Get Out The Vote:
How Encouraging Voting Changes Political Outcomes*

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Abstract

How do measures to increase turnout affect election outcomes? I use a novel approach to analyze how these measures influence both voter turnout and the candidates’ political positions. Generally, lowering the net expense of voting reduces political polarization. If the net expense of voting is made very low, then candidates no longer have an incentive to take partisan positions to motivate turnout and will converge at the median voter’s ideal point. For small changes in the net expense of voting, however, decreasing the cost of voting and penalties for not voting (two common measures) can result in drastically different political outcomes. Counter-intuitively, measures that make voting cheaper might not increase turnout: since these measures decrease the difference between the candidates’ political positions, they also decrease the benefit of voting.

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1 Introduction

Policy makers can influence voter turnout indirectly by decreasing the cost of voting, or directly by making voting mandatory. In the United States, measures that decrease the cost of voting have been introduced by individual States, which are responsible for running elections. Citizens of Oregon vote by mail instead of using polling stations and as a result are more likely to turn out (Southwell and Burchett (2000)), and six other States allow all citizens to register as permanent absentee voters. Other US measures that encourage voting include early voting and federal measures such as the Motor Voter Act, which decreased the cost of registering to vote. Internationally, roughly one out of every five citizens in an electoral democracy is legally obligated to vote (Engelen (2007)), and empirical studies have shown that turnout is systematically higher in these countries (Birch 2009). Voter turnout increased by over 30 percent after Australia introduced mandatory voting in 1924. Conversely, voter turnout in Holland and Venezuela dropped significantly after the abolition of mandatory voting.

But measures to increase turnout are not neutral or innocuous. Encouraging turnout changes citizens’ incentives to vote, and may affect which candidate wins the election. Additionally, candidates may change their political platforms in response to these measures; candidates are also strategic actors in elections and are sensitive to changes in voter behavior. Therefore, to understand the full effect of measures to increase turnout on election outcomes, it is important to understand their effect on both the strategic decisions of the citizens and the strategic decisions of the candidates. To capture the actions of both citizens and candidates I use a two-stage approach: in the first stage, candidates choose between partisan or centrist political positions; in the second stage, citizens decide whether or not to vote.

Using this two-stage model, I compare the consequences of decreasing the cost of voting to the consequences of making voting mandatory (which entails a penalty for not voting). Both measure reduce political polarization: if the cost of voting is made low enough, or no-vote penalties are high enough, then the candidates converge at the median citizen’s ideal point. The intuition is as follows. Candidates choose political positions to maximize relative turnout; they face a tradeoff between taking a partisan political position to increase turnout among partisan voters or taking a centrist position to win centrist votes. Significantly decreasing the cost of voting or a high penalty for not voting, however, ensures high turnout among partisan voters and therefore causes the candidates to converge at the center.

Even though both measures decrease the net expense of voting, decreasing the cost of voting is not equivalent to no-vote penalties. The political impact of the two measures can be drastically different for small changes in the net expense of voting. This difference occurs because the measures affect the distribution of heterogeneous voting costs differently. Decreasing the cost of voting lowers the net expense of voting proportionally, which keeps the net expense positive. No-vote penalties

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1 This paper considers measures to increase general turnout, not party-specific voter turnout drives a la Kramer (1970).
lower the net expense of voting by a fixed amount, which gives some citizens a negative net expense of voting. This can result in an equilibrium where only citizens with a negative net expense of voting vote and the candidates converge at the center. Therefore, it is possible that even small penalties for not voting will have a large impact on political polarization.

Counter-intuitively, my model suggests that voter turnout could decrease as a result of measures that encourage voting. While these measures decrease the net expense of voting, they can also decrease the benefit of voting since the candidates’ political positions move closer together. Funk (2005) finds precisely this phenomenon in Switzerland: relative voter turnout has decreased in cantons that switched to voting by mail. I also examine data on the political positions of US congressional representatives and, consistent with my model, I find that polarization is lower (relative to the national average) for representatives elected in states that allow voting by mail.

A key feature of the model is citizens’ preferences over the left/right political spectrum. Following standard spatial models, citizen utility is decreasing in the distance between their ideal point and the political outcome. I make the additional assumption that citizens have decreasing intensity of political preferences, which entails that citizens care less about political differences that are farther from their ideal point. For example, a Democratic citizen with decreasing intensity of political preferences cares more about the political difference between two Democratic candidates than the difference between two far-right candidates. If citizens had increasing intensity of political preferences, then a Democratic citizen would care more about the political difference between the two far-right candidates.

This assumption is supported by the empirical observation that voters do not “hedge” in primary elections by voting for a moderate candidate in the opposing party’s primary when the primary and general elections are competitive.² If citizens had increasing intensity, we should see significant crossover voting (Democrats voting in Republican primaries and vice versa) in competitive elections. In Appendix B, I present this argument more formally.

In my model equilibria exist in which candidates take partisan positions. This divergence is in stark contrast to the Hotelling-Black median voter result and the findings of Ledyard (1984). Ledyard (1984) shows that in a two-stage model with increasing intensity of citizen’s political preferences there exists a unique equilibrium in which candidates choose the same political position. Ledyard’s seminal paper is one of the only other papers that explicitly models citizens’ decision to vote and the candidates’ choice of political position, but his paper does not considers the impact of decreasing the net expense of voting.

Another important feature of the model is the motivation to vote. Voting in large elections is not rational from an individual cost-benefit calculation. The expected utility an individually-rational citizen receives from voting is equal to the probability that their vote influences the outcome of the election (this event is referred to as a “pivotal” vote) times the benefit of their vote being pivotal, minus the effort and opportunity costs of voting. Since the probability of being pivotal in a large

²Some voters do crossover when one candidate is a “shoe-in” in their party’s primary, or when the opposing party’s nominee is a shoe-in for the general election (Alvarez and Nagler (1999)).
election is very small, voting in large elections is not rational at the individual level (this “paradox of voting” was first presented by Downs (1957)).

While no consensus has been reached, explaining why people vote has proved a rich area for theoretical analysis (see Feddersen (2004), Blais (2000), and Aldrich (1993) for reviews of the extensive literature on voter turnout). Ledyard (1981) introduced a game theoretic analysis, which recognized that the probability of being pivotal is endogenous. It is not an equilibrium for no one to vote, since one vote would determine the election. Instead, citizens turn out at a level such that the expected benefit of voting is equal to the cost of voting for the marginal voter. Experimental studies have shown that this approach, known as the pivotal voter model, does well at explaining the comparative statics of turnout, but that it under-predicts levels of turnout (Levine and Palfrey (2007), Gerardi et al. (2008)).

I use the rule-utilitarian model of voter turnout, which accounts for both high turnout in large elections and the comparative statics of turnout. First suggested by Harsanyi (1980) and extended to non common-value elections by Feddersen and Sandroni (2006), this approach makes the argument that like-minded citizens consider the probability of collectively influencing the outcome of the election given “a rational commitment to a comprehensive joint strategy” (Harsanyi (1980)). Specifically, the rule-utilitarian equilibrium makes the behavioral assumption that citizens choose a voting rule as if all other like-minded citizens will follow the same voting rule. In a structural analysis of turnout in Texas liquor referenda, Coate and Conlin (2004) find that the rule-utilitarian model out-performs the expressive voting model, which models turnout as an exogenous function of the intensity of voter preference.

This paper is related to a recent set of articles that examine the impact of no-vote penalties (Börgers (2004), Krasa and Polborn (2009), and Gerardi et al. (2008)). Börgers (2004) considers a model of elections with two symmetric groups. He shows that, in this setting, voluntary voting gives higher aggregate utility than full turnout. Krasa and Polborn (2009) consider two asymmetric groups and find that no-vote penalties are welfare improving when the size of one group is sufficiently close to one, or when the number of citizens is very large. Both papers focus on the effect of no-vote penalties on citizen behavior, holding the behavior of the candidates fixed. This paper extends the analysis to account for the political positions of the candidates; candidates are also strategic actors in an election and to get a full understanding of the effects of decreasing the net expense of voting, we must account for their actions.

I remain agnostic as to whether increasing turnout has inherent value, but consider normative criteria similar to Börgers (2004). In contrast to Börgers (2004) I find that full turnout can increase aggregate utility, since it both insures that the option preferred by the majority wins the election and results in convergence at the median voter’s ideal point. Convergence is not always welfare improving, however, since it eliminates voter choice.

Gerardi et al (2008) consider a related and important question: how no-vote penalties affect the

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3 Although their microfoundations are quite different, the group-leader models of turnout by Morton (1991) and Uhlaner (1989) have a similar mathematical structure.
information aggregation properties of elections. They model a common-value election, where all 
individuals have identical preferences but receive different signals as to which candidate is “best.” 
In this setting, penalties for not voting have two competing effects: they increase the number of 
people who vote and decrease the average quality of their information. The authors show that 
increasing turnout in small elections can be welfare improving.

This paper makes two methodological contributions. First, I show that the Hotelling-Black 
median voter theorem does not hold in a model with diminishing intensity of political preferences 
and costly voting.\footnote{\textsuperscript{4}I show this result in a setting with complete information and citizens who turn out according to pivotal voter model (Valasek (2010)). Here I extend the result to incomplete information and a rule-utilitarian model of voting.} Second, I extend the proofs of existence and uniqueness from Feddersen and Sandroni (2006a) (who consider a contest between two groups) to a setting with an arbitrary number 
of groups distributed over the political space. By considering more than two groups, I expand the 
analysis of rule-utilitarian equilibria to include the strategic interaction between citizens who prefer 
the same candidate, but have different ideal political positions.

Section 2 introduces the model, Section 3 analyzes the equilibrium, and Section 4 details the 
effects of measures to increase turnout. Section 5 concludes, and all formal proofs are given in 
Appendix A.

2 The Model

This section introduces a two-stage model of elections. In the first stage, two candidates choose 
political positions (partisan or centrist) to maximize the probability of winning the election. In the 
second stage, citizens choose whether or not to vote according to the rule-utilitarian approach.

2.1 Candidates

I consider a majority rule election with 2 candidates, \( k \in \{D,R\} \). As a mnemonic tool rather 
than a political statement, consider \( D \) to be a Democratic candidate, and \( R \) to be a Republican 
candidate. Candidates are purely office-seeking in the tradition of Mayhew (1974) and Ledyard 
(1984) and are not motivated by policy preference. Candidates receive a utility of 1 if elected and 
0 if not. I denote the probability that candidate \( k \) wins as \( P_k \); \( P_k \) is also candidate \( k \)'s expected 
payoff.

Candidates have two characteristics: a political position and a measure of charisma. The politi-
cal position, \( g_k \), is a choice variable. The political space consists of three points: \( \{0, \frac{1}{2}, 1\} \). Candidates 
simultaneously commit to observable political positions prior to the election; the Democratic 
candidate sets \( g_D \) at 0 or \( \frac{1}{2} \) and the Republican candidate sets \( g_R \) at \( \frac{1}{2} \) or 1; take \( g \equiv (g_D, g_R) \).

This model of candidate choice represents the most simple deviation from the standard model 
of voter turnout, which fixes candidate positions at partisan positions. Specifically, each candidate 
can deviate from the partisan position to a third, centrist, position. A three-point policy space 
is useful since it offers (relative) analytical ease and allows any convex utility function over the
political space to be characterized by a single variable. The main results of this model, however, are robust to more complex models of candidate choice; in appendix B, I show that the main results generalize to a continuous policy space.

The charisma of candidate \( k \) is chosen by Nature from the distribution \( q_k \sim U[0, 1] \), however, its value is unobserved. Charisma is not directly valued by citizens, but it does influence their behavior; specifically, a proportion of citizens are disillusioned and will not participate in the election, and the number of citizens that are disillusioned is a function of their preferred candidate’s charisma (I will formally describe disillusioned citizens in the next subsection). Charisma can include such factors as the quality of the campaign, the effectiveness of the party’s voter turnout drive, and the candidate’s ability to engage the electorate. Candidates cannot be certain how effective they are at preventing disillusionment among their supporters. Therefore, candidates know the distribution of \( q_k \), but not its realization.\(^5\)

### 2.2 Citizens

There is a continuum of citizens of measure 1. Each citizen, \( i \), is one of three different types (\( t \)): left (\( t = l \)), middle (\( t = m \)), and right (\( t = r \)). Types correspond to ideal points in the political space; citizen \( i \) of type \( l \) has an ideal point, \( \eta_i \), equal to 0, type \( m \) has \( \eta_i = \frac{1}{2} \), and type \( r \) has \( \eta_i = 1 \). The measure of each type is \( \lambda_t \).

Citizen’s utility over the political space is:

\[
    u_i(\mid \hat{g} - \eta_i \mid) = \begin{cases} 
    0 & \text{if } |\hat{g} - \eta_i| = 0; \\
    -(\frac{1}{2} + v) & \text{if } |\hat{g} - \eta_i| = \frac{1}{2}; \\
    -1 & \text{if } |\hat{g} - \eta_i| = 1; \\
\end{cases}
\]

Where \( \hat{g} \) is the political position of the winning candidate, and \( 0 < v < \frac{1}{2} \), which implies that citizens have diminishing intensity of political preferences.

The cost of voting, \( c_i \), is heterogeneous and is drawn from a uniform distribution with a support of \([0, \bar{c}]\). \( c_i \) is stochastic, but since the population is a continuum the distribution of voting costs is identical for each type and is known by both citizens and candidates.

A proportion of citizens who prefer candidate \( k \), \( (1 - q_k) \), are disillusioned. Disillusioned citizens do not participate in the election unless they have a negative net expense of voting (which is possible with no-vote penalties). \( q_k \) citizens are not disillusioned; I refer to these citizens as “motivated” citizens.

Following Harsanyi (1980), motivated citizens play a rule-utilitarian equilibrium. The equilibrium strategy always takes the form of a cutoff cost, \( \tilde{c}_i \), where citizen \( i \) votes if their cost of voting is below \( \tilde{c}_i \), and abstain if it is above \( \tilde{c}_i \). Motivated citizen \( i \) of type \( t \) chooses a cutoff cost, \( \tilde{c}_i \), to

\(^5\)A stochastic number of disillusioned voters is analogous to the stochastic number of ethical citizens in Feddersen and Sandroni (2006a,b); both introduce uncertainty over the final election outcome. It is not a valence term since it does not enter into citizens’ utility functions. Turnout and election outcomes are uncertain ex ante, perhaps best demonstrated by Truman’s historic and unexpected defeat of Dewey in the 1948 presidential election.
maximize the ex-ante utility function:

\[ W_i = W_i(g, \tilde{c}_i, \{\tilde{c}_j\}) \]

subject to:

- \( \tilde{c}_j = 0 \) if \( j \) is disillusioned.
- \( \tilde{c}_j \) constant if \( j \) is motivated and \( j \)'s type is not \( t \).
- \( \tilde{c}_j = \tilde{c}_i \) if \( j \) is motivated and \( j \)'s type is \( t \).

The rule-utilitarian equilibrium models the voting game as a hybrid between a non-cooperative and a cooperative game. Strategies between types are non-cooperative. Strategies within a given type, however, are not non-cooperative. Instead, the rule-utilitarian equilibrium makes the behavioral assumption that motivated citizens consider the probability of collectively influencing the outcome of the election given “a rational commitment to a comprehensive joint strategy” (Harsanyi (1980)); motivated citizens choose a cutoff cost as if all other motivated citizens of their type will follow the same cutoff cost.\(^6\)

Since all citizens of type \( t \) have identical ex ante utility functions, these strategies imply that all motivated citizens of type \( t \) choose the same cutoff cost, labeled \( c_t \) (\( \tilde{c}_i = c_t \) for all motivated \( i \) of type \( t \)). \( c_t \) maximizes the ex ante utility:

\[ W_i(g, \tilde{c}_i, \{\tilde{c}_j\}) = \beta_i(g)P_k(c_t, c_m, c_r) - E(c_i|c_t). \]

\( \beta_i(g) \) is the benefit \( i \) receives if their preferred candidate wins the election, which is equal to \( i \)'s utility difference between the two candidates (\( \beta_i(g) = u(|g_D|) - u(|g_R|) \)); \( P_k(c_t, c_m, c_r) \) is the probability \( i \)'s preferred candidate wins given the voting rules \( \{c_t, c_m, c_r\} \); and \( E(c_i|c_t) \) is \( i \)'s expected voting cost given the cutoff cost \( c_t \).\(^7\)

Before detailing the probability that each candidate wins (\( P_k(c_t, c_m, c_r) \)), I introduce the following notation: take \( V_k \) to be the proportion of motivated citizens who vote for candidate \( k \). For example, if types \( l \) and \( m \) prefer candidate \( D \), then \( V_D = \lambda_l c_l + \lambda_m c_m \). The probability that candidate \( D \) wins, \( P_D(c_t, c_m, c_r) \), can be written as \( P(q_D V_D > q_R V_R) \) or:

\[ P_D(c_t, c_m, c_r) = P\left( \frac{V_D}{V_R} > \frac{q_R}{q_D} \right) = G\left( \frac{V_D}{V_R} \right) \]

\(^6\)Feddersen and Sandroni (2006) show that the rule-utilitarian equilibrium is a Nash equilibrium if motivated citizens receive a “warm glow” payoff from playing according to their type’s optimal joint strategy.

\(^7\)Harsanyi (1980), Morton (1987, 1991), Uhlaner (1989), Fedderson and Sandroni (2006a,b), and Coate and Conlin (2004) all use similar models of turnout. These models vary in the level of citizen altruism assumed. At one end is Harsanyi, who models voting as a purely social or altruistic activity: citizens care about aggregate utility (aggregate benefits and aggregate costs). My model is at the other end: citizens care only about individual benefits and costs. If citizens are altruistic over aggregate elections costs, as in Fedderson and Sandroni (2006a,b), then a no-vote fine would not impact voting behavior, since the fine nets out of aggregate election costs. This prediction contradicts evidence that mandatory voting increases turnout (see Birch (2009)).
Where $G$ is the distribution of the ratio of two uniform variables distributed between zero and one.

### 2.3 Timing and Information

The timing of the game is as follows:

1. Nature chooses $q_k$, which is unobserved.
2. Candidates simultaneously set $g_k$.
3. Citizens simultaneously set $\tilde{c}_i$.
4. Citizens draw $\{c_i\}$ and choose whether to vote or abstain. The election results are revealed.

Information is incomplete since neither candidates or citizens observe $q_D$ and $q_R$ (although they can be inferred when the election results are revealed). I assume that the beliefs of both candidates and citizens over $q_D$ and $q_R$ are determined according to Bayes’ Rule at each stage of the game. Since no information about $q_D$ and $q_R$ is revealed until stage three of the game, the beliefs of both candidates and citizens over $q_D$ and $q_R$ remain equal to the prior ($q_k \sim U[0,1]$) at all decision nodes.

### 2.4 Equilibrium in the Voting Game

Following standard backward induction, I start with the citizens’ choice to turnout given the candidate positions $g$.

**Definition 1.** An equilibrium in the voting game is a set of cutoff costs, $\{c^*_l, c^*_m, c^*_r\}$, where

$$\beta_l(g) P_k(c^*_l, c^*_m, c^*_r) - E(c_i|c^*_t) \geq \beta_l(g) P_k(\{c^*_l, c^*_m, c^*_r, c_t\}|c^*_t) - E(c_i|c_t) \quad \text{for all } c_t \in [0, \tilde{c}] \text{ and for all } t.$$  

Whenever possible, I drop the asterix and refer to the optimal cutoff cost for type $t$ as simply $c_t$.

### 2.5 Equilibrium in the Candidates’ Game

Take $\{c_t(g)\}$ to be a rule-utilitarian equilibrium of the voting game at $g$. Candidates choose positions to maximize their probability of winning the election given the equilibria of the resulting voting games. I focus on pure strategy equilibria to avoid introducing additional notation (I will prove that candidates play pure strategies in equilibrium in the following section). Also for simplicity, when a candidate receives the same probability of winning as a partisan and as a centrist, I assume the candidate will choose a partisan position.\(^8\)

**Definition 2.** An equilibrium in the candidates’ game is a policy pair $(g^*_D, g^*_R)$ such that

$$U_D(g^*_D, g^*_R, \{c_t(g)\}) \geq U_D(g_D, g^*_R, \{c_t(g)\}), \text{ for } g_D \in \{0, \frac{1}{2}\}, \text{ and } U_R(g^*_D, g^*_R, \{c_t(g)\}) \geq U_R(g^*_D, g_R, \{c_t(g)\}), \text{ for } g_R \in \{\frac{1}{2}, 1\}.$$  

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\(^8\)This tie-breaking rule does not affect the predictions of the model, but gives a unique equilibrium in the knife-edge cases when one candidate is indifferent about running as a partisan or a centrist.
Since I use a non-Nash equilibrium in the voting game, I define an equilibrium of the full election game as the set \( \{ g^*_D, g^*_R; \{ c_l(g) \} \} \) (with beliefs equal to \( q_k \sim U[0, 1] \) for all agents at all decision nodes). The equilibrium specifies that candidates consider equilibrium play in the voting game for \( g \) not reached in equilibrium, and that the beliefs of both candidates and citizens over \( q_D \) and \( q_R \) are determined according to Bayes’ Rule at each stage of the game.

3 Equilibrium

In this section I characterize the equilibrium of the voting game and the candidates’ game. I first look at the voting game and show that a unique equilibrium exists for any \( g \). I also show that the proof generalizes to \( n \) groups, extending the rule-utilitarian equilibrium of Feddersen and Sandroni (2006) to a setting with \( n \) groups distributed over the left-right political space. I then characterize the equilibrium of the candidates’ game. Importantly, the candidates can be polarized in equilibrium; i.e. the median voter result does not hold in a model of elections with turnout and diminishing intensity of political preference.

3.1 Voting game

I first solve for the equilibrium of the voting game. I find that an equilibrium exists and is unique.

Proposition 1. A unique equilibrium of the voting game exists for all \( g \).

A formal proof is given in Appendix A, however, I give some intuition regarding the strategic interaction between citizen types here.

Refer to a type that prefers the candidate with an expected minority as the “minority type,” and a type that prefers the candidate with an expected majority as the “majority type.” The reaction functions (shown for an interior equilibrium only) give insight into the strategic interaction between citizen types:

\[
c_{\text{min}} = \frac{1}{V_{\text{maj}}} \left[ \frac{\beta_{\text{min}} \lambda_{\text{min}} c}{2} \right]
\]

And:

\[
c_{\text{maj}} = V_{\text{min}} \left[ \frac{\beta_{\text{maj}} \lambda_{\text{maj}} c}{2V^2_{\text{maj}}} \right]
\]

The voting game has 2 components: the game between types who prefer the same candidate, and the game between types who prefer opposing candidates. I start with the former. Suppose types \( l \) and \( m \) prefer same candidate, which is true at the political position pair \((\frac{1}{2}, 1)\). Whether the minority or majority type, the ratio of the reaction functions of type \( m \) and type \( l \) shows that the ratio of \( c_m \) and \( c_l \) is constant. Therefore, given an equilibrium value of \( c_l \), the equilibrium value of \( c_m \) is a fixed proportion of \( c_l \). The linear relationship between types who prefer the same candidate

9By using Feddersen and Sandroni’s (2006a) method to make the equilibrium in the voting game a Nash Equilibrium, the equilibrium concept can be changed to PBE.
holds regardless of the number of types. Therefore, no matter the number of types, the voting game can be fully characterized by determining the equilibrium cutoff costs for two types who prefer opposing candidates. For example, in the model with three types $c_m$ is a fixed proportion of either $c_l$ or $c_r$ and the equilibrium of the voting game can be characterized by the equilibrium values of $c_l$ and $c_r$.

I now turn to the game between types who prefer opposing candidates. Suppose $t$ is the minority type. Equation 1 shows that if the majority type increases their cutoff cost, then $t$’s best response is to decrease $c_t$. Equation 2 shows that if $t$ is the majority type, however, the result is reversed: if the minority type increases their cutoff cost, then $t$’s best response is to increase $c_t$.

The intuition is as follows. If the majority type increases their cutoff cost, increasing their expected lead, then the minority type has less of an incentive to vote since they are further behind. Conversely, if the minority type increases their cutoff cost then the election becomes more competitive, increasing the majority type’s incentive to vote. This relationship leads to a unique equilibrium: in two-player games where one player’s strategy is a strategic substitute and the other’s is a strategic complement the best response functions cross only once. The full proof of uniqueness is more involved, however, since $l$ is the majority type for only part of strategy space, and the minority type for the other part.

### 3.2 Candidates’ game

Before giving the general result, I characterize the payoffs when candidates both choose partisan political positions. In this case type $m$ citizens are indifferent between the candidates, the voting game reduces to a game between two types with identical benefits from winning, which corresponds to the game studied in most rational agent models of turnout (Palfrey and Rosenthal (1985), Börgers (2004), and Feddersen and Sandroni (2006a,b)). Also, since the equilibrium of the voting game is unique for each $g$, candidates’ probabilities of winning are also unique for each $g$. Therefore, I will refer to the probability that candidate $k$ wins the election as $P_k(g)$.

**Lemma 1.** $P_D(0, 1) = \frac{\lambda_l}{2\lambda_r}$.

The formal proof is relegated to Appendix A. In line with the results of other papers, the candidate preferred by a larger number of citizens wins an expected plurality.

**Proposition 2.** Assume (without loss of generality) that $\lambda_l \leq \lambda_r$. An equilibrium in the candidates’ game exists and is unique. Moreover, the equilibrium is either: $(0, 1)$, $(\frac{1}{2}, 1)$, or $(\frac{1}{2}, \frac{1}{2})$.

Proposition 2 is a key result of this paper: it shows that polarization can occur in equilibrium. This result formalizes the intuition, dating back to Downs (1957, p. 118), that candidates take partisan positions to motivate turnout among partisan citizens. Again, I leave the formal proof of existence for Appendix A. I detail the intuition behind the result below, but first give a short discussion of the importance of the distribution of citizen types.
The relative number of citizens of each type affects the strategic behavior of candidates. As a benchmark, I use the distribution of types where the median citizen is of type \( m \), and the number of citizens of type \( m \) is smaller than both partisan types. Mathematically:

\[
\lambda_m < \lambda_l < \lambda_r \quad \text{and} \quad \lambda_l + \lambda_m > \lambda_r
\]

Other distributions give potential equilibria that are a strict subset of the benchmark case. In the benchmark case, candidates are fully polarized when \( g_D = 0 \) and \( g_R = 1 \) and converge when \( g_D = g_R = \frac{1}{2} \).

My results hold generally, but the definition of full polarization and convergence change with other distributions. Full polarization is when each candidate sets \( g_k \) at the ideal point of the largest group in their half of the policy space, and convergence is when the political outcome is the median citizen’s ideal point. For example, if \( \lambda_l < \lambda_m < \lambda_r \) then full polarization changes to \( g = \left( \frac{1}{2}, 1 \right) \) and convergence is still \( g = \left( \frac{1}{2}, \frac{1}{2} \right) \).

Returning to the intuition for Proposition 2, if both candidates take partisan positions or converge to the middle, then the expected utility for candidate \( D \) depends only on the relative number of partisan types \((P_D|g = (0,1)) = \frac{\lambda_l}{2\lambda_r} \) and \((P_D|g = (\frac{1}{2}, \frac{1}{2})) = \frac{1}{2}\). Therefore, the equilibrium of the candidates’ game depends crucially on the payoffs at \( g = \left( \frac{1}{2}, 1 \right) \).

The intuition of why partial turnout and decreasing intensity of political preference can result in candidate polarization lies in what happens when candidate \( D \) takes a centrist position, and candidate \( R \) takes a partisan position, illustrated in Figure 1. In this case, a majority of citizens prefer candidate \( D \). Therefore, if all citizens were to vote then candidate \( D \) would win the election. With full turnout, however, \( g = \left( \frac{1}{2}, 1 \right) \) is not an equilibrium: since any candidate who is preferred by the median citizen wins the election, the unique equilibrium would be for both candidates to locate at the median.

Since voting is costly, however, the relative strength of preferences also matter. Referring to
Figure 1, we see that while a majority of citizens prefer candidate $D$, the average benefit of having their preferred candidate win is smaller than the average benefit for citizens who prefer candidate $R$ ($\beta_l = \frac{1}{2} - v$ and $\beta_m = \beta_r = \frac{1}{2} + v$). Therefore, the turnout rate for citizens who prefer candidate $D$ is lower than turnout rate of type $r$ citizens, and the probability candidate $D$ wins can be less than $\frac{1}{2}$, or even less than $\frac{\lambda_l}{2\lambda_r}$. Even though candidate $D$ is preferred by a majority of citizens, candidate $D$ can still lose the election in expectation.

Given the probability that candidate $D$ wins at $g = (\frac{1}{2}, 1)$, the equilibrium is easy to identify: if the probability is less than $\frac{\lambda_l}{2\lambda_r}$, then the equilibrium outcome is $g = (0, 1)$; if it is between $\frac{\lambda_l}{2\lambda_r}$ and $\frac{1}{2}$, then the equilibrium outcome is $g = (\frac{1}{2}, 1)$; and if it is greater than $\frac{1}{2}$ then the equilibrium is $g = (\frac{1}{2}, 1)$.

It is also easy to show why increasing intensity of political preference would lead to candidate convergence. With diminishing intensity of political preferences $v$ is positive and a candidate can be preferred by a majority of citizens and still lose the election in expectation. With increasing intensity, however, $v$ is negative. Looking again at Figure 1, we see that at $g = (\frac{1}{2}, 1)$ increasing intensity implies the average benefit of winning for citizens who prefer candidate $D$ is higher than for citizens who prefer candidate $R$. Therefore, candidate $D$ wins an expected plurality.\textsuperscript{10} Candidates’ payoffs at $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$, however, are the same whether $v$ is negative or positive. Therefore, the unique equilibrium would be for both candidates to converge at $g = (\frac{1}{2}, \frac{1}{2})$; if any candidate chooses a partisan position, then the other candidate can win an expected plurality by setting $g_k = \frac{1}{2}$.

Diminishing intensity of political preferences is a necessary condition for candidate polarization, but it is not sufficient. The probability that candidate $D$ wins given an interior equilibrium is:

$$\frac{1}{2} \left[ \frac{(\frac{1}{2} - v)\lambda_l^2 + (\frac{1}{2} + v)\lambda_m^2}{(\frac{1}{2} + v)\lambda_r^2} \right]^{1/2}. $$

This equation identifies two sufficient conditions for polarization: candidates are polarized if $v$ is high enough, or if $\lambda_m$ low enough. Additionally, when $(\lambda_r - \lambda_l)$ is higher then the probability that candidate $D$ wins at $g = (\frac{1}{2}, 1)$ is lower, making a polarized equilibrium more likely.

4 Measures to Increase Turnout

The previous two sections have described and analyzed a two-stage model of elections. In this section, I introduce measures to increase turnout to this model. Suggested measures to increase turnout usually fall into two categories: decreasing the cost of voting, such as Oregon’s switch to voting by mail; or penalizing nonvoters (called mandatory or compulsory voting), such as Australia’s no-vote fines. In this section I analyze and compare the effect of the two measures on political outcomes. Mechanically, the two differ because decreasing the cost of voting leaves all citizens with positive voting costs, while a fine for nonvoters gives citizens with low voting costs an incentive to vote regardless of whether or not they are disillusioned. If the cost of voting is made low enough, or

\textsuperscript{10}This result corresponds with the results of both Ledyard (1984) and Taylor and Yilderim (2010a).
no-vote penalties are high enough, then the candidates converge at the median citizen’s ideal point. At lower levels, however, the mechanical difference in the two measures could lead to substantially different results.

4.1 Decreasing the cost of voting:

First, I introduce additional motivation for the distribution of voting costs, $U[0, \bar{c}]$. A major component of the cost of voting is the opportunity cost of time. Consider $\bar{c}$ the amount of time it takes to vote and $U[0, 1]$ the distribution of opportunity costs. Measures to decrease the cost of voting target the time it takes to vote. Therefore, they can be represented as a decrease in $\bar{c}$; if switching to voting by mail decreases the time it takes to vote by one-half, then this changes the distribution of voting costs to $U[0, \frac{\bar{c}}{2}]$. This logic also applies to a non-uniform distribution of opportunity costs.

Again, the probability that candidate $D$ wins at $g = (\frac{1}{2}, 1)$ is pivotal for the analysis. For high enough $\bar{c}$, all cutoff costs are interior and the probability that candidate $D$ wins the election is constant and equal to:

$$P_D\left(\frac{1}{2}, 1\right) = \frac{1}{2} \left[ \frac{(\frac{1}{2} - v)\lambda_1^2 + (\frac{1}{2} + v)\lambda_2^2}{(\frac{1}{2} + v)\lambda_2^2} \right]^{\frac{1}{2}}$$

**Proposition 3.** If $P_D\left(\frac{1}{2}, 1\right) \leq \frac{N}{2\lambda_1}$, then there exists $\bar{c}_2 > \bar{c}_1 > 0$ such that the unique equilibrium of the candidates’ game is $(0, 1)$ for $\bar{c} \geq \bar{c}_2$, $(\frac{1}{2}, 1)$ for $\bar{c} \in (\bar{c}_2, \bar{c}_1]$, and $(\frac{1}{2}, \frac{1}{2})$ for $\bar{c} < \bar{c}_1$.

If $P_D\left(\frac{1}{2}, 1\right) \in \left(\frac{N}{2\lambda_1}, \frac{1}{2}\right]$, then there exists $\bar{c}_1$, such that the unique equilibrium of the candidates’ game is $(\frac{1}{2}, 1)$ for $\bar{c} \geq \bar{c}_1$, and $(\frac{1}{2}, \frac{1}{2})$ for $\bar{c} < \bar{c}_1$.

If $P_D\left(\frac{1}{2}, 1\right) > \frac{1}{2}$, then the unique equilibrium of the candidates’ game is $(\frac{1}{2}, \frac{1}{2})$.

The intuition is as follows. The equilibrium of the candidates’ game depends on the value of $P_D\left(\frac{1}{2}, 1\right)$; if $P_D\left(\frac{1}{2}, 1\right)$ is smaller than $\frac{N}{2\lambda_1}$ then $g = (0, 1)$, if it is between $\frac{N}{2\lambda_1}$ and $\frac{1}{2}$ then $g = (\frac{1}{2}, 1)$, and if it is greater than $\frac{1}{2}$ then $g = (1, \frac{1}{2})$. For a high $\bar{c}$, the cutoff costs are interior and $P_D\left(\frac{1}{2}, 1\right) = P_D\left(\frac{1}{2}, 1\right)$. For a lower $\bar{c}$ the proportion of citizens who vote is higher, but if cutoff costs remain interior then the election outcome stays the same.

For some $\bar{c}'$, however, all motivated citizens who prefer candidate $R$ turnout. As $\bar{c}$ decreases further, turnout among citizens who prefer candidate $D$ continues to increase. Therefore, starting at $\bar{c}'$, $P_D\left(\frac{1}{2}, 1\right)$ increases continuously as $\bar{c}$ decreases until all motivated citizens turn out, at which point $P_D\left(\frac{1}{2}, 1\right) > \frac{1}{2}$.

While decreasing the cost of voting results in convergence, increasing the cost of voting cannot decrease the payoff to candidate $D$ at $(\frac{1}{2}, 1)$ past $P_D\left(\frac{1}{2}, 1\right)$. Therefore, raising $\bar{c}$ can never result in greater polarization than is found at an equilibrium with interior cutoff costs.

Note that decreasing the cost of voting does not need to increase turnout. For example, assume $\bar{c}$ is initially greater than $\bar{c}_1$, and is lowered to a value less than $\bar{c}_1$. If candidate positions were fixed at $(0, 1)$, then turnout would increase. The candidates move to $(\frac{1}{2}, 1)$, however, which decreases the incentive for the partisan types to vote and could decrease aggregate turnout.
4.2 Penalties for not voting:

To be consistent with previous literature, I model no-vote penalties as a lump sum payment, \( s \), given to all citizens who vote, rather than a fine on abstaining. Modeling no-vote penalties as a subsidy is without loss of generality since any uniform fine \( f \) is equivalent to some uniform subsidy \( s \).

Take \( s^* = -(\frac{1}{2} + v) \). For \( s \leq s^* \), turnout is zero if candidate \( D \) sets \( g_D = \frac{1}{2} \), since the benefit a citizen receives from having her preferred candidate win is always smaller than the cost of voting. Therefore, when \( s \leq s^* \) candidate \( D \) can force a tie by setting \( g_D = \frac{1}{2} \), which is an equilibrium of the candidates’ game.

**Proposition 4.** There exist \( s^* < s_1 < s_2 < \bar{c} \) such that for \( s \in (s^*, s_1] \) the candidate equilibrium is \( (0, 1) \) and for \( s \in [s_2, \infty) \) the candidate equilibrium is \( (\frac{1}{2}, \frac{1}{2}) \).

When \( s \) is low enough, but not below \( s^* \), a candidate only receives votes from the type whose ideal point coincides with the candidate’s political position. For example, if \( s < -\beta_l(g) \) at the political position \( (\frac{1}{2}, 1) \) then \( c_l \) equals zero even though type \( l \) strictly prefers candidate \( D \). By continuity, there exists some \( s_1 \) larger than \( -\beta_l(g) \) where candidates choose to locate at the two largest groups and the equilibrium \( g \) is \( (0, 1) \).

On the other hand, when \( s \) is high enough, the proportion of citizens with negative voting cost is large enough that any candidate who is preferred by the type \( m \) wins the election with probability one. Again by continuity there exists some \( s_2 \) smaller than \( \bar{c} \) so that both candidates locate at the median voter.

4.3 Comparing Measures

Propositions 3 and 4 show that if the net expense of voting is made low enough, then measures to increase turnout effectively remove turnout as a strategic consideration for candidates; with turnout among the partisan base secured, candidates compete over centrist voters and converge at the median citizen’s ideal point.

While decreasing the cost of voting and no-vote penalties can both cause convergence, there are two important differences between the two measures. First, a subsidy on voting can always achieve full polarization with a low \( s \) and full convergence for a high \( s \). Changing the cost of voting, however, can only achieve the level of polarization found at interior equilibria of the voting game since changing the cost of voting does not change the candidate’s payoffs at interior equilibria of the voting game. Should a social planner desire to increase polarization, then it might require a poll tax.

Second, and perhaps more importantly, the two measures can behave quite differently for intermediate values. As is evident from Proposition 3, decreasing the cost of voting results in a predictable path towards convergence. No-vote penalties, however, can produce a quick shift to convergence.
The reason no-vote penalties can cause a quick shift to convergence is as follows. A subsidy on voting gives candidate \( D \) a “built in” lead at \( g = (\frac{1}{2}, 1) \); some citizens, both disillusioned and motivated, vote simply because their cost of voting is negative; therefore at \( \{c_l, c_m, c_r\} = \{0, 0, 0\} \) candidate \( D \) has a vote share advantage of \( z = (\lambda_l + \lambda_m - \lambda_r)s \). The vote share advantage affects the marginal returns to voting for type \( r \) when \( c_r \lambda_r < z \) there are no returns to voting \( (P_R(0, 0, c_r) = 0) \) and when \( c_r \lambda_r \in (z, (z + c_l \lambda_l + c_m \lambda_m)) \) there are increasing returns to voting for type \( r \) as candidate \( R \) “catches up” to candidate \( D \)’s lead among disillusioned voters.

Formally, if \( c_r \lambda_r \in (z, (z + c_l \lambda_l + c_m \lambda_m)) \) then:

\[
P_R(c_l, c_m, c_r) = \frac{c_r \lambda_r}{(c_l \lambda_l + c_m \lambda_m)} \left[ \frac{1}{2} + \frac{z^2}{2(c_r \lambda_r)^2} - \frac{z}{c_r \lambda_r} \right]
\]

Since \( P_R(c_l, c_m, c_r) \) is convex in \( c_r \), both the benefits and the costs of voting are increasing at an increasing rate for \( c_r \lambda_r \in (z, (z + c_l \lambda_l + c_m \lambda_m)) \). Therefore, the payoff function for type \( r \) is neither concave or quasi-concave. With non-concave payoffs, multiple equilibria of the voting game can exist at \( g = (\frac{1}{2}, 1) \). Moreover, if multiple equilibria exist and \( P_D(\frac{1}{2}, 1) < \frac{1}{2} \) at one equilibrium, then \( P_D(\frac{1}{2}, 1) > \frac{1}{2} \) at all other equilibria.\(^{11}\)

The uncertainty of candidates’ payoffs at \( g = (\frac{1}{2}, 1) \) has important implications for the equilibrium of the candidates’ game. If \( P_D(\frac{1}{2}, 1) < \frac{1}{2} \), then candidate \( R \) does better setting \( g_R = 1 \). If \( P_D(\frac{1}{2}, 1) > \frac{1}{2} \), however, then the unique equilibrium is for both candidates to set \( g_k = \frac{1}{2} \). With multiple equilibria in the voting game, the equilibrium of the candidates’ game is sensitive to candidates’ beliefs over which equilibrium will be played in the voting game. Therefore, even a small fine on non-voting could change the equilibrium of the candidates’ game from an equilibrium where candidates choose partisan positions to an equilibrium where candidates both locate at the center.

In addition to multiple equilibria, no-vote penalties could result in candidate convergence for other reasons. Normative institutions such as voting can be fragile to the introduction of direct monetary incentives.\(^{12}\) Once citizens are paid to vote, the collective action problem of voting disappears. Without the collective action problem, citizens might switch from behaving normatively and playing a rule-utilitarian equilibrium to playing individually-rational strategies. The individually-rational strategy is for citizens with negative net expense of voting to vote, and for all others to abstain. These strategies also lead to convergence, as the candidate who is preferred by the median citizen will win the election.

The distinction between the two measures is important: a politician interested in increasing turnout with minimal effect on partisan outcomes should consider a small decrease in the cost of voting, since this has little effect on which candidate wins. A small fine on not voting, however,

\(^{11}\)Because of the payoff functions are not quasi-concave, the first-order conditions no longer define the best response functions. They do still define local reaction functions and can be used to find local equilibria, which are a necessary condition for an equilibrium. Since the local reaction functions are still concave in the region where candidate \( R \) wins an expected plurality, a local equilibrium with \( P_D(\frac{1}{2}, 1) < \frac{1}{2} \) is unique over that portion of the strategy space. Note that with \( s > 0 \), \( \{c_l, c_m, c_r\} = \{0, 0, 0\} \) is always a local equilibrium.

\(^{12}\)See discussion in Benabou and Tirole (2006); for example Gneezy and Rustichini (2000) show in their article “A Fine is a Price” that instituting a fine can actually increase deviant behavior as it undermines social enforcement.
could result in a large partisan advantage. A social planner, however, might be interested in general citizen welfare; in the following subsection I examine the welfare implications of measures to increase turnout.

4.4 Costs and Benefits of Full Participation

There is no clear normative criterion that captures all relevant political and economic considerations of increasing voter turnout. Scholars such as Hill (2006) argue that voter turnout has independent value, since it is essential to democracy legitimacy. The precise point at which turnout is too low, however, is undefined. Others, such as Lijphart (1997), are concerned with equal representation.

Campbell (1999 p. 1200-1202) suggests two normative criteria: a democratic criterion based on the probability that the option preferred by a majority of citizens wins the election, and an economic criterion based on aggregate welfare. Under the democratic criterion, full turnout clearly outperforms partial turnout: with full turnout the majority always wins. Under the economic criterion it could be welfare optimal for a minority with relatively strong preferences to receive an expected plurality. While each citizen only has one vote, voluntary voting allows citizens with relatively strong preferences to skew the outcome in their favor by voting at a higher rate.

As this paper illustrates, increasing participation also influences the candidates’ choice of political position. Therefore, the relevant welfare comparison is between full turnout and convergence on one hand, and partial turnout and polarization on the other. At first glance full turnout might seem the best of both worlds: decreasing polarization and ensuring majority representation. As Fiorina (1999) notes, however, polarization has only recently gained a negative connotation:

At mid-century popular commentary often derided American politics as “issueless.” Candidates imitated each other with “me too” strategies. Prominent political scientists proposed institutional reforms intended to produce clear partisan differences...Today, popular commentary bemoans the polarization of American politics. Candidates sharply differentiate themselves from each other. Political scientists ponder institutional reforms designed to mute the strident voices that characterize politics today. (p. 28)

This quote illustrates a potential problem with full turnout: convergence removes voter choice by inducing candidates to choose identical political positions.

To analyze the tradeoff formally, I use the economic criterion introduced by Börgers (2004) and consider whether full turnout increases citizens’ ex ante utility (which, in my model, is equivalent to increasing aggregate utility). Using this metric, full turnout and political convergence outperforms partial turnout and polarization if:

\[
\frac{\lambda_m}{2} - (\lambda_l + \lambda_r - \lambda_m)v - \frac{c}{2} > \frac{(\lambda_r - \lambda_l)^2}{2\lambda_r}
\]

Convergence is optimal when the number of centrist citizens is high and when the cost of voting is low. Polarization does better when \(v\) is larger, since partisan voters have a smaller utility difference...
between \( g = \frac{1}{2} \) and the opposing partisan position, and when \((\lambda_r - \lambda_l)\) is larger, since both \(\lambda_r\) and the probability that type \(r\) gets their most preferred outcome are higher.

In Section 3, I showed that candidate convergence occurs in equilibrium when \(\lambda_m\) is large, and when \(v\) and \((\lambda_r - \lambda_l)\) are small, which is precisely when convergence is most likely to increase aggregate utility. It is not true, however, that the convergence conditions and the welfare conditions overlap perfectly. Therefore, measures to encourage voting can still improve economic welfare. This result contrasts starkly with Börgers (2004), who finds that full turnout is never welfare improving; when we also consider the impact on candidates’ political positions, full turnout can improve economic welfare in a large number of cases.

5 Conclusion

In this paper I formally model elections, considering both the candidates’ choices of political position and the citizens’ decisions to vote. I show that the Hotelling-Black median voter theorem, which predicts candidate convergence, does not hold in a model with diminishing intensity of political preferences and costly voting. I then use this model to study how electoral outcomes change with the introduction of measures to increase turnout. In contrast to previous literature on turnout and penalties for not voting, which takes candidate positions as fixed, I analyze the effect of decreasing the net expense of voting on both who wins the election and the candidates’ political positions.

Generally, measures to increase turnout decrease political polarization. If the cost of voting is made low enough, or no-vote penalties are high enough, then candidates converge at the median citizen’s ideal point. When the net expense of voting is made low enough, the strategic effect of turnout on candidates effectively disappears, returning the median voter result. Full turnout ensures that the candidate who is favored by the highest number of citizens wins the election, regardless of how diffuse their support is. Regarding the partisan effect of these measures, current party registration numbers in the United States suggest that the Democrats, who are greater in number but have a lower rate of turnout, would benefit from no-vote penalties. The partisan advantage would not be as large as direct extrapolation of registration numbers suggest, however, since the Republican party would respond by moving to the center and “poaching” some citizens who currently prefer the Democrats.

To ensure equal representation, Lijphart (1997) advocates increasing turnout in the US by either decreasing the cost of voting or penalizing non-voters. Which measure is used to increase turnout can be important; the two measures can have drastically different outcomes for small changes in the net expense of voting. Decreasing the cost of voting leads to a systematic decrease in polarization. Penalties for not voting, however, do not give predictable results; even a small fine could result in multiple equilibria, which in turn could lead to convergence and a large decrease in political polarization.

Penalties for not voting have not been enforced in the US since the 19th century. Instead, many states in the US are considering a switch to voting by mail. The results of this paper suggest that
broader adoption of voting by mail will decrease polarization in US politics. It is unclear, however, whether polarization in the US is too high. While polarization is often portrayed in a negative light by the current popular media, historically the reverse has been true. My analysis suggest that candidate convergence increases aggregate welfare when citizens preferences are close to linear, and when the number of centrist citizens is high. In Europe, the impact of decreasing the net expense of voting is less clear, since a proportional representation system is cannot always be approximated by a two-party model. My model does predict, however, that the parties with diffuse ideological support will benefit from the introduction of mandatory voting, or a decrease in the cost of voting.

A rigorous empirical test of whether decreasing the cost of voting decreases polarization would be desirable. A change at the federal level might allow such a test; the federal Motor Voter Act of 1993 stipulated that all states adopt certain measures to make voter registration easier. While this act did not have a discernible effect, since the change occurred at the federal level, it does not suffer from the same potential endogeneity as voting by mail, which has been introduced primarily in western states.

This paper adds to the formal literature on the effect of decreasing the net cost of voting on political outcomes by analyzing the impact on the candidates’ political positions. This approach could also be applied more broadly to the literature on voter turnout. For example, Goeree and Grosser (2007) and Taylor and Yilderim (2010b) analyze the effect of public information on voter turnout. It is quite possible, however, that the level of public information affects candidates’ positions as well; therefore, extending these models to include the candidate’s choice of political positions could provide a fuller understanding of the impact of measures such as banning pre-election polls.

6 Appendix A: Proofs

Proof of Proposition 1
This proof extends the proof of existence in Feddersen and Sandroni (2006a) to \( n \) groups, and proves uniqueness of the voting equilibrium.

Proof of Existence: The equilibrium at \( (\frac{1}{2}, \frac{1}{2}) \) is trivially \( \{c_l, c_m, c_r\} = \{0, 0, 0\} \). What follows deals with existence at other political positions.

Using the vote share notation \( (V_k) \) allows the probability that candidate \( D \) wins the election to be written as:

\[
P_D(c_l, c_m, c_r) = G\left(\frac{V_D}{V_R}\right).
\]

Where \( G \) is the distribution of \( \frac{q_R}{q_D} \):

\[
G\left(\frac{V_D}{V_R}\right) = \begin{cases} 
\frac{V_D}{2V_R} & \text{if } \frac{V_D}{V_R} \leq 1, \\
1 - \frac{V_R}{2V_D} & \text{if } \frac{V_D}{V_R} \geq 1.
\end{cases}
\]
The ex ante payoff function for type $l$ is:

$$\beta_l(g) G\left(\frac{V_D}{V_R}\right) - E[c_l|c_l]$$

And ex ante payoff function for type $r$ is:

$$\beta_r(g) \left(1 - G\left(\frac{V_D}{V_R}\right)\right) - E[c_r|c_r]$$

The payoff functions are concave over $(0, \bar{c}]$, but are not continuous at 0; for example, at the policy pair $(0, 1)$ assume $V_D > 0$ and $V_R = 0$. The payoff for type $l$ is 1 for all $V_D > 0$, but falls discontinuously to $\frac{1}{2}$ at $V_D = 0$.

To deal with this discontinuity, take a modification of the game where some motivated citizens always vote. In this game, called “$\epsilon$-election game” in Feddersen and Sandroni (2006a), $q_k \lambda_t(1 - \epsilon)$ citizens follow the optimal cutoff rule, $c_t$, and $q_k \lambda_t \epsilon$ vote with certainty; this modification is only used to prove existence in the voting game. In the $\epsilon$-election game, the payoff and best response functions are continuous over the convex and compact action space $[0, \bar{c}]$, which implies the existence of an equilibrium $\{c_{l,}\, c_{m,}\, c_{r,}\}$ for all policy pairs.

For any given policy pair, take an infinite sequence of $\epsilon$ values converging to zero. There exists a corresponding subsequence of $\{c_{l,}\, c_{m,}\, c_{r,}\}$ that converges to $\{c_{l,}^*, c_{m,}^*, c_{r,}^*\}$. $\{c_{l,}^*, c_{m,}^*, c_{r,}^*\}$ will be an equilibrium in the election game as long as $V_D$ and $V_R$ are greater than zero at $\{c_{l,}^*, c_{m,}^*, c_{r,}^*\}$. Assume for convenience that $c_{l,}^*$ is the largest cutoff cost among types that prefer candidate $D$, and $c_{r,}^*$ is the largest among types that prefer candidate $R$. A sufficient condition for the existence of equilibrium in the voting game is that $c_{l,}^*$ and $c_{r,}^*$ are always strictly greater than 0. This is a sufficient condition for existence even when there are $n$ types: as long as $V_D$ and $V_R$ are greater than zero in the limit as $\epsilon \to 0$, then an equilibrium exists.

Since the payoff functions are concave, the reaction functions for types $l$ and $r$ are characterized by the first-order conditions:

$$\beta_l(g) g\left(\frac{V_D}{V_R}\right) \frac{\lambda_l}{\bar{c}} \begin{cases} \leq 0 & \text{if } c_{l,}^* = 0, \\ = 0 & \text{if } c_{l,}^* \in (0, \bar{c}), \\ \geq 0 & \text{if } c_{l,}^* = \bar{c}. \end{cases}$$

And:

$$\beta_r(g) g\left(\frac{V_D}{V_R}\right) \frac{\lambda_r V_D}{V_R^2} \frac{c_{r,}^*}{\bar{c}} \begin{cases} \leq 0 & \text{if } c_{r,}^* = 0, \\ = 0 & \text{if } c_{r,}^* \in (0, \bar{c}), \\ \geq 0 & \text{if } c_{r,}^* = \bar{c}. \end{cases}$$

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Where:
\[
g\left(\frac{V_D}{V_R}\right) = \begin{cases} 
\frac{1}{2} & \text{if } \frac{V_D}{V_R} \leq 1, \\
\frac{V_R^2}{2V_D} & \text{if } \frac{V_D}{V_R} \geq 1.
\end{cases}
\]

Assume that \( c_l^* \) and \( c_r^* \) are equal to 0. Assume there exists an infinite subsequence where \( V_D \leq V_R \) for all values of \( \epsilon \); in this case the FOC\([c_l^*]\) converges to \( \infty \) as \( \epsilon \) approaches zero. This, however, contradicts the assumption that \( c_l^* \) converges to 0, since FOC\([c_l^*]\) > 0 in equilibrium only if \( c_l^* = \bar{c} \).

If no such subsequence exists, then there must exist a subsequence where \( V_D V_R \geq 1 \), which implies that FOC\([c_r^*]\) converges to \( \infty \) as \( \epsilon \) approaches zero and contradicts the assumption that \( c_r^* \) converges to 0.

Assume that \( c_l^* = 0 \) and \( c_r^* > 0 \). There is an infinite subsequence such that \( \frac{V_D}{V_R} \leq 1 \). This implies that \( \lim_{\epsilon \to 0} \text{FOC}[c_l^*] \) < 0. This is also a contradiction, since FOC\([c_l^*]\) < 0 implies that \( c_r^* \) converges to 0. The analogous argument holds for \( c_l^* > 0 \) and \( c_r^* = 0 \).

Together these show that \( V_D \) and \( V_R \) are bounded from zero in the limit as \( \epsilon \to 0 \).

**Proof of Uniqueness:**

[\( g = (0, 1) \):] \( P_D(c_l^*, c_m^*, c_r^*) = \frac{\lambda_l}{2n} \).

[\( g = \left(\frac{1}{2}, \frac{1}{2}\right) \):] \( P_D(c_l^*, c_m^*, c_r^*) = \frac{1}{2} \).

[\( g = (\frac{1}{2}, 1) \): First I show that there cannot exist multiple equilibria that give the same candidate an expected plurality. Second, I show that if candidate \( k \) wins an expected plurality in equilibrium, then no equilibrium exists where the opposing candidate wins an expected plurality.

The structure of \( g(.) \) entails that the left-hand-side of the reaction function is the same for types who prefer the candidate with an expected minority, and likewise for an expected majority. That is, the reaction function for a type who prefers the candidate with an expected minority is:

\[
\frac{\beta_t(g)}{2} \frac{\lambda_t}{V_{maj}} - c_t \begin{cases} 
\leq 0 & \text{if } c_t = 0, \\
= 0 & \text{if } c_t \in (0, \bar{c}), \\
\geq 0 & \text{if } c_t = \bar{c}.
\end{cases}
\]

And the reaction function for a type who prefers the candidate with an expected majority is:

\[
\frac{\beta_t(g)}{2} \frac{\lambda_t V_{min}}{V_{maj}^2} - c_t \begin{cases} 
\leq 0 & \text{if } c_t = 0, \\
= 0 & \text{if } c_t \in (0, \bar{c}), \\
\geq 0 & \text{if } c_t = \bar{c}.
\end{cases}
\]

The voting game has 2 components: the game between types who prefer the same candidate, and the game between types who prefer opposing candidates. Equations 3 and 4 characterize both these games. At \( g = (\frac{1}{2}, 1) \) types \( l \) and \( m \) both prefer candidate \( D \). Whether in the majority or
minority, for interior values, dividing the reaction functions of type \( m \) and type \( l \) gives:

\[
\frac{c_m}{c_l} = \frac{\beta_m(g) \lambda_m}{\beta_l(g) \lambda_l}
\]

This implies that the equilibrium value of \( c_m \) is a fixed proportion of \( c_l \). Corner solutions must be considered as well, but importantly the equilibrium value of \( c_m \) can be expressed as a function of the equilibrium value of \( c_l \) and the parameters of the model. Also importantly, the relationship between the cutoff costs of types who prefer the same candidate is weakly positive.

Since \( c_m \) is a function of \( c_l \), the equilibrium of the voting game can be characterized by the values of \( c_l \) and \( c_r \) that satisfy Equations 3 and 4. That is, the equilibrium of the game between types who prefer opposing candidates can be characterized by the strategies of two types who prefer opposing candidates. Equations 3 and 4 show that when candidate \( R \) has an expected majority, then \( c_l \) is a strategic complement, and \( c_r \) is a strategic substitute (and vice versa when candidate \( D \) has an expected majority).

Assume an equilibrium exists, \( \{c_l^*, c_{m}^{**}, c_{r}^{**}\} \), that gives candidate \( R \) an expected plurality (\( V_R^* > V_D^* \)). By contradiction, assume there exists another equilibrium, \( \{c_l^{**}, c_{m}^{**}, c_{r}^{**}\} \), with corresponding vote shares \( V_R^{**} \) and \( V_D^{**} \).

First I will show that \( V_R^{**} \) cannot be greater than or equal to \( V_D^{**} \). If \( c_r^{**} > c_r^* \), then \( V_D^{**} < V_D^{*} \). By equation 4, however, \( c_r^{**} \) cannot be a best response since \( c_r^{**} > c_r^* \) and \( V_D^{**} < V_D^{*} \). If \( c_r^{**} < c_r^* \), then \( V_D^{**} > V_D^{*} \). Again, by equation 4 \( c_r^{**} \) cannot be a best response. This shows that multiple equilibria cannot exist that give the same candidate an expected plurality.

Therefore, it must be that \( V_R^{**} < V_D^{**} \) and \( c_r^{**} < c_r^* \). Since \( V_R \) and \( V_D \) are continuous in the cutoff costs, there exists a \( c_r^{**} < c_r' < c_r^* \) such that \( V_R = V_D \), assuming \( c_l' \) and \( c_{m}' \) are best responses to \( c_r' \).

At \( \{c_l', c_{m}', c_r'\} \):

\[
\frac{\beta_r(g) \lambda_r}{2} \frac{\lambda_r}{V_D} - \frac{c_r'}{c} > 0
\]

For \( c_r < c_r' \), however, candidate \( R \) is the minority candidate, and type \( r \)'s best response function is defined by equation 3. Therefore, if \( c_r^{**} < c_r' \) and \( V_R^{**} < V_D^{**} \) then:

\[
\frac{\beta_r(g) \lambda_r}{2} \frac{\lambda_r}{V_D} - \frac{c_r^{**}}{c} > \frac{\beta_r(g) \lambda_r}{2} \frac{\lambda_r}{V_D} - \frac{c_r'}{c} > 0.
\]

By equation 3, \( c_r^{**} \) cannot be a best response. This shows that an equilibrium cannot exist that gives the opposing candidate an expected plurality and completes the proof.

The same proof hold with \( n \) types. The voting game can always be characterized by the cutoff costs of two types who prefer opposing candidates, and the relationship between the cutoff costs of types who prefer the same candidate is always weakly positive. Therefore, even with \( n \) types, the equilibrium of the voting game can be defined by two cutoff costs that satisfy equations 3 and 4.

Case \( (0, \frac{1}{2}) \): Analogous to \( (\frac{1}{2}, 1) \).
Proof of Lemma 1: First assume that the equilibrium cutoff costs are interior, so that the reaction functions are:

\[ \beta_l(g) \left( \frac{V_D}{V_R} \right) \frac{1}{V_R} \lambda_l - c_l = 0 \]  

(5)

And

\[ \beta_r(g) \left( \frac{V_D}{V_R} \right) \frac{V_D}{V_R^2} \lambda_r - c_r = 0 \]  

(6)

Dividing equation 6 by equation 5 gives:

\[ \frac{V_D}{V_R} = \frac{\beta_l(g) \lambda_l c_r}{\beta_r(g) \lambda_r c_l} \]  

(7)

Rearranging and plugging in for \( \beta_l(g) = \beta_r(g) \), \( V_D = \lambda_l c_l \), and \( V_R = \lambda_r c_r \) at \((0, 1)\) gives:

\[ \frac{c_l}{c_r} = \frac{c_r}{c_l} \]

Which implies that \( c_l = c_r \).

Now, assume that in equilibrium \( c_l = \bar{c} \), but that \( c_r < \bar{c} \). This implies that the left-hand-side of Equation 5 is greater than zero. Again, dividing the reaction functions and simplifying gives:

\[ \frac{c_l}{c_r} < \frac{c_r}{c_l} \]

This implies that \( c_l < c_r \), which is a contradiction. The same logic holds if we assume that \( c_l < \bar{c} \) and \( c_r = \bar{c} \).

Since \( c_l \) is always equal to \( c_r \) at \((0, 1)\):

\[ P_D(0, 1) = \frac{V_D}{2V_R} \equiv \frac{\lambda_l c_l}{2\lambda_r c_r} = \frac{\lambda_l}{2\lambda_r} \]

Proof of Proposition 2: I prove Proposition 2 by proving the following two results:

Result 1. If \( P_D(\frac{1}{2}, 1) \leq \frac{\lambda_l}{2\lambda_r} \) then \( P_R(0, \frac{1}{2}) \leq 1 - \frac{\lambda_l}{2\lambda_r} \).

Result 2. If \( P_D(\frac{1}{2}, 1) > \frac{1}{2} \) then \( P_D(0, \frac{1}{2}) < \frac{1}{2} \).

Given Lemma 1 and Results 1 and 2, the equilibrium of the candidates’ game can be determined by the value of \( P_D(\frac{1}{2}, 1) \):

- If \( P_D(\frac{1}{2}, 1) \leq \frac{\lambda_l}{2\lambda_r} \) then the equilibrium of the candidates’ game is \( g = (0, 1) \); candidate \( D \)’s probability of winning is higher at \( g = (0, 1) \) than at \( g = (\frac{1}{2}, 1) \) and, by Result 1, candidate \( R \)’s probability of winning is higher at \( g = (0, 1) \) than at \( g = (0, \frac{1}{2}) \).
• If $P_D(\frac{1}{2}, 1) \in (\frac{\lambda}{2x_r}, \frac{1}{2}]$ then the equilibrium of the candidates’ game is $g = (\frac{1}{2}, 1)$; candidate $D$’s probability of winning is higher at $g = (\frac{1}{2}, 1)$ than at $g = (0, 1)$ and candidate $R$’s probability of winning is higher at $g = (\frac{1}{2}, 1)$ than at $g = (\frac{1}{2}, \frac{1}{2})$

• If $P_D(\frac{1}{2}, 1) > \frac{1}{2}$ then the equilibrium of the candidates’ game is $g = (\frac{1}{2}, \frac{1}{2})$; candidate $R$’s probability of winning is higher at $g = (\frac{1}{2}, \frac{1}{2})$ than at $g = (\frac{1}{2}, 1)$ and, by Result 2, candidate $D$’s probability of winning is higher at $g = (\frac{1}{2}, \frac{1}{2})$ than at $g = (0, \frac{1}{2})$.

This shows that, given Results 1 and 2, a unique equilibrium of the candidates game exists for all values of $P_D(\frac{1}{2}, 1)$.

**Interior equilibria:** First I will prove both Results when all cutoff costs are interior at $g = (\frac{1}{2}, 1)$ and $g = (0, \frac{1}{2})$.

Since all cutoff costs are interior:

$$c_m = \frac{\beta_m(g)\lambda_m}{\beta_l(g)\lambda_l} c_t,$$

where $t = l$ for $g = (\frac{1}{2}, 1)$, and $t = r$ for $g = (0, \frac{1}{2})$.

Taking the case of $g = (\frac{1}{2}, 1)$, plugging in for $c_m$ in equation 7 allows us to solve for $\frac{c_r}{c_l}$:

$$\frac{c_r}{c_l} = \frac{\beta_r(g)^{1/2}}{\beta_l(g)\lambda_l} [\beta_l(g)\lambda_l^2 + \beta_m(g)\lambda_m^2]^{1/2}.$$

Plugging this solution back into Equation 7 (and plugging in for the values of $\beta_l(g)$ gives:

$$\frac{V_D}{V_R} = \left[\frac{(\frac{1}{2} - v)\lambda_l^2 + (\frac{1}{2} + v)\lambda_m^2}{(\frac{1}{2} + v)\lambda_l^2}\right]^{1/2} \text{ at } g = (\frac{1}{2}, 1)$$

Equations 8 and 9 allow us to prove Results 1 and 2 for interior equilibria.

**Result 1:** I will show that if $P_R(0, \frac{1}{2}) > 1 - \frac{\lambda}{2x_r}$ then $P_D(\frac{1}{2}, 1) > \frac{\lambda}{2x_r}$, which implies Result 1. By contradiction, assume that $P_R(0, \frac{1}{2}) > 1 - \frac{\lambda}{2x_r}$ and $P_D(\frac{1}{2}, 1) \leq \frac{\lambda}{2x_r}$.

By equation 9, since $P_R(0, \frac{1}{2}) > \frac{1}{2}$:

$$P_R(0, \frac{1}{2}) = 1 - \frac{1}{2} \left[\frac{(\frac{1}{2} + v)\lambda_l^2}{(\frac{1}{2} - v)\lambda_l^2 + (\frac{1}{2} + v)\lambda_m^2}\right]^{1/2}.$$
And since $P_R(0, \frac{1}{2}) > 1 - \frac{\lambda_l}{2\lambda_r}$ then:

$$1 - \frac{1}{2} \left[ \frac{\left( \frac{1}{2} + v \right) \lambda_l^2}{\left( \frac{1}{2} - v \right) \lambda_l^2 + \left( \frac{1}{2} + v \right) \lambda_m^2} \right]^{1/2} > 1 - \frac{\lambda_l}{2\lambda_r},$$

which simplifies to:

$$\left[ \frac{\left( \frac{1}{2} + v \right)}{\left( \frac{1}{2} - v \right) + \left( \frac{1}{2} + v \right) \frac{\lambda_m^2}{\lambda_l^2}} \right]^{1/2} < 1 \quad (10)$$

By equation 8, since $P_D(\frac{1}{2}, 1) < \frac{1}{2}$:

$$P_D(\frac{1}{2}, 1) = \frac{1}{2} \left[ \frac{\left( \frac{1}{2} - v \right) \lambda_l^2 + \left( \frac{1}{2} + v \right) \lambda_m^2}{\lambda_l^2} \right]^{1/2}$$

And since $P_D(\frac{1}{2}, 1) \leq \frac{\lambda_l}{2\lambda_r}$ then:

$$\frac{1}{2} \left[ \frac{\left( \frac{1}{2} - v \right) \lambda_l^2 + \left( \frac{1}{2} + v \right) \lambda_m^2}{\lambda_l^2} \right]^{1/2} \leq \frac{\lambda_l}{2\lambda_r},$$

which simplifies to:

$$\left[ \frac{\left( \frac{1}{2} + v \right)}{\left( \frac{1}{2} - v \right) + \left( \frac{1}{2} + v \right) \frac{\lambda_m^2}{\lambda_l^2}} \right]^{1/2} \geq 1$$

Since $\lambda_l < \lambda_r$, this equation implies that:

$$\left[ \frac{\left( \frac{1}{2} + v \right)}{\left( \frac{1}{2} - v \right) + \left( \frac{1}{2} + v \right) \frac{\lambda_m^2}{\lambda_l^2}} \right]^{1/2} \geq 1,$$

which contradicts equation 10.

**Result 2:** By contradiction, assume that $P_R(\frac{1}{2}, 1) < \frac{1}{2}$ and $P_D(0, \frac{1}{2}) \geq \frac{1}{2}$. By Equation 8, $P_R(\frac{1}{2}, 1) < \frac{1}{2}$ implies that:

$$P_R(\frac{1}{2}, 1) = \frac{1}{2} \left[ \frac{\left( \frac{1}{2} + v \right) \lambda_l^2}{\left( \frac{1}{2} - v \right) \lambda_l^2 + \left( \frac{1}{2} + v \right) \lambda_m^2} \right]^{1/2}. $$

Since $P_D(0, \frac{1}{2}) > \frac{1}{2}$, $P_D(0, \frac{1}{2}) = 1 - \frac{V_r}{2V_D} < \frac{V_r}{2V_D}$ which, by Equation 9, implies that:

$$P_D(0, \frac{1}{2}) < \frac{1}{2} \left[ \frac{\left( \frac{1}{2} + v \right) \lambda_l^2}{\left( \frac{1}{2} - v \right) \lambda_l^2 + \left( \frac{1}{2} + v \right) \lambda_m^2} \right]^{1/2}. $$

And since $\lambda_l < \lambda_r$, these equations imply that $P_D(0, \frac{1}{2}) < P_R(\frac{1}{2}, 1)$, which is a contradiction.

**Non-interior equilibria:** Now I must show that the same result holds for non-interior solutions. First, I show that when two types strictly prefer the same candidate, these two types set lower cutoff costs in equilibrium than the opposing type. Take $t_1 = r$ at $g = (\frac{1}{2}, 1)$ and $t_1 = l$ at
$g = (0, \frac{1}{2})$, so that $t_1$ prefers one candidate, and $t_2$ and $t_3$ prefer the opposing candidate.

**Fact 1:** If the equilibrium of the voting game at $g = (\frac{1}{2}, 1)$ or $g = (0, \frac{1}{2})$ is interior, then $c_{t_1} > c_{t_2}, c_{t_3}$.

Dividing the reaction function for type $t_1$ by the reaction function for type $t_j$ with $j \in \{2, 3\}$ and rearranging gives:

$$\frac{c_{t_1}}{c_{t_j}} = \left[ \frac{\beta_{t_1}(g)(1 + \beta_{t_j}(g)\lambda_{t_j}^2)}{\beta_{t_j}(g)\lambda_{t_j}^2} \right]^{\frac{1}{2}},$$

where $-j = \{1, 2\} \setminus j$. Since $\frac{\beta_{t_1}(g)}{\beta_{t_j}(g)}$ is greater or equal to one, $\frac{c_{t_1}}{c_{t_j}}$ is strictly greater than 1, which proves Fact 1.

For following part of the proof, it will be useful to consider comparative statics on $\bar{c}$. Therefore, for the remainder of the proof of Proposition 2, I label the parameter value of $\bar{c}$ as $\bar{c}_p$, and consider $\bar{c}$ as variable.

**Fact 2:** For $g = (\frac{1}{2}, 1)$ or $g = (0, \frac{1}{2})$: if the equilibrium of the voting game is not interior at $\bar{c}_p$, there exists a some $\bar{c}_p' > \bar{c}_p$ such that the equilibrium of the voting game is interior; if the equilibrium of the voting game is interior at $\bar{c}_p$, there exists a some $\bar{c}_p'' < \bar{c}_p$ such that $c_{t_1} = \bar{c}_p''$.

First I will show the existence of $\bar{c}_p'$. By the best response functions, $c_{t_1} = \bar{c}_p$ only if:

$$1 \leq \frac{\beta_{t_1}(g)V_{\min}}{2\lambda_{t_1}\bar{c}_p^2}$$

Since $V_{\min}$ bounded above by $\bar{c}(\lambda_{t_2} + \lambda_{t_3})$, there exists some $\bar{c}_p' > \bar{c}$ where the left-hand side of the inequality is smaller than one and the equilibrium of the voting game is interior.

$$1 \leq \frac{\beta_{t_1}(g)\lambda_{t_1}}{2V_{\maj}}$$

Since $V_{\maj} > \lambda_{t_1}\bar{c}$ ($t_1$ is the minority type), there exists some $\bar{c}_p' > \bar{c}$ where the left-hand side of the inequality is smaller than one and the equilibrium of the voting game is interior.

Next I will show the existence of $\bar{c}_p''$. If the equilibrium is interior then $\frac{c_{t_1}}{\bar{c}} \leq 1$, and by the best response functions:

$$1 \geq \frac{\beta_{t_1}(g)\lambda_{t_1}}{2V_{\maj}} \left[ \frac{V_{\min}}{V_{\maj}} \right]$$

Since $\left[ \frac{V_{\min}}{V_{\maj}} \right]$ is constant for an interior equilibrium and $V_{\maj} < \lambda_{t_1}$, however, there exists a $\bar{c}_p'' < \bar{c}$ such that the right-hand side of the inequality is greater than one and $c_{t_1} = \bar{c}_p''$. Similarly:

$$1 \geq \frac{\beta_{t_1}(g)\lambda_{t_1}}{2V_{\maj}}$$

Since $V_{\maj}$ bounded above by $\bar{c}(\lambda_{t_2} + \lambda_{t_3})$, there exists a $\bar{c}_p'' < \bar{c}$ such that the righthand side of the inequality is greater than one and $c_{t_1} = \bar{c}_p''$. This proves Fact 2.

Since the equilibrium cutoff costs are continuous in $\bar{c}$, Facts 1 and 2 show that if the equilibrium
of the voting game is non-interior, then \( t_1 = \bar{c} \). Returning to the proof for non-interior solutions, if \( c_r = \bar{c} \) at \( (\frac{1}{2}, 1) \), then at the equilibrium of the voting game:

\[
\frac{V_D}{V_R} = \left[ \frac{(\frac{1}{2} - v)\lambda_r^2 + (\frac{1}{2} + v)\lambda_m^2}{2c^2\lambda_r^2} \right]^{1/2} \quad \text{at } g = (\frac{1}{2}, 1) \tag{11}
\]

And if \( c_l = \bar{c} \) at \( (0, \frac{1}{2}) \), then:

\[
\frac{V_D}{V_R} = \left[ \frac{(\frac{1}{2} - v)\lambda_r^2 + (\frac{1}{2} + v)\lambda_m^2}{2c^2\lambda_r^2} \right]^{1/2} \quad \text{at } g = (0, \frac{1}{2}) \tag{12}
\]

Equations 11 and 12 have a very similar structure to the equations 8 and 9 (which were used to prove Results 1 and 2 for interior equilibria), swapping only \((\frac{1}{2} + v)\) for \(2c^2\) in the denominator of the term in brackets. It is therefore possible manipulate equations 11 and 12 to prove Results 1 and 2 just as Equations 8 and 9 were used to prove the Results for interior equilibria.

It remains to be shown that Results 1 and 2 hold when one equilibrium of the voting game is interior. Fact 2 shows that for some \( \bar{c} \), the equilibria of the voting game will be non-interior for both \( g = (\frac{1}{2}, 1) \) and \( g = (0, \frac{1}{2}) \). Take \( \bar{c}_y \) to be the largest \( \bar{c} \) such that both equilibria are non-interior. Also, for some \( \bar{c} \) greater than \( \bar{c}_y \), the equilibria of the voting game is interior for both \( g = (\frac{1}{2}, 1) \) and \( g = (0, \frac{1}{2}) \). Take \( \bar{c}_x \) to be the smallest \( \bar{c} \) such that both equilibria are interior.

For \( \bar{c} \in (\bar{c}_y, \bar{c}_x) \) the equilibrium of the voting game is either interior for \( g = (\frac{1}{2}, 1) \) and non-interior for \( g = (0, \frac{1}{2}) \), or non-interior for \( g = (\frac{1}{2}, 1) \) and interior for \( g = (0, \frac{1}{2}) \). Consider the case where the equilibrium of the voting game is interior for \( g = (\frac{1}{2}, 1) \) and non-interior for \( g = (0, \frac{1}{2}) \). For \( \bar{c} \in (\bar{c}_y, \bar{c}_x) \), \( P_D(\frac{1}{2}, 1) \) is constant and by equation 12, \( P_D(0, \frac{1}{2}) \) is decreasing. Therefore, since Results 1 and 2 hold for \( \bar{c}_y \) and \( \bar{c}_x \), they also hold for \( \bar{c} \in (\bar{c}_y, \bar{c}_x) \). The case where the equilibrium of the voting game is non-interior for \( g = (\frac{1}{2}, 1) \) and interior for \( g = (0, \frac{1}{2}) \) is analogous.

\[\Box\]

**Proof of Proposition 3:** As I show in the proof of Proposition 2, the equilibrium of the candidates’ game depends on the value of \( P_D(\frac{1}{2}, 1) \): if \( P_D(\frac{1}{2}, 1) \leq \frac{\lambda_r + \lambda_m}{2\lambda_r} \), then \( g^* = (0, 1) \); if \( P_D(\frac{1}{2}, 1) \in (\frac{\lambda_r + \lambda_m}{2\lambda_r}, \frac{\lambda_r}{2\lambda_r}) \), then \( g^* = (\frac{1}{2}, 1) \); and if \( P_D(\frac{1}{2}, 1) > \frac{\lambda_r}{2\lambda_r} \), then \( g^* = (\frac{1}{2}, \frac{1}{2}) \).

First, I will show that \( P_D(\frac{1}{2}, 1) = \frac{\lambda_r}{2\lambda_r} \) for some \( \bar{c}_x \), and that \( P_D(\frac{1}{2}, 1) = \hat{P}_D(\frac{1}{2}, 1) \) for some \( \bar{c}_y > \bar{c}_x \). Then I will show that \( \partial P_D(\frac{1}{2}, 1)/\partial \bar{c} \leq 0 \). Together, these imply that \( P_D(\frac{1}{2}, 1) \) decreases continuously from \( \frac{\lambda_r}{2\lambda_r} \) to \( \hat{P}_D(\frac{1}{2}, 1) \) as \( \bar{c} \) increases from \( \bar{c}_x \) to \( \bar{c}_y \), which proves the Proposition.

Fact 2 from the proof of Proposition 2 implies that the equilibrium of the voting game is interior for some \( \bar{c}' \) high enough. Since the equilibrium at \( \bar{c}' \) is interior, \( P_D(\frac{1}{2}, 1) = \hat{P}_D(\frac{1}{2}, 1) \) (this proves the existence of a \( \bar{c}_y \), but I define \( \bar{c}_y \) more precisely below). Also by Fact 2, there exists a \( \bar{c} < \bar{c}' \) such that \( c_r = \bar{c} \). Take \( \bar{c}_y \) to be the maximum \( \bar{c} \) such that \( c_r = \bar{c} \). By continuity, at \( \bar{c}_y \) \( P_D(\frac{1}{2}, 1) = \hat{P}_D(\frac{1}{2}, 1) \).
When \( c_r = \bar{c} \) the best response functions of types \( l \) and \( m \) simplify to:

\[
c_t = \max \left\{ \bar{c}, \frac{\beta_l(g)\lambda_l}{\lambda_r} \right\}
\]

This equation shows that \( c_l = c_m = \bar{c} \) for some \( \bar{c}_x < \bar{c}_y \). Since all motivated citizens vote at \( \bar{c}_x \),

\[
P_D(\frac{1}{2}, 1) = \frac{\lambda_l + \lambda_m}{2\lambda_r}.
\]

When \( c_r = \bar{c} \), equation 11 shows that \( \partial P_D(\frac{1}{2}, 1)/\partial \bar{c} \leq 0 \). Therefore, \( \partial P_D(\frac{1}{2}, 1)/\partial \bar{c} \leq 0 \) for \( \bar{c} < \bar{c}_y \). Lastly, since \( \bar{c}_y \) is the maximum \( \bar{c} \) such that \( c_r = \bar{c} \), all cutoffs costs are interior and \( P_D(\frac{1}{2}, 1) = P_D(\frac{1}{2}, 1) \) for \( \bar{c} > \bar{c}_y \).

\[\diamondsuit\]

Proof of Proposition 4: \([s_1]\) Propositions 1 and 2 and Lemma 1 hold for \( s < 0 \). Take \( s^* < s \leq -\left(\frac{1}{2} - v\right) \). If \( g = (\frac{1}{2}, 1) \) then \( c_l = 0 \), since \( \beta_l(\frac{1}{2}, 1) = (\frac{1}{2} - v) \) is lower than the lowest net cost of voting. Since no type \( l \) votes, \( P_D(\frac{1}{2}, 1) = \frac{\lambda_m}{2\lambda_r} \). Therefore, for \( s^* < s \leq -\left(\frac{1}{2} - v\right) \) \( P_D(\frac{1}{2}, 1) \) is strictly lower than \( P_D(0, 1) \), and the equilibrium of the candidates’ game is \( g = (0, 1) \). Since \( P_D(\frac{1}{2}, 1) \) is continuous in \( s \), there exists some maximum \( s_1 > -\left(\frac{1}{2} - v\right) \) such that for all \( s \in (s^*, s_1) \) the equilibrium of the candidates’ game is \( g = (0, 1) \).

\[\diamondsuit\]

\([s_2]\) Take \( s' < \bar{c} \) such that \( (\lambda_l + \lambda_m)s' = \lambda_r \bar{c} \). Such an \( s \) exists since \( \lambda_l + \lambda_m > \lambda_r \). For all \( s \in (s', \bar{c}) \) \( P_D(\frac{1}{2}, 1) \) is strictly greater than \( P_D(\frac{1}{2}, \frac{1}{2}) \), and the equilibrium of the candidates’ game is \( g = (\frac{1}{2}, \frac{1}{2}) \). Again by continuity, there exists some minimum \( s_2 < s' \) such that for all \( s \in (s_2, \infty) \) the equilibrium of the candidates’ game is \( g = (\frac{1}{2}, \frac{1}{2}) \).

\[\diamondsuit\]

7 Appendix B

7.1 The case for diminishing intensity of political preferences:

Diminishing intensity of political preferences translates into citizen utility functions over the political space that are convex, while increasing intensity of political preferences implies concavity. Increasing intensity of political preference (concave utility) implies that citizens with extreme preferences are very sensitive to differences in moderate candidates. Because of this thought experiment, leading scholars in the area of voting such as Osborne (1995) have expressed doubt as to whether concave utility is the appropriate assumption. The distinction between concave and convex utility (given fixed candidate positions), however, is empirically mute in most elections since voters choose between only two viable candidates.\(^{13}\) Congressional and presidential elections in US provide an exception, however, since parties use primary elections to choose which candidates will stand in the general election. With this two-stage election procedure, the shape of utility is empirical relevant to voting patterns, even assuming fixed candidate positions.

\(^{14}\)See Cox (1994) for a formal discussion of Duverger’s law.
I construct a thought experiment which asks whether voting patterns in primary elections are consistent with convex or concave utility. Consider the following stylized example of a citizen, \( i \), with a political ideal point in the left of the political space who is participating in the primary elections. The voter can vote in either the Republican or the Democratic primary, but only in one.\(^\text{14}\) The Democratic candidates, \( \{A, B\} \), are the same distance apart as the Republican candidates, \( \{C, D\} \). Assume, for the purpose of illustration, that regardless of who contests the general election, the Democratic and Republican candidates have the same chance of winning. This example is illustrated below in Figure 1:

**Figure 1:**

![Diagram](image)

If voter \( i \) has concave utility, as illustrated above, then the outcome of the Republican primary is more important to \( i \) than the outcome of the Democratic primary (\( x < y \) above). Therefore, if \( i \)’s vote carries equal weight in both primaries, then \( i \) will choose to vote in the Republican primary, and vote for the moderate Republican candidate.

While this is a very stylized example, the same logic would hold in a more fully specified model of elections with primaries. If citizens have increasing intensity of political preferences, then a significant proportion of partisan citizens would “hedge” in primary elections by voting for a more moderate candidate in the opposing partisan primary election when both the primary election and the general election is competitive. This result is not consistent with the empirical evidence on crossover voting. Voters in primary elections crossover when their first choice candidate is in the opposing party’s primary, or when a vote in the opposing primary is considered to carry greater weight (Alvarez and Nagler (1997)). For example, crossover voting by Democrats occurs if a candidate is a “shoe-in” for the Democratic nomination, or if the Republican party’s candidate is considered a shoe-in for the general election, making a vote in the Democratic primary superfluous (a recent legal decision in Idaho, where the Republican candidate almost always wins, closed primaries to prevent this type of crossover voting). Therefore, while there is evidence of strategic crossover voting, it is not consistent with the hedging in competitive elections that concave preferences

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\(^\text{14}\) Which is the case in most US states; some states even hold open primary elections, which do not require a citizen to be registered for a party to vote in that party’s primary.
7.2 Model Robustness:

In the above analysis, I have assumed that candidates choose to run either as partisans or centrists. A more realistic model would include a more dense political space, but sacrifices analytical ease and the ability to unambiguously compare convexity between different utility functions. Additionally, a more dense political space results in nonexistence of an equilibrium in pure strategies in the candidates’ game for certain parameter values. The proofs of Propositions 3 and 4, however, show that the main result of political convergence holds even with a continuous political space.

Take the following modifications to the model: candidate $D$ sets $g_D \in [0, \frac{1}{2}]$, and candidate $R$ sets $g_R \in [\frac{1}{2}, 1]$. Also, $u_i(|\hat{g} - \eta_i|)$ is defined over $[0, 1]$, and is decreasing, differentiable and convex.

**Corollary 1.** There exists a $\bar{c}_1$ such that the unique equilibrium of the candidates’ game is $g = (\frac{1}{2}, \frac{1}{2})$ for all $\bar{c} \in [0, \bar{c}_1]$.

There exists a subsidy, $s_2$, such that the unique equilibrium of the candidates’ game is $g = (\frac{1}{2}, \frac{1}{2})$ for all $s \geq s_2$.

The proof of Proposition 3 shows that as long as $g_D = \frac{1}{2}$ and $g_R > \frac{1}{2}$ there exists a $\bar{c}_1$ so that candidate $D$ wins an expected plurality for all $\bar{c}$ lower than $\bar{c}_1$. The same is true for if $g_D < \frac{1}{2}$ and $g_R = \frac{1}{2}$. This implies that $g = (\frac{1}{2}, \frac{1}{2})$ is an equilibrium for $\bar{c} < \bar{c}_1$, and that for any $g \neq (\frac{1}{2}, \frac{1}{2})$ at least one candidate has a best response to deviate to $g_k = \frac{1}{2}$. The proof of $s_2$ follows directly from the proof in Proposition 4.