial.

The idea then presents itself of replacing the m equations (2) by the single equation (4) even in the case in which condition (3) is not satisfied. For this it is obviously necessary to assume that equation (4) is independent of the choice of index h, that is, that the signs ε_{ij} are compatible. The necessary and sufficient condition for this is that

$$\varepsilon_{ij} \varepsilon_{jk} \varepsilon_{ki} = -1 \qquad (i, j, k = 1, 2, \ldots, m).$$

It should be noted that this procedure for determining a mean regression line is an entirely mechanical one which cannot be justified by *a priori* considerations. But this is a remark that can also be applied to the very application of the method of least squares to problems that do not fall within the proper scope of the theory of errors of observation.

The method can be generalized to the case where the signs are not compatible, but I shall not dwell on this because the material to which I shall apply the method is such that the signs are compatible.

Here is how one can obtain explicit expressions for the b coefficients by means of the second order moments of the given distribution.[14]

In terms of the usual notation

$$c_{ij} = [x_i x_j] = \sum_{\nu=1}^{n} x_i^{(\nu)} x_j^{(\nu)},$$

the condition for the m sums

$$\sum_{\nu=1}^{n} \left[\sum_{j=1}^{m} b_{ij} x_j^{(\nu)} \right]^2 \qquad (i = 1, 2, \ldots, m)$$

to be minimized can be expressed by the $m(m-1)$ equations

$$(5) \qquad \sum_{j=1}^{m} b_{ij} c_{kj} = 0 \qquad \begin{aligned} i &= 1, 2, \ldots, m \\ k &= 1, 2, \ldots (\neq i), \ldots m. \end{aligned}$$

But it is easy to see that these equations are satisfied by setting the b_{ij} proportional to the minors Δ_{ij} of the determinant

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ c_{m1} & c_{m2} & \cdots & c_{mm} \end{vmatrix}.$$

In fact, setting $b_{ij} = \alpha_i \Delta_{ij}$, the α_i being arbitrary, the left side of equations (5) will be

$$\alpha_i \sum_{j=1}^{m} c_{kj} \Delta_{ij} \qquad (k \neq i)$$

which vanishes according to a well known theorem.[15]

We may therefore write the m regression equations as

$$\sum_{j=1}^{m} \Delta_{ij} x_j = 0 \qquad (i = 1, 2, \ldots, m)$$

or else in the abridged form

$$\mathbf{A}^{-1}\mathbf{x} = 0,$$

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[14] [Translator's note: The reference is evidently to the empirical distribution since only sample moments are considered.]

[15] See for example G. Kowalewsky, *Determinantentheorie*, p. 41.

where A^{-1} denotes the reciprocal tensor[16] of the tensor A whose components are the elements of Δ, and x denotes the vector with components $x_1, x_2, \ldots, x_m$.

In the case of compatible signs the unique mean regression equation will be

$$(6) \qquad \sum_{j=1}^{m} \varepsilon_{hj} \sqrt[m]{|\Delta_{1j}, \Delta_{2j}, \ldots, \Delta_{mj}|}\, x_j = 0,$$

where ε_{hj} is the sign $\Delta_{hj}/|\Delta_{hj}|$ of Δ_{hj} (h arbitrary).

Applying the unique regression method to the three isoquants which provided the point of departure of the investigation I obtained results that were in good agreement. Having thus established the success of the method with the incomplete material relating to the isoquants $x = 1250$, $x = 1500$, and $x = 1750$, I applied it to the entire material.

I assumed as before that the marginal utility of money can be expressed by the interpolation formula $g(\rho) = c/(\log \rho - \log a)$.

For the marginal utility of sugar I adopted the formula $f(x) = b/(x + d)$, where b and d are constants. Introducing these functions in the equation of the consumption surface we have

$$x + \frac{b \log a}{c} w - \frac{b}{c} w \log \rho + d = 0,$$

that is,

$$x + Aw + B(w \log \rho) + C = 0,$$

where A, B, and C are the constants to be determined. The means of the three variables x, w, and $(w \log \rho)$ are, according to Table 1,

$$x_0 = 1660, \qquad w_0 = 1066, \qquad (w \log \rho)_0 = 2578,$$

respectively.

Take the means as origin and let

$$x_1 = x - x_0, \qquad x_2 = w - w_0, \qquad x_3 = (w \log \rho) - (w \log \rho)_0.$$

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[16] [Translator's note: For this terminology, see, for example, R. Courant and D. Hilbert, *Methods of Mathematical Physics*, New York: Interscience Publishers Inc., 1953, Vol. 1, p. 7). Evidently A is meant to refer to the matrix of which Δ is the determinant. The apparently contradictory equation $A^{-1}x = 0$ seems to deserve some explanation, for on the face of it, if it is to hold exactly it has only the trivial solution $x = 0$; but this would imply $\Delta = 0$ and the nonexistence of A^{-1}. Evidently the quantities Δ_{ij}, referred to as minors, are intended instead to denote the cofactors of A. Then the equations $\sum_{j=1}^{m} \Delta_{ij} x_j = 0$ can be represented in matrix form by $Gx = 0$ where G is the adjoint of A. Note that this situation is not really resolved if equation (2) is allowed to hold with error, for Δ would still be close to zero and A^{-1} would be ill-conditioned; indeed this point was very much stressed by Professor Frisch in his subsequent writings, "Correlation and Scatter in Statistical Variables," *Nordic Statistical Journal*, 1, 1929, 36–102, and *Statistical Confluence Analysis by Means of Complete Regression Systems*, Oslo: Universitetets Økonomiske Institutt, 1934. As is clear from (6), Professor Frisch used the adjoint rather that the inverse of A in making the calculations leading to (7) below.]

We are to determine the three linear regressions

$$\sum_{j=1}^{3} b_{ij}x_j = 0 \qquad (i = 1, 2, 3).$$

Calculating the second-order moments of the observed distribution we obtain in terms of the previous notation

$$\begin{aligned} c_{11} &= 7{,}458{,}890, & c_{12} &= -\ \ 786{,}768,\\ c_{22} &= \ \ 728{,}596, & c_{13} &= -1{,}366{,}711,\\ c_{33} &= 3{,}458{,}200, & c_{23} &= \ \ 1{,}580{,}193. \end{aligned}$$

The calculation shows that the signs of the coefficients b_{ij} are compatible. We may therefore proceed to the calculation of the unique regression

$$B_1x_1 + B_2x_2 + B_3x_3 = 0$$

with the help of formula (6). The result of this calculation is

$$x_1 + 35.75\,x_2 - 16.02\,x_3 = 0.$$

Returning finally to the variables x, w, and $\log \rho$ we obtain

$$w = \frac{(x + 1550)/16.02}{\log \rho - 2.232}. \tag{7}$$

This is the equation of the consumption surface. For x = constant = $\bar{x}$ it reduces to the equation of the isoquant corresponding to $x = \bar{x}$.

To check the result obtained I calculated for each month of Table 1 the value of x (consumption of sugar) given by formula (7). The comparison between theoretical and observed values of x revealed an interesting property of the x series. I found a very pronounced annual periodic fluctuation in the figures for the deviation of the observed value of x from the theoretical value, consumption being relatively small in the first half of the year and very large during the months June–September.

The question then arises as to whether one should eliminate these fluctuations as was done in the case of periodic fluctuations in sales turnover. It should be noted that a periodic fluctuation in the quantity of an article consumed can be due to two different factors. Either it is supply that varies with the season (vegetables in summer, etc.) or else it is the need, that is, the choice field itself varies (beverages in summer, fuel in winter, etc.). Fluctuations of the first kind should not be eliminated if the statistical data are used to study the variation of marginal utilities. On the contrary it is precisely the reaction of the price to these fluctuations that must be watched. On the other hand, fluctuations of the second kind must surely be eliminated since they correspond to fluctuations in a datum that we have assumed constant: the constitution of the choice field. In the present case it cannot be doubted that the observed fluctuations are of the second kind. The fact that they persist after account has been taken of price variability already shows them to be of the second kind.

To investigate the way in which the observed values are grouped around the

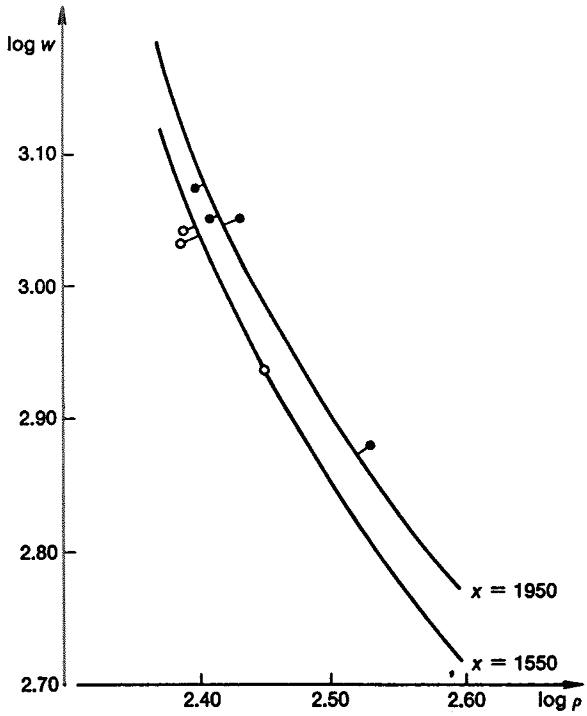

FIGURE 2

isoquants given by formula (7) it is therefore necessary to eliminate the periodic fluctuations in x. For the general shape of the isoquants given by the formula these fluctuations are probably not very important.

Having eliminated the periodic fluctuations in sugar consumption by the same method as that used for the sales turnover, I determined the observation points corresponding to the isoquants $x = 1550$, $x = 1750$, and $x = 1950$ by the time interpolation method. It was found that the observed points are grouped closely around the isoquants in question. Figure 2 gives some idea of this. It seems to me that the agreement of results obtained by such different methods provides a check of no small importance. So as not to clutter the diagram I have drawn only the isoquants $x = 1550$ and $x = 1950$ together with the corresponding observed points. The figure carries the logarithmic scale on both axes, so that the slope is equal to the percentage increase.

Table 2 shows the percentage variation in the marginal utility of money as well as the percentage decrease as a function of income r. The income percentage is relative to the mean income for the period in question. Values of r falling outside the interval of actually observed values are enclosed in parentheses.

Table 2

VARIATION IN MARGINAL UTILITY OF MONEY WITH VARIATIONS IN INCOME, PRICE LEVEL BEING HELD CONSTANT

Income	Marginal utility of money	Percentage decrease of marginal utility
Percent	Percent	
(75)	(284.0)	(6.40)
(80)	(201.0)	(4.52)
85	158.0	3.55
90	131.0	2.96
95	113.0	2.55
100	100.0	2.25
105	90.4	2.03
110	82.3	1.85
115	76.1	1.71
120	70.8	1.59
125	66.6	1.50
130	62.7	1.41
135	59.7	1.34
140	56.8	1.28
(145)	(54.3)	(1.22)
(150)	(52.3)	(1.18)

SOURCE: *L'Union des Coopérateurs Parisiens* and *Statistique Générale de France.*

It goes without saying that formula (7) also provides information about the marginal utility of the particular good (sugar) that we have taken as unit of comparison.

Before closing I shall indicate a few points which I believe would be interesting to focus on if one should wish to pursue the investigation more deeply.

First, it would be interesting to look for a correction for the observed difference between the sugar price recorded by the Union des Coopérateurs and the average retail price collected by the Statistique Générale de la France. It is true that most of the time the difference in question is minimal. But there are nevertheless some months when it rises to 5 percent or even higher. A difference of such magnitude certainly has the effect of producing an irregular deviation in the quantity x we are considering.

Secondly, it would be interesting to look for a correction for the trend which may have taken place in the relation between the total income of the typical member of a cooperative and the portion of income devoted to purchases in the cooperative. In the case in which this relation has not been appreciably constant, the index we have adopted for the variation in ρ should be corrected. Nevertheless

I do not believe that the correction to be made would be large for the relatively short period we have considered. It should be noted that a study of the relation in question would have to be combined with a study of the variation in the relation between sales to the public and sales to the membership.

Thirdly, it would be very interesting to introduce more commodities into the analysis. If the same result is obtained concerning the variation in the marginal utility of money when taking, for example, wine as the unit of comparison instead of sugar, this would obviously be a most reliable check. The introduction of a good such as meat would certainly lead to very great complications. First it would be necessary to distinguish several qualities, and then it would be necessary to drop the hypothesis of independent goods, since the marginal utility of meat is certainly influenced appreciably by, say, the consumption of fish.

It is to be hoped that there will be an ever increasing amount of research on utility curves as well as on the laws of production. I believe that economic theory has arrived at a point in its development where the appeal to quantitative empirical data has become more necessary than ever. At the same time its analyses have reached a degree of complexity that require the application of a more refined scientific method than that employed by the classical economists.

APPENDIX

INTEGRABILITY CONDITIONS AND THE EXISTENCE OF AN ORDINAL UTILITY INDICATOR[17]

In the theory of choice we must distinguish carefully between the choice *situation* (i.e., the things that are not subject to choice) and the choice *objects*

[17] Editor's note: This "appendix" is being reproduced here at Professor Frisch's suggestion. It originally appeared as the appendix (pp. 192–196) to his paper, "A Complete Schema for Computing All Direct and Cross Demand Elasticities in a Model with Many Sectors," *Econometrica*, 27, 1959, 177–196. It is reprinted with the kind permission of the Econometric Society.

Professor Frisch writes as follows (in a letter dated August 8, 1966): "I have often regretted that I presented this simply as an 'appendix' printed with small characters. This may have conveyed the idea to the econometric fraternity that this only pertains to some technical details in the paper to which it was appended.

"In fact this 'appendix' lays bare a very fundamental point concerning the meaning of the integrability conditions of an ordinal utility indicator. I have found several misunderstandings on this score and would be very happy if I could contribute to clearing away the fog in this field.

"The essence of the situation can be formulated by saying that the existence of an ordinal utility indicator is *sufficient* for integrability. But the reverse is not true: integrability is not sufficient to ensure the existence of an ordinal utility indicator.

"True enough, integrability is sufficient to ensure the existence of a *potential function*. But this potential function (or any monotonically increasing transformation of it) *may not* be an ordinal utility indicator for the marginal vector field in question. In the 'appendix' I have given a counter example."

Professor Frisch adds: "As the years have passed by I have not found any reason to change my fundamental views in this field."

(i.e., the things amongst which the individual can choose). Let the choice objects be points $x = (x_1, x_2, \ldots, x_n)$ in an n-dimensional object space.

Choice questions may relate to objects at large in the object space, or they may be marginal, that is, relate to objects that are only infinitesimally different.

Through marginal choice questions or, what amounts to the same, observations of the quantities demanded in a quantity-adapting market under different prices, it will in principle be possible to observe the *direction* of the vector $(u_1, u_2, \ldots, u_n)$ associated with each quantity point $(x_1, x_2, \ldots, x_n)$ by virtue of the equations

$$\text{(A1)} \qquad u_i = u_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

that define the marginal utilities $u = \partial U(x_1, x_2, \ldots, x_n)/\partial x_i$ as functions of the quantity point, and the tangency conditions

$$\frac{u_1}{p_1} = \frac{u_2}{p_2} = \cdots = \frac{u_n}{p_n},$$

where $p_1, p_2, \ldots, p_n$ are the prices at which the representative consumer can buy the goods. It is of considerable theoretical and some practical interest to clear up how much information about the utility indicator can be deduced from the knowledge of the *directional* field $(u_1, u_2, \ldots, u_n)$. I shall not go into a detailed analysis of the extensive literature on this question,[18] but shall simply state that on some fundamental points I believe the matter has not been treated satisfactorily. I shall indicate briefly one of these points.

Suppose that functions of the form (A1) are given and that no assumptions are made to the effect that these functions are the partial derivatives of some function $U(x_1, x_2, \ldots, x_n)$. The integrability condition for the vector field u_i can be written

$$\text{(A2)} \qquad u_k(u'_{ij} - u'_{ji}) - u_j(u'_{ik} - u'_{ki}) + u_i(u'_{jk} - u'_{kj}) = 0$$
$$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; k = 1, 2, \ldots, n),$$

where $u'_{ij} = \partial u_i/\partial x_j$. If there exists a function U such that u_i is its partial derivative with respect to x_i, then each of the three terms in (A2) is obviously zero; but for the moment u_i is not assumed to be a partial derivative. We have gone further in the direction of relaxing assumptions in that we have assumed only that the *direction* of the vector field u_i is given. Then (A2) expresses a condition on this direction. It is easily seen that if (A2) is fulfilled for the field u_i, it is also fulfilled for Ψu_i where Ψ is an arbitrary nonvanishing function of $x_1, x_2, \ldots, x_n$ with continuous partial derivatives. And vice versa.

It is a classical fact that when and only when (A2) is fulfilled, is it possible to find a function $U(x_1, x_2, \ldots, x_n)$ such that its partial derivatives, U_i, are everywhere proportional to the u_i. $U(x_1, x_2, \ldots, x_n)$ is then called a *potential*. The degree of indeterminateness in U—when only the *direction* of the field u_i is given—is expressed by the fact that we may perform on the potential U an arbitrary mono-

[18] A survey is given by Professor Samuelson in *Economica*, N.S., 17, 1950, 355–385.

tonic transformation. That is, any such transformation yields new partial derivatives which are proportional to the u_i; and any function that shall be of this sort can be obtained by a monotonic transformation of U.

From this there follows one important consequence: If an ordinal (total) utility function exists with continuous partial derivatives (and this utility function applies to choice objects at large, as well as to objects that are only infinitesimally different), then the direction of the observed vector field u_i must satisfy the integrability condition. On the other hand, total choice questions, with answers that satisfy not only the determinateness but also the transitivity axiom, lead to an ordinal total utility function. In this connection transitivity in itself is *sufficient*. Transitivity implies, indeed, determinateness. There would be no meaning in speaking of transitivity if we did not have determinateness. Therefore transitivity in the object space at large leads to integrability.[19] *Hence from nonintegrability we can conclude nontransitivity*. This, it seems, is the simplest way in which we can arrive at H. S. Houthakker's theorem,[20] the essence of which is that nonintegrability entails nontransitivity.[21]

At this junction it is very tempting to draw the conclusion that from integrability we can deduce transitivity, and hence ordinal utility. I have a feeling that this erroneous idea is—at least unconsciously—implied in parts of the literature on the subject. It means confounding necessity and sufficiency.

The truth is that when we have integrability, it is always possible to construct a *potential function*. And this potential function is determined apart from an arbitrary monotonic transformation. But we can *not* conclude that this potential is an indicator of choice at large. If in addition to the information about the direction of the marginal preference field, we also had the information that an ordinal utility indicator *exists*, then we could say that the potential constructed from the knowledge of the integrable u_i is actually an ordinal utility indicator. The class of these ordinal utility functions is, indeed, *the same* as the class of all potentials that can be derived from an integrable u_i direction. One of the functions in the class is obtained from any other by a monotonic transformation. But we cannot

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[19] Strictly speaking, a word should be added on the Debreu condition. Gerard Debreu has proved that if the points x in object space are preference ordered (i.e., transitivity fulfilled), a utility function expressing this ordering will exist if, for every point x^0, the set of all points x which are at least as good as x^0 and the set of all points which are not preferred to x^0 are both closed sets. (Compare for instance the survey by Arrow in *Econometrica*, **26**, 1958, 1–23, see p. 17). The case where we have ordering, but where the Debreu condition is *not* fulfilled, will in our context be excluded by the fact that we assume the direction of the marginal utility vector is given everywhere with the derivative properties which are implied when we speak of integrability as distinct from nonintegrability (in the sense of (A2)), and the meaning of the marginal utility vector in any point x is that $x + dx$ will be preferred to, be indifferent to, or be deferred to x accordingly as $\sum_i u_i \, dx_i \gtreqless 0$.

[20] A discussion along different lines is given by Professor Samuelson in the 1950 *Economica* paper quoted. His "index number" assumptions imply ordinal utility.

[21] In this connection it does not make any difference whether we think of the direction of u_i as being observed through marginal choice questions or through observations of a quantity-adapting market under different prices.

prove the existence of an ordinal utility indicator by the mere fact we have a vector u_i satisfying the integrability conditions. It takes much more axiomatic apparatus to get from the marginal preferences to total utility—ordinal or cardinal.[22]

The following is a simple example of the fact that we may have observable integrability but no ordinal utility function and no ordinal transitivity.

Let $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)$ and $x^2 = (x_1^2, x_2^2, \ldots, x_n^2)$ be two points in the object space. We shall define the distribution of the preferences of the individual between any two such objects by a *general indicator* Ω, a function of $x_1^1, x_2^1, \ldots, x_n^1$ and $x_1^2, x_2^2, \ldots, x_n^2$. We shall denote it by $\Omega(x^1, x^2)$ and assert the following rule:

$$\text{(A3)} \qquad x^2 \text{ is} \begin{Bmatrix} \text{preferred to} \\ \text{indifferent to} \\ \text{deferred to} \end{Bmatrix} x^1 \text{ accordingly as } \Omega(x^1, x^2) \gtreqless \Omega(x^2, x^1).$$

The question of how such a general indicator can be observed through choice questions is irrelevant in this connection; we are only constructing a precise example which will show that integrability does not entail ordinal utility.

As an example we choose

$$\text{(A4)} \qquad \Omega(x^1, x^2) = f(x^2) + A(x^1, x^2),$$

where $f(x^2)$ may, for instance, be assumed to have the usual properties of a cardinal utility. The function $A(x^1, x^2)$ is assumed to be continuous and *antisymmetric*, that is, for all x^1 and x^2,

$$\text{(A5)} \qquad A(x^1, x^2) = -A(x^2, x^1)$$

and hence $A(x, x) = 0$ for all x.

We further assume that $A(x^1, x^2)$ has continuous partial derivatives,

$$\text{(A6)} \qquad A_k^1(x^1, x^2) = \frac{\partial A(x^1, x^2)}{\partial x_k^1} \quad \text{and} \quad A_k^2(x^1, x^2) = \frac{\partial A(x^1, x^2)}{\partial x_k^2} \qquad (k = 1, 2, \ldots, n)$$

satisfying

$$\text{(A7)} \qquad A_k^1(x, x) = 0; \qquad A_k^2(x, x) = 0 \qquad \text{for all } x = (x_1, x_2, \ldots, x_n) \quad (k = 1, 2, \ldots, n).$$

(Because of (A5) one of the two sets of conditions in (A7) follows from the other. It is therefore sufficient to assume but one of them).

Is the assumption (A4) "plausible"? This is again an irrelevant question from the viewpoint of our example. If it should not be found "plausible," it would only

[22] In my 1929 paper (in Norwegian) in *Nationaløkonomisk Tidsskrift*, Copenhagen (in particular pp. 370–377), I discussed the process by which we may pass from *one choice object to another* through a series of infinitesimal steps. These infinitesimal steps constitute the logical bridge (p. 372) leading from the marginal to the total viewpoint. And each step must be discussed axiomatically.

help to single out *some of the additional assumptions* which may be sufficient to proceed from marginal observations to an ordinal total utility.

In the case (A4) we have

(A8) $$\Omega(x^1, x^2) - \Omega(x^2, x^1) = f(x^2) - f(x^1) + 2A(x^1, x^2)$$

so that

(A9) $$x^2 \text{ is} \begin{Bmatrix} \text{preferred to} \\ \text{indifferent to} \\ \text{deferred to} \end{Bmatrix} x^1 \text{ accordingly as } f(x^2) - f(x^1) + 2A(x^1, x^2) \gtreqless 0.$$

In particular if x^1 is any point (denoted x) and x^2 is only infinitesimally different from x, that is,

(A10) $$x_k^2 = x_k + dx_k \qquad (k = 1, 2, \ldots, n)$$

we get

(A11) $$\begin{aligned} \Omega(x, x + dx) - \Omega(x + dx, x) &= f(x + dx) - f(x) + 2A(x, x + dx) \\ &= \sum_{k=1}^{n} f_k'(x)\, dx_k + 2\left[A(x, x) + \sum_{k=1}^{n} A_k^2(x, x)\, dx_k\right], \end{aligned}$$

where $f_k'(x)$ is the partial derivative of $f(x)$ with respect to x_k.

The bracket in the right member of (A11) vanishes by virtue of (A5) and (A7), so that

(A12) $$(x + dx) \text{ is} \begin{Bmatrix} \text{preferred to} \\ \text{indifferent to} \\ \text{deferred to} \end{Bmatrix} x \text{ accordingly as } \sum_{k=1}^{n} f_k'(x)\, dx_k \gtreqless 0.$$

The *direction* of the vector f_k' can therefore be observed in the usual way by marginal choice questions or by observing a quantity-adapting market under different prices. This direction will have all the usual properties of the direction of a vector of marginal utilities and in particular *will satisfy the integrability conditions.* This follows simply from the fact that the f_k' are the partial derivatives of a function of the point. A potential is easily determined. It is the function $f(x)$ itself. But this function is *not* an indicator of choice between two points in the object space at large. The choice between two such points is indeed given by (A3), and it is easily seen that this rule will in general not be equivalent to the rule

(A13) $$x^2 \text{ is} \begin{Bmatrix} \text{preferred to} \\ \text{indifferent to} \\ \text{deferred to} \end{Bmatrix} x^1 \text{ accordingly as } f(x^2) \gtreqless f(x^1).$$

Suppose for instance that we move from a point x^1 to another point x^2 that is on the same f surface as x^1, i.e., $f(x^2) = f(x^1)$. According to the rule (A13) the new point should be indifferent to x^1. But in reality the new point—for which $f(x^2) = f(x^1)$—will be indifferent to x^1 only if

(A14) $$A(x^1, x^2) = 0.$$

In general, it will obviously be possible to choose many points x^2 where (A14) is not fulfilled (under a given x^1).

Take as a simple example:

$$A(x^1, x^2) = \sum_{k=1}^{n} a_k(x_k^2 - x_k^1)^3, \tag{A15}$$

where the a_k are constants. In the case $n = 2$, (A14) will then give the straight line in (x_1^2, x_2^2) coordinates

$$x_2^2 - x_2^1 = -\sqrt[3]{a_1/a_2}(x_1^2 - x_1^1). \tag{A16}$$

It would only be by pure coincidence that the point x^2 which we have chosen, that is, for which $f(x^2) = f(x^1)$, is such as to satisfy (A16). And if it happened to satisfy (A16), we could take some other point x^2. Or we could even choose the function $f(x)$ differently.

In the example considered, the transitivity axiom cannot be satisfied for arbitrary objects in the object space. Indeed, if it were (and if the necessary continuity conditions were assumed), an ordinal utility function could be derived by choice questions in the object space *at large*. Moreover, the *direction* of the vector of the partial derivatives of this utility function would have to coincide with the *direction* of the vector field of marginal preferences that would be derived by *marginal* choice questions, or by observing a quantity-adapting market under changing prices. Hence the potential obtained by integrating the observed marginal preference vector would be an ordinal utility function, that is, obtainable by a monotonic transformation from the ordinal utility function derived directly by choice questions in the object space at large, assuming transitivity. All this is simply restating the introductory remark about what can be concluded from marginal observations if, in addition to these marginal observations, we *assume* the existence of a total ordinal utility (or, which essentially amounts to the same, assume transitivity in the object field at large).

The essential thing to retain from the above analysis is that from marginal observations *alone*—even if the observed marginal preference vector satisfies the integrability condition—we can *not* draw conclusions about the distribution of preferences among objects at large in the object space. The potential which is determined by integrating such a marginal preference field may *not* tell us how these preferences at large are distributed.

Note on want independence. If we admit total axioms of choice under one given choice situation, an indicator of total utility exists, and it is determined apart from an arbitrary increasing transformation. Suppose that it is possible to make a transformation that will make the goods independent in the sense that $u_{ik} = u_{ki} = 0$ for $i \neq k$, that is, the marginal utility of good i depends only on the quantity of good i and not on any other quantity (and equivalently, the quantity of good i influences only the marginal utility of good i and not any of the other marginal utilities). This transformation contains as an arbitrary element an increasing *linear* transformation. On the basis of one of the indicators thus obtained, we can derive most of the above formulae. But we would not know that the marginal utilities, and so on, thus derived actually correspond to those we would get from direct choice questions elaborated on the assumption that *interlocal* marginal choice axioms hold (i.e., questions involving more than one choice situation). This conclusion is of the same nature as the fundamental fact discussed in the main part of the Appendix.

I feel that choice questions of the interlocal sort come close to everyday experience.